Ballisticity for Random walks in degenerate random environments

Mark Holmes
(Joint work with Tom Salisbury)

Example: Orthant model in 2 dimensions

At each intersection of a grid like city, put a sign post, and toss a p-coin

Example: Orthant model in 2 dimensions

At each intersection of a grid like city, put a sign post, and toss a p-coin

- if the coin shows heads, add signs/arrows pointing north and east, i.e. $\uparrow\!\!\!\!\lrcorner\!\!\rightarrow$

Example: Orthant model in 2 dimensions

At each intersection of a grid like city, put a sign post, and toss a p-coin

- if the coin shows heads, add signs/arrows pointing north and east, i.e. $\uparrow\rightarrow$
- otherwise add signs pointing south and west, i.e. $\leftarrow\downarrow$

Example: Orthant model in 2 dimensions

At each intersection of a grid like city, put a sign post, and toss a $\mathfrak{p}$-coin

- if the coin shows heads, add signs/arrows pointing north and east, i.e. $\uparrow\!\!\!\rightarrow$
- otherwise add signs pointing south and west, i.e. $\leftarrow\!\!\!\downarrow$
- what parts $\mathcal{C}_o$ of the city can you reach by following signs?

Example: Orthant model in 2 dimensions

At each intersection of a grid like city, put a sign post, and toss a p-coin

- if the coin shows heads, add signs/arrows pointing north and east, i.e. $\uparrow\!\!\!\lrcorner\!\rightarrow$
- otherwise add signs pointing south and west, i.e. $\leftarrow\!\!\urcorner\!\downarrow$
- what parts $\mathcal{C}_o$ of the city can you reach by following signs? (connectivity of random directed graph)

Example: Orthant model in 2 dimensions

At each intersection of a grid like city, put a sign post, and toss a p-coin

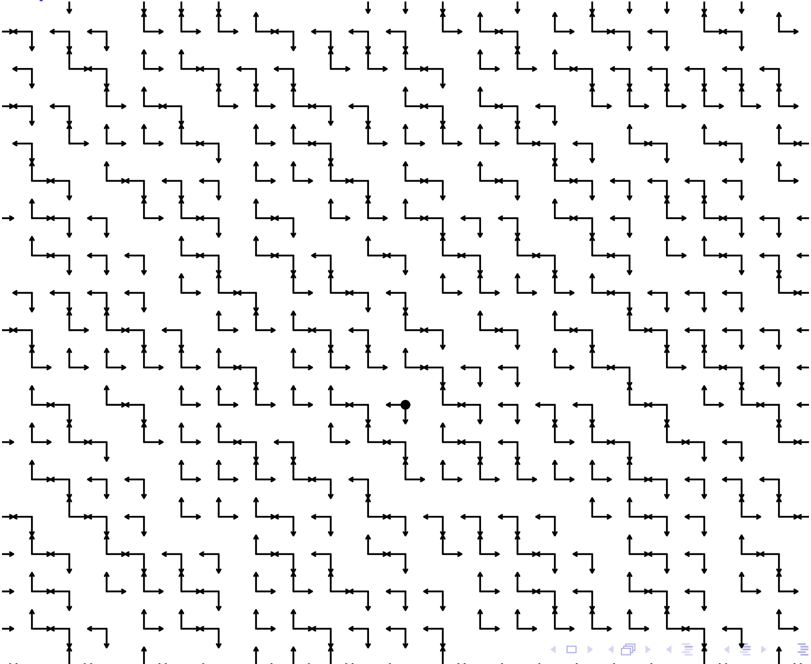
- if the coin shows heads, add signs/arrows pointing north and east, i.e. $\uparrow\rightarrow$
- otherwise add signs pointing south and west, i.e. $\leftarrow\downarrow$
- what parts $\mathcal{C}_o$ of the city can you reach by following signs? (connectivity of random directed graph)
- if you choose to follow signs uniformly at random, what will your journey look like?

Example: Orthant model in 2 dimensions

At each intersection of a grid like city, put a sign post, and toss a p-coin

- if the coin shows heads, add signs/arrows pointing north and east, i.e. $\uparrow\!\!\rightarrow$
- otherwise add signs pointing south and west, i.e. $\leftarrow\!\!\downarrow$
- what parts $\mathcal{C}_o$ of the city can you reach by following signs? (connectivity of random directed graph)
- if you choose to follow signs uniformly at random, what will your journey look like? (random walk in random environment)

Example continued



Example continued

- Fix $p > p_c \approx .5956$ and $\ell = (1, 1)$.

