On the speed of the one-dimensional polymer in the large range regime

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Self-avoiding walk

- An N-step self-avoiding walk ω on $\mathbb{Z}^d$, beginning at the site x, is defined as a sequence of sites $(\omega(0), \omega(1), ..., \omega(N))$ with $\omega(0) = x$, satisfying $|\omega(j+1) - \omega(j)| = 1$, and $\omega(i) \neq \omega(j) \ \ \forall i \neq j$.
- Let c_N be the number of N-step self-avoiding walks beginning at the origin.
- Mean-square displacement

$$\langle |\omega(N)|^2 \rangle = \frac{1}{c_N} \sum_{\omega : |\omega| = N} |\omega(N)|^2.$$

- $\langle |\omega(N)|^2 \rangle \sim D N^{2\nu}$. Conjecture: $d = 2$, $\nu = \frac{3}{4}$; $d = 3$, $\nu = 0.588...$; $d \geq 4$, $\nu = \frac{1}{2}$.
- $d \geq 5$ proved by Hara and Slade 1992.

Weakly self-avoiding walk

- A sequence of random variable $(S_n)_{n\in\mathbb{N}\cup 0}$ with $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$, where $(X_i)_{i\in\mathbb{N}}$ is a sequence of IID random variables. The distribution of X_i's is

$$P(X_1 = x) = \begin{cases} \frac{1}{2d}, & x \in \mathbb{Z}^d \text{ with } ||x|| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

  The random process $(S_n)_{n\in\mathbb{N}\cup 0}$ is called the *simple symmetric random walk* (SSRW) on $\mathbb{Z}^d$.
- Fix $n \in \mathbb{N}$ and a parameter $\beta \in (0, \infty)$, we define the polymer measure P_n^β on $S = (S_0, S_1, ..., S_{n-1})$ by

$$P_n^H(S) := \frac{1}{Z_n^H} e^{-\beta H_n(S)} P(S), \tag{1}$$

  where

$$Z_n^H := E(e^{-\beta H_n}) \text{ and } H_n(S) := \sum_{i,j=0;\ i\neq j}^{n-1} \mathbf{1}_{S_i = S_j}. \tag{2}$$

- $$P_n^H(S) := \frac{e^{-\beta H_n(S)}}{E(e^{-\beta H_n})} P(S), \tag{3}$$

  where

  $$H_n(S) := \sum_{i,j=0;\ i \neq j}^{n-1} \mathbf{1}_{S_i = S_j} = \sum_{x \in \mathbb{Z}^d} \ell_n^2(x) - n \tag{4}$$

  is the self-intersection local time up to time n, and

  $$\ell_n(x) = \#\{0 \leq i \leq n-1 : S_i = x\}, \quad x \in \mathbb{Z}^d,$$

  is the local time at site x up to time n. β is called the strength of the self-repellence. The path receives a penalty $e^{2\beta}$ when the path self-intersects itself. This model is also called the *weakly self-avoiding walk*. $\beta = 0$, SSRW; $\beta = \infty$, SAW.
- $\hat{H}_n := \sum_{x \in \mathbb{Z}} \ell_n^2(x)$.

Flory's argument 1949

- The heuristic of this model is that if the end-point of the path S has the scale α_n,

$$H_n \approx \sum_{x\in\mathbb{Z}^d} \left(\frac{n}{\alpha_n^d}\right)^2 \approx \alpha_n^d \times \left(\frac{n}{\alpha_n^d}\right)^2. \tag{5}$$

On the other hand, by the local limit theorem of SSRW,

$$P(|S_n| = \alpha_n) \approx \exp(-C\alpha_n^2/n). \tag{6}$$

Combine this with (5),

$$\log Z_n^H \approx -\beta\frac{n^2}{\alpha_n^d} - C\frac{\alpha_n^2}{n}.$$

Let $\frac{n^2}{\alpha_n^d} = \frac{\alpha_n^2}{n}$, we get $\alpha_n = n^{\frac{3}{d+2}}$. It is expected that $E_n^H(|S_n|) \sim n^{\frac{3}{d+2}}$ for $d = 1,2,3$, and $E_n^H(|S_n|) \sim n^{1/2}$ for $d \geq 4$ with a logarithmic correction when $d = 4$. $d \geq 5$ proved by Brydges and Spencer 1985 (Lace expansion for weak interaction).

