On the reals weakly low for K

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April 27, 2015

(Weak) lowness for K

Definition

- A real x is *low for K* if $\overline{\lim}_{n\to+\infty} K(n) - K^x(n) < +\infty$.
- A real x is *weakly low for K* if $\underline{\lim}_{n\to+\infty} K(n) - K^x(n) < +\infty$.

Preliminary results (I)

Theorem (Hirschfedlt, Nies and Stephan)

A real x is low for K if and only if $\overline{\lim}_{n \to +\infty} K(x \upharpoonright n) - K(n) < +\infty$ *(or x is K-trivial).*

Preliminary results (I)

Theorem (Hirschfedlt, Nies and Stephan)

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Theorem (Miller)

A real x is weakly low for K if and only if Ω is 1-x-random (or x is low for Ω).

Preliminary results (II)

Ample Excess Lemma

Lemma (Miller and Y)

A real r is 1-random if and only if $\exists c \forall n (K(r \upharpoonright n) - n \geq K^r(n) - c)$.

Coding Theorem

Theorem (Chaitin and Levin)

$\overline{\lim}_{n \to +\infty} \sum_{\sigma \in 2^n} 2^{-K(\sigma)} - 2^{-K(n)} < +\infty.$

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Are they really powerless over a real?

Low for K along a real

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Is there a non K-trivial real low for K along a real?

Triviality

Theorem (Merkle and Y)

x is K-trivial if and only if x is low for K along a real.

Proof.

If $K(n) - K^x(n) > d$ at stage s, then put $(K(\sigma)[s] - d, \sigma)$ into M^x for any $\sigma \in 2^n$ unless $\sum_{\sigma \in 2^n} 2^{-K(\sigma)[s]+d} > 2^{-K^x(n)[s]}$.
By the coding theorem, $K_{M^x}(\sigma) < K(\sigma) - d$ for any $\sigma \in 2^n$. □

Weakly low for K along a real

Definition

A real x is weakly low for K along a real z if $\underline{\lim}_{n\to+\infty} K(z \upharpoonright n) - K^x(z \upharpoonright n) < +\infty$.

Theorem (Merkle and Y)

A real x is weakly low for K if and only if the set of reals z so that x is weakly low for K reals along z has measure 1.

Proof.

Applying the Coding Theorem. □

A case study

Let $S_n = \{\sigma \in 2^\omega \mid \exists\tau(\sigma = \tau^\frown 1^\frown 0^{n\frown} 1)\}$.

Theorem (Merkle and Y)

x is not K-trivial if and only if for any c and m, there are some $n \geq m$ and $\sigma \prec \Omega$ in S_n so that $K^x(\sigma) \leq K(\sigma) - c$.

Lemma

If $T \subseteq 2^{<\omega}$ is a $\emptyset'$-recursive tree having infinitely many infinite paths, then for any c and uniformly recursive sequence disjoint infinite sets $\{S_n\}_{n\in\omega}$, there is some m such that for any $n \geq m$ there is some i such that for any $k \geq i$ in S_n, there is some $x \in [T]$ so that $K^x(k) < K(k) - c$.

A question

Question

Is it true that for any weakly low for K real x,
$\underline{\lim}_{n \to +\infty}(K(\Omega \upharpoonright n) - K^x(\Omega \upharpoonright n) < +\infty)$*?*

On scatted sets (1)

Theorem (Merkle and Y)

1. *For any infinite set* A, $\{x \mid \lim_{m\in A} K(m) - K^x(m) = +\infty\}$ *is null.*
2. *For any infinite set* A, $\{x \mid \overline{\lim}_{n\in A} K(n) - K^x(n) = +\infty\}$ *is conull.*

Proof.

For (1). Let $\tilde{K}(m) = \min\{n \mid \mu(\{x \mid K^x(m) \geq n\}) < \frac{1}{4}\}$. Then $\overline{\lim}_m K(m) - \tilde{K}(m) = +\infty$. Let $B_m = \{(x,y) \mid \exists k \leq \tilde{K}(m)(U^x(y \upharpoonright k) = m)\}$ and $B = \bigcup_m B_m$. For $m_0 \neq m_1$, $B_{m_0} \cap B_{m_1} = \emptyset$. Moreover by the Fubini theorem, for every m, $\mu(B_m) \geq \int_{K^x(m)<\tilde{K}(m)} 2^{-\tilde{K}(m)} dx \geq \frac{3}{4} \cdot 2^{-\tilde{K}(m)}$.

Then $1 \geq \mu(B) = \sum_m \mu(B_m) \geq \sum_m \frac{3}{4} \cdot 2^{-\tilde{K}(m)}$. So $2^{-\tilde{K}(m)} < +\infty$, a contradiction.

□

On scatted sets (2)

Proof.

For (2). A simpler 2^c-proof plus Cantor–Cantelli lemma. □

On uncountability

A natural question is whether the set in (2) can be co-countable?

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Theorem (Merkle and Y)

There is an infinite set A so that the set $R_A = \{x \mid \exists c \forall n \in A (K^x(n) \geq K(n) - c)\}$ is uncountable.

Proof.

By (1) and Shoenfield absolutenss via Mathias Forcing. □

WLK-reduction

For any x and c, let $A_{x,c} = \{n \mid K^x(n) \geq K(n) - c\}$.

Definition

$x \geq_{WLK} y$ if for any constant c, there is a constant d so that $A_{x,c} \subseteq A_{y,d}$.

So if x is weakly low for K, then $y \leq_{LK} x \implies y \leq_{WLK} x$.

Countability of *WLK*-degrees

Theorem (Merkle and Y)

If $x \geq_{WLK} y$ *and is weakly low for* K*, then* $x' \geq_{LK} y'$*.*

A question

Question

Is it true that if x is weakly low for K, then $y \leq_{WLK} x \implies y \leq_{LK} x$?

An application to non-gap theory

Theorem (Miller)

x is 2-random if and only if $\underline{\lim}_{n \to +\infty} n + K(n) - K(x \upharpoonright n) < +\infty$.

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Theorem (Merkle and Y)

For any f with $\overline{\lim}_n K(n) - f(n) = +\infty$, there is a weakly-2-random real x which is not 2-random but $\overline{\lim}_{n\to+\infty} K(x\upharpoonright n) - n - f(n) > -\infty$.

Proof.

Applying random forcing and (1). □

Finish