Example continued

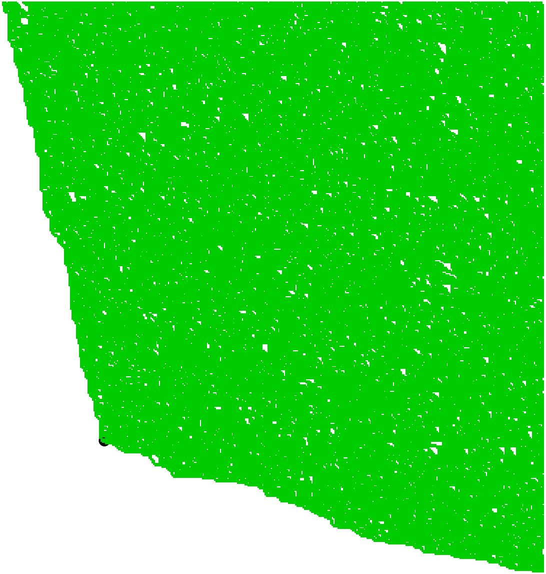
- Fix $p > p_c \approx .5956$ and $\ell = (1, 1)$. Then $\exists$ cone

$$H_{\kappa,\ell} = \{u \in \mathbb{R}^d : u \cdot \ell \geqslant \kappa \|u\|\}$$

Example continued

- Fix $p > p_c \approx .5956$ and $\ell = (1,1)$. Then $\exists$ cone
$$H_{\kappa,\ell} = \{u \in \mathbb{R}^d : u \cdot \ell \geqslant \kappa \|u\|\}$$
such that $\mathbb{Q}(\cup_n \{\mathcal{C}_o \subset -n\ell + H_{\kappa,\ell}\}) = 1$.

Example, $(\upharpoonleft\!\!\llcorner, \urcorner\!\!\downarrow)$, $p = .8$



Techniques for studying the graph:

- Networks of coalescing random walks

Techniques for studying the graph:

- Networks of coalescing random walks
- Duality

Techniques for studying the graph:

- Networks of coalescing random walks
- Duality
- Coupling:

Techniques for studying the graph:

- Networks of coalescing random walks
- Duality
- Coupling:
  Let U_x be i.i.d. standard uniform random variables,

Techniques for studying the graph:

- Networks of coalescing random walks
- Duality
- Coupling:
  Let U_x be i.i.d. standard uniform random variables, and let

$$\mathcal{G}_x(p) = \begin{cases} \begin{smallmatrix}\uparrow & \\ & \rightarrow\end{smallmatrix}, & \text{if } U_x < p \\ \begin{smallmatrix}\leftarrow & \\ & \downarrow\end{smallmatrix}, & \text{otherwise.} \end{cases}$$

Techniques for studying the graph:

- Networks of coalescing random walks
- Duality
- Coupling:
  Let U_x be i.i.d. standard uniform random variables, and let

$$\mathcal{G}_x(p) = \begin{cases} \uparrow\!\!\!\!\lrcorner\!\!\rightarrow, & \text{if } U_x < p \\ \leftarrow\!\!\!\!\urcorner\!\!\downarrow, & \text{otherwise.} \end{cases}$$

  This defines a random graph $\mathcal{G}$ for all values of p simultaneously.

Random walk in random environment:

- $\mathcal{E} = \{\pm e_1, \dots, \pm e_d\}$

Random walk in random environment:

- $\mathcal{E} = \{\pm e_1, \ldots, \pm e_d\}$
- $\omega = \{\omega(x, e) : x \in \mathbb{Z}^d, e \in \mathcal{E}\}$ environment

Random walk in random environment:

- $\mathcal{E} = \{\pm e_1, \ldots, \pm e_d\}$
- $\omega = \{\omega(x, e) : x \in \mathbb{Z}^d, e \in \mathcal{E}\}$ environment
- $\omega(x, e) \geqslant 0$ and $\sum_{e \in \mathcal{E}} \omega(x, e) = 1$