Known results in 1D in the discrete setting

$P_n^H(S) := \frac{e^{-\beta H_n(S)}}{Z_n^H} P(S)$, $Z_n^H := E(e^{-\beta H_n})$.

Theorem (Ballistic behavior Bolthausen 1990)

For small $\beta > 0$, there exists a $c(\beta) > 0$ such that

$$\lim_{n\to\infty} P_n^H\left(c \le |\frac{S_n}{n}| \le 1/c\right) = 1.$$

Theorem (LDP Geven and den Hollander 1993)

$\theta > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log P_n^H(\{S_n/n \sim \theta\}|S_n > 0) = \begin{cases} J_\beta(\theta), & \theta^{**}(\beta) \le \theta, \\ I_\beta(\theta), & \theta < \theta^{**}(\beta), \end{cases}$$

$\theta^*(\beta)$ is the unique zero of J_β. $0 < \theta^{**}(\beta) < \theta^*(\beta)$.

Known results in 1D in the discrete setting

Theorem (CLT Konig 1996)

$\forall\ C \in \mathbb{R}$, there exists $\theta^*(\beta) > 0$, $\sigma^*(\beta) > 0$ such that

$$\lim_{n\to\infty} P_n^H \left(\frac{S_n - \theta^*(\beta) n}{\sigma^*(\beta)\sqrt{n}} \leq C | S_n > 0 \right) = \Phi(C),$$

where $\frac{1}{\sigma^{*2}(\beta)} = \frac{\partial^2}{\partial \theta^2} J_\beta(\theta)|_{\theta=\theta^*(\beta)}$.

Known results in the continuous setting

Let B_t be the $d-$dimensional Brownian motion, the Hamiltonian is

$$H_t(B) := \int_0^t ds \int_0^t du \ \ \delta_0(B_s - B_u). \qquad (7)$$

However, H_t is infinity when the dimension is higher than one. Past results used truncations to obtain the polymer measure as a weak limit.
$d = 2$, Varadhan 1969, d=3 Westwater 1984 and Bolthausen 1993.
For d=1, LLN Westwater 1980; CLT van der Hofstad, den Hollander and Konig 1997; LDP van der Hofstad, den Hollander and Konig 2003.

New Hamiltonian

We discuss the model with a weaker Hamiltonian

$$G_n := \frac{n^2}{R_n}, \tag{8}$$

where R_n is the number of sites occupied by the walk up to time $n-1$, that is,

$$R_n := \#\{x : \exists\, i,\ S_i = x,\ 0 \le i \le n-1\}. \tag{9}$$

For "weaker" we mean that

$$\left[\sum_{x\in\mathbb{Z}^d} \ell_n^2(x)\right] \cdot \left[\sum_{x\in\mathbb{Z}^d} \mathbf{1}_{\ell_n(x)>0}\right] \ge \left[\sum_{x\in\mathbb{Z}^d} \ell_n(x)\right]^2 = n^2. \tag{10}$$

We have

$$\hat{H}_n \ge \frac{n^2}{R_n} = G_n. \tag{11}$$

Theorem (Hamana and Kesten 2002)

$$I(x) = \lim_{n\to\infty} \frac{-1}{n} \log P\{R_n \geq xn\} \tag{12}$$

exists in $[0,\infty]$ *for all* x. $I(x)$ *is continuous on* $[0,1]$ *and strictly increasing on* $[\gamma_d, 1]$, *and for* $d \geq 2$, $I(x)$ *is convex on* $[0,1]$. *Furthermore,*

$$\begin{aligned} I(x) &= 0 \;\; \textit{for} \;\; x \leq \gamma_d, \\ 0 < I(x) &< \infty \;\; \textit{for} \;\; \gamma_d < x \leq 1, \\ I(x) &= \infty \;\; \textit{for} \;\; x > 1. \end{aligned} \tag{13}$$

Note that $I(1) = \log 2d$. When $d = 1$ and S is the SSRW, $I(x)$ can be found explicitly. For $0 \leq x \leq 1$

$$I(x) = \frac{1}{2}(1+x)\log(1+x) + \frac{1}{2}(1-x)\log(1-x). \tag{14}$$

- $G_n := \frac{n^2}{R_n}$, $Z_n^G := E\left(\exp\left(-\beta G_n\right)\right)$.
- The polymer measure is then defined by $P_n^G(S) := \frac{1}{Z_n^G} e^{-\beta G_n(S)} P(S)$.

