# Zero Range Process with Sitewise Disorder: IMS Workshop on Stochastic Processes in Random Media May 7 2015

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# Motivation

(Ferrari, Krug,96) Consider a row of cars moving in $L$ sites in $\mathbf{Z}$ with periodic boundary conditions moving to the right. Suppose the first $L-1$ cars move at rate 1 and the car at $L$ moves at rate $c<1$. Then there are stationary distributions for car spacings if $\rho<c$ where $\rho$ is the spacing density for speed 1 cars. $\rho_1=\frac{\rho}{c}$ is the spacing density for the slow car. If $\rho>c$ there are no product measure equilibria. Can be transformed to zero range process with site wise disorder.

# The Model

Let $\bar{\mathbf{N}} := \mathbf{N} \cup \{\infty\}$ and let $\mathbf{X} = \{\bar{\mathbf{N}}\}^{\mathbf{Z}}$ be the state space of the process. Let $c > 0$ and define the set of admissible environments as $\alpha \in \mathbf{A} = (c, 1]^{\mathbf{Z}}$. Let $g : \mathbf{N} \to [0, 1]$ be a nondecreasing function with $0 = g(0) < g(1) \leq \lim_{k\to\infty} g(k) := g(\infty) = 1$. Let $1/2 < p \leq 1$ and $q = 1 - p$.

For $\eta \in \mathbf{X}$ and any local function $f$ and $\alpha \in \mathbf{A}$ let the quenched Markov process $(\eta_t^\alpha), t \geq 0$ on $\mathbf{X}$ be defined by the generator

$$L^\alpha f(\eta) = \sum_{x\in\mathbf{Z}} \alpha(x) \left[ pg(\eta(x))(f(\eta^{x,x+1}) - f(\eta)) \right.$$
$$\left. + qg(\eta(x)x)(f(\eta^{x-1,x}) - f(\eta)) \right]$$

# The invariant measures

For $\lambda < 1$, we define the probability measure $\theta_\lambda$ on $\mathbb{N}$ by

$$\theta_\lambda(n) := Z(\lambda)^{-1} \frac{\lambda^n}{g(n)!}, \quad n \in \mathbb{N} \tag{1}$$

where $g(n)! = \prod_{k=1}^{n} g(k)$ for $n \geq 1$, $g(0)! = 1$, and $Z(\lambda)$ is the normalizing factor:

$$Z(\lambda) := \sum_{n=0}^{+\infty} \frac{\lambda^n}{g(n)!} \tag{2}$$

We extend $\theta_\lambda$ into a probability measure on $\overline{\mathbb{N}}$ by setting $\theta_\lambda(\{+\infty\}) = 0$. For $\lambda \leq c$, we denote by $\mu_\lambda^\alpha$ the invariant measure of $L^\alpha$ defined as the product measure on $\mathbf{X}$ with one-site marginal $\theta_{\lambda/\alpha(x)}$.

$\mu_\lambda^\alpha$ is weakly continuous and stochastically increasing with respect to $\lambda$.

Let

$$R(\lambda) := \sum_{n=0}^{+\infty} n\theta_\lambda(n) \tag{3}$$

denote the mean value of $\theta_\lambda$. The quenched mean particle density at $x$ under $\mu_\lambda^\alpha$ is defined by

$$R^\alpha(x,\lambda) = \mathbb{E}_{\mu_\lambda^\alpha}[\eta(x)] = R\left(\frac{\lambda}{\alpha(x)}\right) \tag{4}$$

For our main theorem, we need to assume that the environment $\alpha$ has the following properties. First, the set of slow sites should not be too sparse. To this end we require that

$$\forall \varepsilon \in (0,1), \quad \lim_{n\to+\infty} \min\{\alpha(x) : x \in \mathbb{Z} \cap [-n, n(1-\varepsilon)]\} = c \tag{5}$$