Random walk in random environment:

- $\mathcal{E} = \{\pm e_1, \ldots, \pm e_d\}$
- $\omega = \{\omega(x, e) : x \in \mathbb{Z}^d, e \in \mathcal{E}\}$ environment
- $\omega(x, e) \geqslant 0$ and $\sum_{e \in \mathcal{E}} \omega(x, e) = 1$
- Random walk X in environment ω is Markov chain with $p(x, x + e) \equiv \omega(x, e)$

Random walk in random environment:

- $\mathcal{E} = \{\pm e_1, \dots, \pm e_d\}$
- $\omega = \{\omega(x, e) : x \in \mathbb{Z}^d, e \in \mathcal{E}\}$ environment
- $\omega(x, e) \geqslant 0$ and $\sum_{e \in \mathcal{E}} \omega(x, e) = 1$
- Random walk X in environment ω is Markov chain with $p(x, x + e) \equiv \omega(x, e)$
- Make ω random: $\omega(o, \bullet) \sim \mu$ and

Random walk in random environment:

- $\mathcal{E} = \{\pm e_1, \dots, \pm e_d\}$
- $\omega = \{\omega(x, e) : x \in \mathbb{Z}^d, e \in \mathcal{E}\}$ environment
- $\omega(x, e) \geqslant 0$ and $\sum_{e \in \mathcal{E}} \omega(x, e) = 1$
- Random walk X in environment ω is Markov chain with $p(x, x + e) \equiv \omega(x, e)$
- Make ω random: $\omega(o, \bullet) \sim \mu$ and
- Make $\omega(x, \bullet)$ i.i.d. in x: $\mathbb{Q} = \mu^{\otimes \mathbb{Z}^d}$.

Random walk in random environment:

- $\mathcal{E} = \{\pm e_1, \ldots, \pm e_d\}$
- $\omega = \{\omega(x, e) : x \in \mathbb{Z}^d, e \in \mathcal{E}\}$ environment
- $\omega(x, e) \geqslant 0$ and $\sum_{e \in \mathcal{E}} \omega(x, e) = 1$
- Random walk X in environment ω is Markov chain with $p(x, x + e) \equiv \omega(x, e)$
- Make ω random: $\omega(o, \bullet) \sim \mu$ and
- Make $\omega(x, \bullet)$ i.i.d. in x: $\mathbb{Q} = \mu^{\otimes \mathbb{Z}^d}$.
- X not a MC under $\mathbb{P}(X \in \bullet) = \int P_\omega(X \in \bullet) d\mathbb{Q}(\omega)$

Ellipticity/degeneracy

By definition, $\omega(x, e) \geqslant 0$.

Ellipticity/degeneracy

By definition, $\omega(x, e) \geqslant 0$. ω is

- *elliptic* if $\omega(x, e) > 0 \; \forall x, e$

Ellipticity/degeneracy

By definition, $\omega(x, e) \geqslant 0$. ω is

- *elliptic* if $\omega(x, e) > 0 \ \forall x, e$
- *uniformly elliptic* if $\omega(x, e) > \varepsilon > 0 \ \forall x, e$

Ellipticity/degeneracy

By definition, $\omega(x, e) \geqslant 0$. ω is

- *elliptic* if $\omega(x, e) > 0\ \forall x, e$
- *uniformly elliptic* if $\omega(x, e) > \varepsilon > 0\ \forall x, e$

Our interest is in degenerate (non-elliptic) random environments.

Induced directed graph

Let $\mathcal{G}_x = \{(x, x+e) : \omega(x, e) > 0\}$ be a set of directed edges. Then

- $\mathcal{G} = (\mathcal{G}_x)_{x \in \mathbb{Z}^d}$ is a random directed graph.

Induced directed graph

Let $\mathcal{G}_x = \{(x, x+e) : \omega(x,e) > 0\}$ be a set of directed edges. Then

- $\mathcal{G} = (\mathcal{G}_x)_{x \in \mathbb{Z}^d}$ is a random directed graph.
- $\mathcal{G} = E(\mathbb{Z}^d)$ if and only if ω is elliptic.