Theorem

(i) For $\beta > 0$, $\lim_{n\to\infty} \frac{1}{n} \log Z_n^G = g^*(\beta)$, where

$$g^*(\beta) := -\inf_{c\in[\tilde{c}(\beta),1]} \left\{ \frac{\beta}{c} + I(c) \right\} \tag{15}$$

and $\tilde{c}(\beta) = \frac{\beta}{\beta+\log 2d}$.
(ii) d=1, the infimum is obtained at $c^*(\beta)$, where $c^*(\beta)$ is the solution of

$$\beta = c^2 I'(c) = \frac{c^2}{2} \log\left(\frac{1+c}{1-c}\right). \tag{16}$$

Note that c^* is strictly monotone ($dc^*(\beta)/d\beta = \sigma^*(\beta)^2/c^*(\beta)^2 > 0$), $\beta^{-1/3} c^*(\beta) \to 1$ as $\beta \to 0$ and $e^{2\beta}(1 - c^*(\beta)) \to 2$ as $\beta \to \infty$. Furthermore, $\beta^{-2/3} g^*(\beta) \to -\frac{3}{2}$ as $\beta \to 0$.

Theorem

(LLN and LDP) $d = 1$ *and* $\theta > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log P_n^G(\{\frac{S_n}{n} \sim \theta\} | S_n > 0) = \begin{cases} -\frac{\beta}{\theta} - I(\theta) - g^*(\beta), & c^*(\frac{\beta}{2}) \leq \theta, \\ -\frac{\beta}{\tilde{r}} - I(2\tilde{r} - \theta) - g^*(\beta), & \theta < c^*(\frac{\beta}{2}), \end{cases}$$

where $\tilde{r} = \tilde{r}_\beta(\theta)$ *is the solution of* $\beta = 2r^2 I'(2r - \theta)$.

Theorem

(CLT) $d = 1$, $\forall\ C \in \mathbb{R}$,

$$\lim_{n\to\infty} P_n^G\left(\frac{S_n - c^*(\beta)n}{\sigma^*(\beta)\sqrt{n}} \leq C | S_n > 0\right) = \Phi(C),$$

where $\frac{1}{\sigma^{*2}(\beta)} = \left(\frac{\beta}{\theta} + I(\theta)\right)'' |_{\theta = c^*(\beta)} = \frac{2\beta}{c^{*3}(\beta)} + \frac{1}{1 - c^*(\beta)^2}$. $\sigma^*(\beta) \to \frac{1}{\sqrt{3}}$ *as* $\beta \to 0$ *and* $e^\beta \sigma^*(\beta) \to 2$ *as* $\beta \to \infty$.

- $G_t := \frac{t^2}{R_t^1}$, $Z_t^G := E\left(\exp\left(-\beta G_t\right)\right)$. $R_t^\epsilon := |\bigcup_{s\leq t} B_\epsilon(B_t)|$.
- The polymer measure is then defined by $dP_t^G := \frac{e^{-\beta G_t}}{Z_t^G} dP$.

Theorem

(i) For $\beta > 0$ and any d,

$$\lim_{n\to\infty} \frac{1}{t} \log Z_t^G = g^{**}(\beta), \tag{17}$$

where

$$g^{**}(\beta) := - \inf_{c\in[\tilde{\beta_d},\infty)} \left\{ \frac{\beta}{c} + J(c) \right\} \tag{18}$$

and $\tilde{\beta_d} := \frac{1}{2}\left(\frac{\beta}{w_{d-1}}\right)^{1/3}$. w_{d-1} is the volume of the unit ball in $(d-1)$-dimension. Set also $w_0 = 1$.
(ii) For d=1, the infimum is obtained at $c^{**}(\beta) = \beta^{1/3}$ and $g^{**}(\beta) = -\frac{3}{2}\beta^{2/3}$. Moreover, $Z_t^G \sim \frac{8}{\sqrt{3}} e^{g^{**}(\beta)t}$.

