Brownian loops and conformal fields

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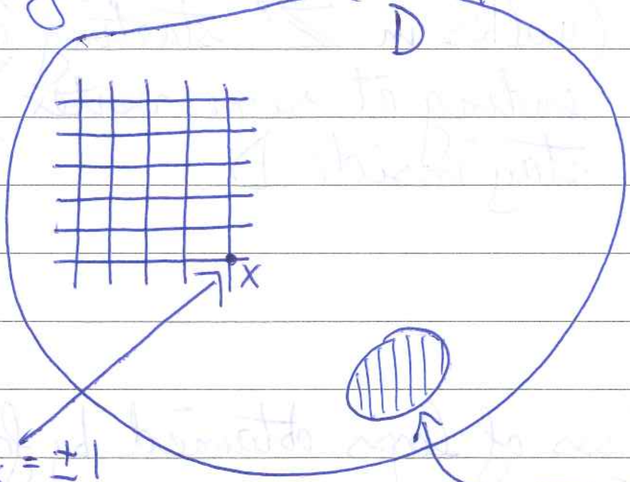
Conformal field theories are special field theories which enjoy a particular type of symmetry called conformal symmetry.

Field theories are very useful in physics
- → Particle physics (Standard Model)
- → Condensed matter physics (e.g., large scale properties of critical and off-critical systems)

Example from statistical mechanics: 2D critical Ising model (on square lattice, $\mathbb{Z}^2$)



$S_x = \pm 1$ spin at x

Continuum scaling limit: replace $\mathbb{Z}^2$ by $\delta\,\mathbb{Z}^2$ and let $\delta \to 0$.

$$\{S_x\}_{x \in \delta\mathbb{Z}^2 \cap D} \xrightarrow{\delta \to 0} ?$$

$\sum_x S_x$ local magnetization (sum of all spins in a "mesoscopic" region)

$$\delta^{15/8} \sum_{x \in \delta\mathbb{Z}^2} S_x \delta_x \xrightarrow{\delta \to 0} \text{"conformal field" (random distribution)}$$

↑ Dirac delta at x

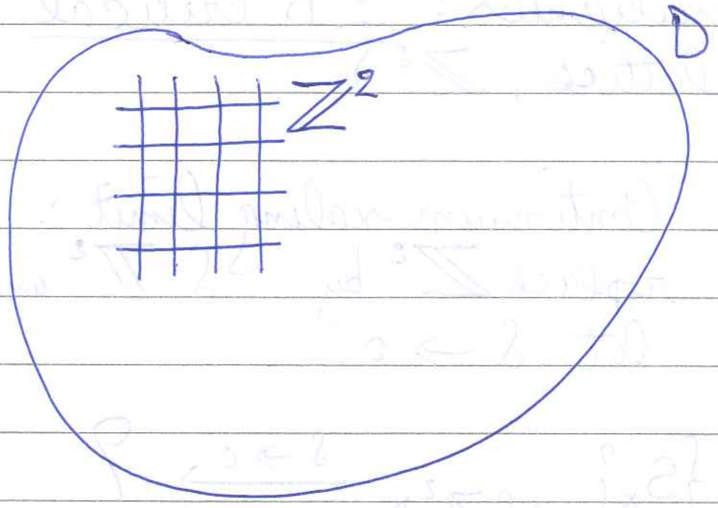
(Camia, Garban, Newman; 2012)

Rest of talk: family of conformal fields built from Brownian motion.

Why are they interesting?

- Natural construction involving Brownian motion.
- New fields with unusual properties.
- Connections with Conformal Loop Ensembles (CLEs) and Schramm-Loewner Evolution (SLE).
- Amenable to rigorous mathematical analysis.

Random Walk Loop Soup



Consider all loops $\tilde{\gamma}$ (walks in $\mathbb{Z}^2$ starting and ending at same vertex) that stay inside D.