Induced directed graph

Let $\mathcal{G}_x = \{(x, x+e) : \omega(x,e) > 0\}$ be a set of directed edges. Then

- $\mathcal{G} = (\mathcal{G}_x)_{x \in \mathbb{Z}^d}$ is a random directed graph.
- $\mathcal{G} = E(\mathbb{Z}^d)$ if and only if ω is elliptic.
- Need to consider set $\mathcal{C}_o$ of reachable sites

Induced directed graph

Let $\mathcal{G}_x = \{(x, x+e) : \omega(x,e) > 0\}$ be a set of directed edges.
Then

- $\mathcal{G} = (\mathcal{G}_x)_{x \in \mathbb{Z}^d}$ is a random directed graph.
- $\mathcal{G} = E(\mathbb{Z}^d)$ if and only if ω is elliptic.
- Need to consider set $\mathcal{C}_o$ of reachable sites
- Want $\mathbb{Q}(|\mathcal{C}_o| = \infty) = 1$

RWRE Main questions:

- Getting stuck/recurrence/transience

RWRE Main questions:

- Getting stuck/recurrence/transience
- Directional transience: $A_\ell = \{X_n \cdot \ell \to +\infty\}$

RWRE Main questions:

- Getting stuck/recurrence/transience
- Directional transience: $A_\ell = \{X_n \cdot \ell \to +\infty\}$
- Ballisticity/LLN: $\liminf n^{-1} X_n \cdot \ell > 0$

RWRE Main questions:

- Getting stuck/recurrence/transience
- Directional transience: $A_\ell = \{X_n \cdot \ell \to +\infty\}$
- Ballisticity/LLN: $\liminf n^{-1} X_n \cdot \ell > 0$
- CLT: (averaged/quenched)

$$\frac{X_n - nv}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} Z.$$

RWRE Main questions:

- Getting stuck/recurrence/transience
- Directional transience: $A_\ell = \{X_n \cdot \ell \to +\infty\}$
- Ballisticity/LLN: $\liminf n^{-1} X_n \cdot \ell > 0$
- CLT: (averaged/quenched)

$$\frac{X_n - nv}{\sigma\sqrt{n}} \stackrel{\mathcal{D}}{\to} Z.$$

- Monotonicity of v.

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$
- $\mathbb{P}(A_\ell) = 1$ implies $\exists v$ such that $\mathbb{P}(v = \lim n^{-1} X_n) = 1$.

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$
- $\mathbb{P}(A_\ell) = 1$ implies $\exists v$ such that $\mathbb{P}(v = \lim n^{-1} X_n) = 1$.
- v exists in 2 dimensions.

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$
- $\mathbb{P}(A_\ell) = 1$ implies $\exists v$ such that $\mathbb{P}(v = \lim n^{-1} X_n) = 1$.
- v exists in 2 dimensions.
- Ballisticity conditions: Kalikow, Sznitman, Ramirez et al

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$
- $\mathbb{P}(A_\ell) = 1$ implies $\exists v$ such that $\mathbb{P}(v = \lim n^{-1} X_n) = 1$.
- v exists in 2 dimensions.
- Ballisticity conditions: Kalikow, Sznitman, Ramirez et al (all require ellipiticity).

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$
- $\mathbb{P}(A_\ell) = 1$ implies $\exists v$ such that $\mathbb{P}(v = \lim n^{-1} X_n) = 1$.
- v exists in 2 dimensions.
- Ballisticity conditions: Kalikow, Sznitman, Ramirez et al (all require ellipiticity).
- existence of v and CLT require control of *regeneration times*.

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$
- $\mathbb{P}(A_\ell) = 1$ implies $\exists v$ such that $\mathbb{P}(v = \lim n^{-1} X_n) = 1$.
- v exists in 2 dimensions.
- Ballisticity conditions: Kalikow, Sznitman, Ramirez et al (all require ellipiticity).
- existence of v and CLT require control of *regeneration times*.
- Rassoul-Agha and Seppäläinen: Ballisticity and quenched CLT when there is a *forbidden direction*.