Theorem (Hamana and Kesten 2002)

$$J(x) = \lim_{n\to\infty} \frac{-1}{t} \log P\{R_t^1 \geq xt\} \quad (19)$$

exists in $[0,\infty)$ *for all* x. $J(x)$ *is continuous on* $[0,\infty)$ *and strictly increasing on* $[C_d,\infty)$, *and for* $d \geq 2$, $J(x)$ *is convex on* $[0,\infty)$. *Furthermore,*

$$\begin{aligned} J(x) = 0 \ \ \textit{for} \ \ x \leq C_d, \\ 0 < J(x) < \infty \ \ \textit{for} \ \ C_d < x. \end{aligned} \quad (20)$$

For $d = 1$, $J(x) = \frac{x^2}{2}$ for $x \geq 0$. C_d is the heat capacity of the unit ball for the d-dimension Brownian motion.

Theorem

(LLN and LDP) $d = 1$ and $\theta > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log P_t^G(\{B_t/t \sim \theta\}|B_t > 0) = \begin{cases} -\frac{\beta}{\theta} - \frac{\theta^2}{2} - g^{**}(\beta), & \sqrt[3]{\frac{\beta}{2}} \le \theta, \\ -\frac{\beta}{\bar{r}} - \frac{(2\bar{r}-\theta)^2}{2} - g^{**}(\beta), & \theta < \sqrt[3]{\frac{\beta}{2}}, \end{cases}$$

where $\bar{r} = \bar{r}_\beta(\theta)$ is the solution of $\beta = 2r^2(2r - \theta)$.

Theorem

(CLT) $d = 1$, $\forall\ C \in \mathbb{R}$,

$$\lim_{t\to\infty} P_t^G \left(\frac{B_t - c^{**}(\beta)t}{\sigma^{**}(\beta)\sqrt{t}} \le C|B_t > 0 \right) = \Phi(C), \tag{21}$$

where $c^{**}(\beta) = \beta^{1/3}$ and $\sigma^{**}(\beta) = \frac{1}{\sqrt{3}}$.

In the case $d = 1$, we have the explicit formula of the density of

$$R_t := \max_{0\le s\le t} B_s - \min_{0\le s\le t} B_s = R_t^1 - 2.$$

From Feller 1951,

$$P(R_t \in dr) = \frac{8}{\sqrt{t}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \phi\left(\frac{kr}{t^{1/2}}\right),$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$Ee^{-\beta \frac{t^2}{R_t}} \approx \exp(-\beta \frac{t^2}{r}) \exp(-\frac{k^2 r^2}{t})$.

We can also compute the joint distribution of B_t and R_t under the condition $B_t > 0$. For $0 < x < r$,

$$P(B_t \in dx, R_t \in dr, B_t > 0) = \frac{r-x}{t\sqrt{t}} \cdot \left\{ \sum_{k=-\infty}^{\infty} 4k^2 \left[-1 + \left(\frac{2kr-x}{\sqrt{t}} \right)^2 \right] \phi\left(\frac{2kr-x}{\sqrt{t}} \right) \right\}$$

$$+ \sum_{k=1}^{\infty} \left\{ 4k(k-1) \left(\frac{2kr-x}{t\sqrt{t}} \right) \phi\left(\frac{2kr-x}{\sqrt{t}} \right) - 4k(k+1) \left(\frac{2kr+x}{t\sqrt{t}} \right) \phi\left(\frac{2kr+x}{\sqrt{t}} \right) \right\}.$$

Comparisons and Remarks

- van der Hofstad 1998 had numerical results for the speed and the variance of the speed in Edwards model, that is, $c^{**}(\beta)\beta^{-1/3} \in [1.104, 1.124]$ and $\sigma^{**}(\beta) \in [0.60, 0.66]$, while we have 1 and $1/\sqrt{3} \fallingdotseq 0.577$ in our model.
- Our Hamiltonian is well-defined.
- Recently, SAW in d=2,3 is sub-ballistic Duminil-Copin and Hammond 2013.
- Recently, WSAW in d=4 is diffusive with $\sqrt{T}(\log T)^{1/8}$ Bauerschmidt, Brydges and Slade.