Unrooted loop γ: equivalence class of loops obtained by forgetting the starting point

Measure ν_D on unrooted loops: $\nu_D(\gamma) = \ell_\gamma \frac{1}{|\gamma|}\left(\frac{1}{4}\right)^{|\gamma|} \mathbb{1}_{\{\gamma \subset D\}}$

$|\gamma|$: length of loop (number of steps)

ℓ_γ: number of loops in γ

RWLS in D with intensity λ:

Take a collection $\{N_\gamma\}_\gamma$ of random variables, one for each unrooted loop γ in D.

N_γ has Poisson distribution with intensity $\lambda \nu_D(\gamma)$:

$$P(N_\gamma = n) = \frac{1}{n!} e^{-\lambda \nu_D(\gamma)} (\lambda \nu_D(\gamma))^n.$$

N_γ represents the multiplicity of γ in the Random Walk Loop Soup.

Brownian Loop Soup in D with intensity λ:

Take scaling limit of RWLS (Brownian scaling):

$$\begin{array}{c} \text{RWLS in} \\ D \cap \delta \mathbb{Z}^2 \text{ with int. } \lambda \end{array} \xrightarrow{\delta \to 0} \begin{array}{c} \text{BLS in D} \\ \text{with int. } \lambda \end{array}$$

Let Γ denote the collection of loops of a BLS in D with intensity λ.

Goal: I want to study the "field" $e^{i\beta N_\omega^{(\delta)}(z)}$, where

$z \in D$, $\beta \in (0, 2\pi)$ and $N_\omega^{(\delta)}(z) = \sum_{\gamma \in \Gamma} \theta_\gamma(z)$

$\delta > 0$ cutoff $\longrightarrow$ diam$(\gamma) > \delta$

winding number of γ around z

Scaling limit? $(\delta \to 0)$

Comments:

- BLS closely related to SLE
- winding of BM in 2D is classical problem with long history

Theorem (C., Gandolfi, Kleban)

If $n \in \mathbb{N}$, $D \subset \mathbb{C}$ is bounded and $\beta_1, \dots, \beta_n \in (0, 2\pi)$, then the limit

$$\lim_{\delta \to 0} \frac{\langle e^{i\beta_1 N_w^\delta(z_1)} \cdot \ldots \cdot e^{i\beta_n N_w^\delta(z_n)} \rangle}{\prod_{j=1}^{n} \delta^{2\Delta(\beta_j)}} \equiv \phi_D(z_1, \dots, z_n; \beta_1, \dots, \beta_n)$$

exists and is finite and nontrivial iff $\Delta(\beta_j) = \frac{\lambda \beta_j (2\pi - \beta_j)}{8\pi^2}$.
($\langle \ldots \rangle$ denotes expectation w.r.t. the BLS.)
Moreover, if D' is another bounded domain and $f: D \to D'$ is a conformal map, then

$$\phi_{D'}(f(z_1), \dots, f(z_n); \beta_1, \dots, \beta_n) = \prod_{j=1}^{n} |f'(z_j)|^{-2\Delta(\beta_j)} \phi_D(z_1, \dots, z_n; \beta_1, \dots, \beta_n).$$

Theorem (C., Lis)

Let $V_\beta^\delta(z) = \delta^{-2\Delta(\beta)} e^{i\beta N_w^\delta(z)}$, with $\Delta(\beta) = \frac{\lambda\beta(2\pi - \beta)}{8\pi^2}$,

and $\langle V_\beta^\delta, \varphi \rangle = \int_D V_\beta^\delta(z)\varphi(z)\, dz$. If D is bounded, ∂D is smooth, $\Delta < \frac{1}{2}$, then $\forall \alpha > 1$, as $\delta \to 0$, V_β^δ converges in second mean in the Sobolev space $\mathcal{H}^{-\alpha}(D)$ to a random distribution $V_\beta \in \mathcal{H}^{-\alpha}(D)$.