RWRE Main answers:

- Example does not get stuck since e.g. can always go $\uparrow$ or $\leftarrow$
- $\mathbb{P}(A_\ell \cup A_{-\ell}) \in \{0, 1\}$
- $\mathbb{P}(A_\ell) = 1$ implies $\exists v$ such that $\mathbb{P}(v = \lim n^{-1} X_n) = 1$.
- v exists in 2 dimensions.
- Ballisticity conditions: Kalikow, Sznitman, Ramirez et al (all require ellipiticity).
- existence of v and CLT require control of *regeneration times*.
- Rassoul-Agha and Seppäläinen: Ballisticity and quenched CLT when there is a *forbidden direction*.
- Monotonicity of v holds for 2-valued environments but not more generally.

Theorem [B]:

Let $d \geqslant 2$ and $\alpha, \beta, \kappa > 0$, $\ell \neq o$. Assume:

Theorem [B]:

Let $d \geqslant 2$ and $\alpha, \beta, \kappa > 0$, $\ell \neq o$. Assume:

(a) There exist $C_1, \gamma_1 > 0$ such that $\forall n$,

$$\mathbb{Q}(\mathcal{C}_o \nsubseteq -n\ell + H_{\kappa,\ell}) \leqslant C_1 e^{-\gamma_1 n^\beta};$$

Theorem [B]:

Let $d \geqslant 2$ and $\alpha, \beta, \kappa > 0$, $\ell \neq o$. Assume:

(a) There exist $C_1, \gamma_1 > 0$ such that $\forall n$,

$$\mathbb{Q}(\mathcal{C}_o \nsubseteq -n\ell + H_{\kappa,\ell}) \leqslant C_1 e^{-\gamma_1 n^\beta};$$

(b) For every $C > 0$, there exist $C_2, \gamma_2 > 0$ such that $\forall n$,

$$\mathbb{P}\Big(|R_n| \leqslant (Cn)^\alpha\Big) \leqslant C_2 e^{-\gamma_2 n^\beta}.$$

Theorem [B]:

Let $d \geqslant 2$ and $\alpha, \beta, \kappa > 0$, $\ell \neq o$. Assume:

(a) There exist $C_1, \gamma_1 > 0$ such that $\forall n$,

$$\mathbb{Q}(\mathcal{C}_o \not\subseteq -n\ell + H_{\kappa,\ell}) \leqslant C_1 e^{-\gamma_1 n^\beta};$$

(b) For every $C > 0$, there exist $C_2, \gamma_2 > 0$ such that $\forall n$,

$$\mathbb{P}\Big(|R_n| \leqslant (Cn)^\alpha\Big) \leqslant C_2 e^{-\gamma_2 n^\beta}.$$

Then $\exists v \neq o$ such that $\mathbb{P}(n^{-1}X_n \to v) = 1$.

Theorem [B]:

Let $d \geqslant 2$ and $\alpha, \beta, \kappa > 0$, $\ell \neq o$. Assume:

(a) There exist $C_1, \gamma_1 > 0$ such that $\forall n$,

$$\mathbb{Q}(\mathcal{C}_o \not\subseteq -n\ell + H_{\kappa,\ell}) \leqslant C_1 e^{-\gamma_1 n^\beta};$$

(b) For every $C > 0$, there exist $C_2, \gamma_2 > 0$ such that $\forall n$,

$$\mathbb{P}\Big(|R_n| \leqslant (Cn)^\alpha\Big) \leqslant C_2 e^{-\gamma_2 n^\beta}.$$

Then $\exists v \neq o$ such that $\mathbb{P}(n^{-1}X_n \to v) = 1$.

Work in progress: FCLT under $\mathbb{P}$

Remarks

(a) holds when $\mathbb{Q}\left(\sum_{i=1}^{d} \omega(o, e_i) = 1\right) > p_d$.

Remarks

(a) holds when $\mathbb{Q}\left(\sum_{i=1}^{d} \omega(o, e_i) = 1\right) > p_d$.

(b) holds e.g. when $(b)'$: some projection of the walk is a submartingale (Menshikov, Popov, Ramírez, and Vachkovskaia)

Remarks

(a) holds when $\mathbb{Q}\left(\sum_{i=1}^{d} \omega(o, e_i) = 1\right) > p_d$.