Assumption (5) implies in particular

$$\liminf_{x\to-\infty} \alpha(x) = c \tag{6}$$

Next, we assume existence of an annealed mean density to the left of the origin:

$$\overline{R}(\lambda) := \lim_{n\to+\infty} n^{-1} \sum_{x=-n}^{0} R\left(\frac{\lambda}{\alpha(x)}\right) \quad \text{exists for every } \lambda \in [0, c) \tag{7}$$

It can be shown that $\overline{R}$ is an increasing $C^\infty$ function on $[0, c)$. We define the critical density by

$$\rho_c := \overline{R}(c) := \lim_{\lambda\uparrow c} \overline{R}(\lambda) \in [0, +\infty] \tag{8}$$

We assume $\rho_c < \infty$. Finally, we need the following convexity assumption:

(H) For every $\lambda \in [0, c)$, $\overline{R}(\lambda) - \overline{R}(c) - (\lambda - c)\overline{R}'^{+}(c) > 0$
where

$$\overline{R}'^{+}(c) := \limsup_{\lambda\to c} \frac{\overline{R}(c) - \overline{R}(\lambda)}{c - \lambda} \tag{9}$$

# Previous Results

Benjamini, Ferrari and Landim [bfl 96] considered an asymmetric simple exclusion process where each particle has a random jump rate and the corresponding zero range process with $g(\eta) = 1\{\eta > 0\}$. Under the same condition on the site-wise disorder they proved the existence of a critical density $\rho_c$ above which there were no product invariant measures for the above zero range process and also proved quenched hydrodynamics in the subcritical regime. Andjel, Ferrari, Guiol and Landim [afgl 2000] proved for the totally asymmetric zero range process with site-wise disorder and $g(\eta) = 1\{\eta > 0\}$ the following: almost every initial configuration with lower left empirical density greater than $\rho_c$ converges to as time goes to infinity to the upper invariant measure with density $\rho_c$.

# Main Theorem

**Theorem**

*Let $\eta_0 \in \mathbb{N}^{\mathbb{Z}}$ be such that*

$$\liminf_{n\to\infty} n^{-1} \sum_{x=-n}^{0} \eta_0(x) \geq \rho_c \tag{10}$$

*Then the quenched process $(\eta_t^\alpha)_{t\geq 0}$ with initial state $\eta_0$ converges in distribution to $\mu_c^\alpha$ as $t \to \infty$.*

# Proof, Ingredients

Upper bound is obtained modifying the argument used in [afgl];
Let $\epsilon > 0$ and suppose $l < 0 < r$ are two points in $\mathbf{Z}$ such that $l = \max\{x \leq 0 | \alpha(x) \leq c + \epsilon\}$ and $r = \min\{\epsilon^{-1}, r_\epsilon\}$ where $r_\epsilon = \min\{x \geq 0 | \alpha(x) \leq c + \epsilon\}$. If we make $\eta(l) = \eta(r) = +\infty$ then using arguments from queueing theory and attractivity of the system we can show that the asymptotic distribution of the process restricted to $(l, r)$ is bounded above by $\mu_\epsilon$ with a flux $c + \epsilon$. Letting $\epsilon \to 0$ we obtain the upper bound.

The strategy of proof is to compare $\eta_t^\alpha$ in the neighborhood of 0 to the process $(\eta_s^{\alpha,t})_{s\geq 0}$ whose initial configuration is (with the convention $(+\infty)\times 0 = 0$)

$$\eta_0^{\alpha,t}(x) = (+\infty)1_{\{x\leq x_t\}} \qquad (11)$$

Main ingredients in the proof of lower bound are 1) flux estimates based on initial configurations; 2) Interface property 3) Hydrodynamics for a semi infinite system with source/sink to the left of origin (or $x_0$ in general); 4) Derivation of local equilibrium.