(b) holds e.g. when $(b)'$: some projection of the walk is a submartingale (Menshikov, Popov, Ramírez, and Vachkovskaia)

(b)' holds for every 2-valued model (for which the walk doesn't get stuck)

Regeneration levels (($\uparrow\!\!\!\lrcorner\!\rightarrow$, $\leftarrow\!\!\!\!\downarrow$) example)

Let $L_n = X_n \cdot (1,1)$. On the event $L_n \to +\infty$

Regeneration levels ($(\uparrow\!\rightarrow, \leftarrow\!\downarrow)$ example)

Let $L_n = X_n \cdot (1,1)$. On the event $L_n \to +\infty$

- $\exists$ random infinite set $\mathcal{D} \subset \mathbb{Z}$ defined by

Regeneration levels $((\uparrow\!\!\lrcorner_{\rightarrow}, \leftarrow\!\!\urcorner_{\downarrow})$ example)

Let $L_n = X_n \cdot (1,1)$. On the event $L_n \to +\infty$

- $\exists$ random infinite set $\mathcal{D} \subset \mathbb{Z}$ defined by

$$k \in \mathcal{D} \iff L_n \geqslant k \text{ for every } n \geqslant \inf\{m : L_m \geqslant k\},$$

Regeneration levels $((\uparrow\!\!\llcorner_{\rightarrow}, \leftarrow\!\!\urcorner_{\downarrow})$ example)

Let $L_n = X_n \cdot (1,1)$. On the event $L_n \to +\infty$

- $\exists$ random infinite set $\mathcal{D} \subset \mathbb{Z}$ defined by

$$k \in \mathcal{D} \iff L_n \geqslant k \text{ for every } n \geqslant \inf\{m : L_m \geqslant k\},$$

- with positive probability $0 \in \mathcal{D}$

Regeneration levels (($\Lsh\!\rightarrow$, $\leftarrow\!\Rsh$) example)

Let $L_n = X_n \cdot (1,1)$. On the event $L_n \to +\infty$

- $\exists$ random infinite set $\mathcal{D} \subset \mathbb{Z}$ defined by

$$k \in \mathcal{D} \iff L_n \geqslant k \text{ for every } n \geqslant \inf\{m : L_m \geqslant k\},$$

- with positive probability $0 \in \mathcal{D}$
- behaviour of the walker between regeneration times $T_{k_{i+1}} - T_{k_i}$ is i.i.d.

Regeneration levels ((↾↳, ↼↓) example)

Let $L_n = X_n \cdot (1,1)$. On the event $L_n \to +\infty$

- $\exists$ random infinite set $\mathcal{D} \subset \mathbb{Z}$ defined by

$$k \in \mathcal{D} \iff L_n \geqslant k \text{ for every } n \geqslant \inf\{m : L_m \geqslant k\},$$

- with positive probability $0 \in \mathcal{D}$
- behaviour of the walker between regeneration times $T_{k_{i+1}} - T_{k_i}$ is i.i.d.
- by the LLN the velocity of the walk is given by

$$\frac{\mathbb{E}[X_T 1_{\{0\in\mathcal{D}\}}]}{\mathbb{E}[T 1_{\{0\in\mathcal{D}\}}]}.$$

Idea of proof of Theorem [B]

- Directional transience easy

Idea of proof of Theorem [B]

- Directional transience easy
- Show that $\mathbb{P}(T > n) \leqslant Ce^{-\gamma n^{\delta}}$

Idea of proof of Theorem [B]

- Directional transience easy
- Show that $\mathbb{P}(T > n) \leqslant Ce^{-\gamma n^{\delta}}$
- $\{T > n\} \Rightarrow$ one of:
  - not contained in a convenient cone

Idea of proof of Theorem [B]

- Directional transience easy
- Show that $\mathbb{P}(T > n) \leqslant Ce^{-\gamma n^{\delta}}$
- $\{T > n\} \Rightarrow$ one of:
  - not contained in a convenient cone
  - contained, and don't go far to regenerate (but still $T > n$)

Idea of proof of Theorem [B]

- Directional transience easy
- Show that $\mathbb{P}(T > n) \leqslant Ce^{-\gamma n^\delta}$
- $\{T > n\} \Rightarrow$ one of:
  - not contained in a convenient cone
  - contained, and don't go far to regenerate (but still $T > n$)
  - need a lot of regeneration attempts

Idea of proof of Theorem [B]

- Directional transience easy
- Show that $\mathbb{P}(T > n) \leqslant Ce^{-\gamma n^{\delta}}$
- $\{T > n\} \Rightarrow$ one of:
  - not contained in a convenient cone
  - contained, and don't go far to regenerate (but still $T > n$)
  - need a lot of regeneration attempts
  - contained, go a long way to attempt few regenerations *

Monotonicity

- The velocity $v(p)$ in any 2-valued model is monotone in p.

Monotonicity

- The velocity $v(p)$ in any 2-valued model is monotone in p.
- Not true for 3-valued environments.

Monotonicity

- The velocity $v(p)$ in any 2-valued model is monotone in p.
- Not true for 3-valued environments.
- **Conjecture:** For the orthant model example $v(p) \cdot (1, 1)$ is strictly increasing and continuous in $p \in [0, 1]$.

Monotonicity

- The velocity $v(p)$ in any 2-valued model is monotone in p.
- Not true for 3-valued environments.
- **Conjecture:** For the orthant model example $v(p) \cdot (1, 1)$ is strictly increasing and continuous in $p \in [0, 1]$.When $p = 1/2$ there are points visited infinitely often.

Monotonicity

- The velocity $v(p)$ in any 2-valued model is monotone in p.
- Not true for 3-valued environments.
- **Conjecture:** For the orthant model example $v(p) \cdot (1,1)$ is strictly increasing and continuous in $p \in [0,1]$.When $p = 1/2$ there are points visited infinitely often.

Work in progress: Strict monotonicity of $v(p) \cdot (1,1)$ for $(\uparrow_{\rightarrow}, {}^{\leftarrow}\!\downarrow)$ for $p > p_c$.

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \,\uparrow\!\!\hookrightarrow$ when $U_x < p$)

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \uparrow\!\!\!\lrcorner_{\rightarrow}$ when $U_x < p$)
- Let $(Y_k)_{k\in\mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \uparrow\!\!\lrcorner\!\!\rightarrow$ when $U_x < p$)
- Let $(Y_k)_{k\in\mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$
- Let $(Z_k)_{k\in\mathbb{N}}$ be i.i.d. with $P(Z_k = \downarrow) = P(Z_k = \leftarrow) = 1/2$

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \overset{\uparrow}{\hookrightarrow}$ when $U_x < p$)
- Let $(Y_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$
- Let $(Z_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Z_k = \downarrow) = P(Z_k = \leftarrow) = 1/2$
- Set $X_0(p) = 0$. Define $X = X(p)$ by taking step Y_k on its kth departure from a $\overset{\uparrow}{\hookrightarrow}$ site and step Z_k on its kth departure from a $\overset{\leftarrow}{\downarrow}$ site.

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \,\uparrow\!\!\rightarrow$ when $U_x < p$)
- Let $(Y_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$
- Let $(Z_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Z_k = \downarrow) = P(Z_k = \leftarrow) = 1/2$
- Set $X_0(p) = 0$. Define $X = X(p)$ by taking step Y_k on its kth departure from a $\uparrow\!\!\rightarrow$ site and step Z_k on its kth departure from a $\leftarrow\!\!\downarrow$ site.
- Let $N_n(p)$ be the number of $\uparrow\!\!\rightarrow$ sites visited before time n. Prove that for $p' > p$:

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \uparrow_{\rightarrow}$ when $U_x < p$)
- Let $(Y_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$
- Let $(Z_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Z_k = \downarrow) = P(Z_k = \leftarrow) = 1/2$
- Set $X_0(p) = 0$. Define $X = X(p)$ by taking step Y_k on its kth departure from a $\uparrow_{\rightarrow}$ site and step Z_k on its kth departure from a $^{\leftarrow}\downarrow$ site.
- Let $N_n(p)$ be the number of $\uparrow_{\rightarrow}$ sites visited before time n. Prove that for $p' > p$:
  - The first time that $N_n(p') \neq N_n(p)$, $N_n(p') = N_n(p) + 1$.