If $x_s^{\alpha,t}$ denotes the location of the interface between $\eta_s^{\alpha}$ and $\eta_s^{\alpha,t}$ at time $s$ then if we can prove prove the following the result follows from local equilibrium for the source/sink process.

$$\lim_{t\to\infty} \mathbb{P}_0 \otimes \mathbb{P}\left(\left\{x_t^{\alpha,t} < A_\varepsilon(\alpha)\right\}\right) = 1 \tag{12}$$

This follows from the following lemma

**Lemma**

*For $\mathcal{P}$-a.e. environment $\alpha \in \mathbf{A}$, the following limits hold in $\mathbb{P}_0 \otimes \mathbb{P}$-probability as $t \to +\infty$:*

$$\lim_{t\to\infty} \left[ t^{-1} \sum_{x=1+\lfloor bt \rfloor}^{A_\varepsilon(\alpha)} \eta_t^{\alpha}(x) + b\rho_c + \varepsilon \right]^{-} = 0 \tag{13}$$

$$\lim_{t\to+\infty} \left[ t^{-1} \sum_{x=1+\lfloor bt \rfloor}^{A_\varepsilon(\alpha)} \eta_t^{\alpha,t}(x) + b\rho_c + 2\varepsilon \right]^{+} = 0 \tag{14}$$

We define the flux for our system as

$$f(\rho) := (p-q)\overline{R}^{-1}(\rho) \tag{15}$$

To state our results, we define $\lambda^{-}(v)$ as the smallest maximizer of $\lambda \mapsto (p-q)\lambda - v\overline{R}(\lambda)$ We also define the Legendre transform of the current :

$$f^{*}(v) := \sup_{\rho \in [0,\rho_c]} [f(\rho) - v\rho] = \sup_{\lambda \in [0,c]} [(p-q)\lambda - v\overline{R}(\lambda)] \tag{16}$$

## Proposition

*Assume $x_t$ in (11) is such that $\beta := \lim_{t\to+\infty} t^{-1}x_t$ exists and $\beta < 0$. Then statements (17) and (18) below hold for $v \in (0, -\beta]$, and statement (19) below holds for $v_0 < v < -\beta$ and $h : \mathbb{N}^Z \to \mathbb{R}$ a bounded local increasing function:*

$$\limsup_{t\to\infty} \left\{ \mathbb{E} \left| t^{-1} \sum_{x>x_t} \eta_t^{\alpha,t}(x) - (p-q)c \right| - p[\alpha(x_t) - c] \right\} \leq 0 \tag{17}$$

$$\lim_{t\to\infty} \mathbb{E} \left| t^{-1} \sum_{x>x_t+\lfloor vt \rfloor} \eta_t^{\alpha,t}(x) - f^*(v) \right| = 0 \tag{18}$$

$$\liminf_{t\to\infty} \left\{ \mathbb{E}h\left(\tau_{\lfloor x_t+vt \rfloor}\eta_t^{\alpha,t}\right) - \int_{\mathbf{X}} h(\eta) d\mu_{\lambda^-(v)}^{\tau_{\lfloor x_t+vt \rfloor}\alpha}(\eta) \right\} \geq 0 \tag{19}$$

### Proposition

*Make all the assumptions of the theorem except super criticality. Assume further that $\eta_0$ satisfies*

$$\rho = \liminf_{n\to\infty} n^{-1} \sum_{x=-n}^{0} \eta_0(x) < \rho_c \tag{20}$$

*Then $\eta_t^\alpha$ does not converge in distribution to $\mu_c^\alpha$ as $t \to +\infty$.*

### Proposition

*Make the same assumptions as for the the theorem and assume further that the jump kernel $p(.)$ is totally asymmetric and $p(1) < 1$. Then there exists $\eta_0 \in \mathbb{N}^{\mathbb{Z}}$ satisfying (10), such that $\eta_t^\alpha$ does not converge in distribution to $\mu_c^\alpha$ as $t \to +\infty$.*

**Thank You!**

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