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \uparrow\!\!\rightarrow$ when $U_x < p$)
- Let $(Y_k)_{k\in\mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$
- Let $(Z_k)_{k\in\mathbb{N}}$ be i.i.d. with $P(Z_k = \downarrow) = P(Z_k = \leftarrow) = 1/2$
- Set $X_0(p) = 0$. Define $X = X(p)$ by taking step Y_k on its kth departure from a $\uparrow\!\!\rightarrow$ site and step Z_k on its kth departure from a $\leftarrow\!\!\downarrow$ site.
- Let $N_n(p)$ be the number of $\uparrow\!\!\rightarrow$ sites visited before time n. Prove that for $p' > p$:
  - The first time that $N_n(p') \neq N_n(p)$, $N_n(p') = N_n(p) + 1$.
  - If later $N_m(p') = N_m(p)$ then $X_m(p') = X_m(p)$.

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \uparrow_{\rightarrow}$ when $U_x < p$)
- Let $(Y_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$
- Let $(Z_k)_{k \in \mathbb{N}}$ be i.i.d. with $P(Z_k = \downarrow) = P(Z_k = \leftarrow) = 1/2$
- Set $X_0(p) = 0$. Define $X = X(p)$ by taking step Y_k on its kth departure from a $\uparrow_{\rightarrow}$ site and step Z_k on its kth departure from a ${}^{\leftarrow}\!\downarrow$ site.
- Let $N_n(p)$ be the number of $\uparrow_{\rightarrow}$ sites visited before time n. Prove that for $p' > p$:
  - The first time that $N_n(p') \neq N_n(p)$, $N_n(p') = N_n(p) + 1$.
  - If later $N_m(p') = N_m(p)$ then $X_m(p') = X_m(p)$.
  - This proves that $N_n(p') \geqslant N_n(p)$ for all n.

Sketch Proof of Monotonicity

- couple environments as before ($\mathcal{G}_x(p) = \uparrow_{\rightarrow}$ when $U_x < p$)
- Let $(Y_k)_{k\in\mathbb{N}}$ be i.i.d. with $P(Y_k = \uparrow) = P(Y_k = \rightarrow) = 1/2$
- Let $(Z_k)_{k\in\mathbb{N}}$ be i.i.d. with $P(Z_k = \downarrow) = P(Z_k = \leftarrow) = 1/2$
- Set $X_0(p) = 0$. Define $X = X(p)$ by taking step Y_k on its kth departure from a $\uparrow_{\rightarrow}$ site and step Z_k on its kth departure from a ${}^{\leftarrow}\!\downarrow$ site.
- Let $N_n(p)$ be the number of $\uparrow_{\rightarrow}$ sites visited before time n. Prove that for $p' > p$:
  - The first time that $N_n(p') \neq N_n(p)$, $N_n(p') = N_n(p) + 1$.
  - If later $N_m(p') = N_m(p)$ then $X_m(p') = X_m(p)$.
  - This proves that $N_n(p') \geqslant N_n(p)$ for all n.
  - By the law of large numbers, $\lim_{n\to\infty} n^{-1} X_n = -1 + 2 \lim_{n\to\infty} n^{-1} N_n$.

Strict monotonicity for $(\uparrow\!\!\rightarrow, \leftarrow\!\!\downarrow)$ model above p_c

Modify coupling proof above so that:

Strict monotonicity for $(\uparrow_{\rightarrow}, {}^{\leftarrow}\downarrow)$ model above p_c

Modify coupling proof above so that:

- There is a collection of mutual regeneration levels (in direction $(1,1)$)

Strict monotonicity for $(\uparrow_{\rightarrow}, {}^{\leftarrow}\downarrow)$ model above p_c

Modify coupling proof above so that:

- There is a collection of mutual regeneration levels (in direction $(1, 1)$)
- The "slower" walk does not reach them quicker

Strict monotonicity for $(\uparrow\!\!\rightarrow, \leftarrow\!\!\downarrow)$ model above p_c

Modify coupling proof above so that:

- There is a collection of mutual regeneration levels (in direction $(1,1)$)
- The "slower" walk does not reach them quicker
- The "faster" walk reaches them quicker with prob.> 0

Strict monotonicity for $(\uparrow_{\rightarrow}, {}^{\leftarrow}\!\downarrow)$ model above p_c

Modify coupling proof above so that:

- There is a collection of mutual regeneration levels (in direction $(1,1)$)
- The "slower" walk does not reach them quicker
- The "faster" walk reaches them quicker with prob.> 0
- Similar to recent proof of strict monotonicity for excited random walks in 1 dimension.