On The Development of Innovation Diffusion Model Using Stochastic Differential Equation

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Outline

- Introduction
  - Innovation-Diffusion model
  - Stochastic Differential Equation
- Model Development using Stochastic Differential Equation
- Data Analysis and Model Validation
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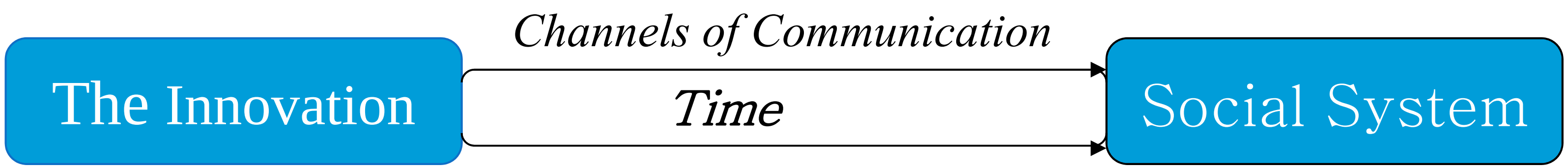
Diffusion of Innovation

- Diffusion theory is most widely used for modelling the life cycles of innovative products.
- The theory of innovation diffusion relates how a new idea, a new product, a service is accepted into a social system over time.
- One of the earliest and widely used model describing the diffusion of innovation is that of Bass.

The Diffusion Process

ELEMENTS OF DIFFUSION

There are four key element of the diffusion



1. The Innovation

Innovation is a powerful, organized, risk-taking change introduced by a marketer in order to maximize economic opportunity.

Characteristics of an innovation

- ✓ Relative advantage
- ✓ Compatibility
- ✓ Complexity
- ✓ Trail ability
- ✓ Observability

The Diffusion Process

2. Channels Of Communication

Represents the means by which information about an innovation reaches to the members of its social system.

Types of communication channels

- ✓ Mass media
- ✓ Interpersonal communication channels

3. Time

Time relate to the rate at which the innovation is diffused or the relative speed with which members of the social system adopt it.

Another aspect related to time is the life cycle of the product

4. The Social System

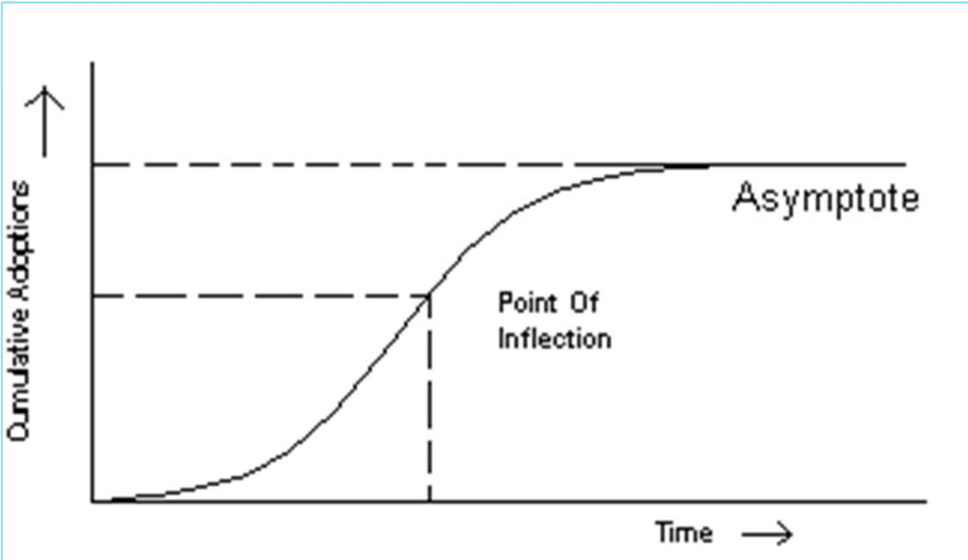
- ✓ Is the population of potential adopters of an innovation
- ✓ Members of the social system are divided as
  - ✓ Innovators
  - ✓ Imitators

The Adoption Process

Adoption

- The decision to utilize an innovation as the best course of action available.
- Stages of customer for accepting and adopting a new product.
  - Knowledge
  - Persuasion
  - Decision
  - Implementation
  - Confirmation

The S-shaped Adoption Curve



Measuring the Adoption Process

- Mathematical modeling is widely applicable and successful tool.
- A large number of innovation diffusion models have been developed in the literature
- An innovation diffusion model produce a life cycle sales curve based on certain parameters such as adopter behavior, promotion, market variables etc. of the diffusion process
- The purpose of a diffusion model is to illustrate the successive increase in the number of adopters and predict the continued development of a diffusion process already in progress.

Innovation Diffusion Model Classification

- **External influence model**
- **Internal influence model**
- **Mixed influence model**
- **Flexible models**
- **Technology substitution models**

Literature-Innovation Diffusion

- The origin of marketing literature in diffusion models seem to be as old as 1960 with Fourt and Woodlook, (1960) pure internal influence model.
- The most widely cited applications of internal influence model was those of Mansfield, (1961) who investigated the diffusion of several innovations such as pallet loaders, diesel locomotives, etc. and Griliches, (1957) who studied the diffusion of hybrid seed corn in 31 states.
- Most widely accepted an earliest model of innovation diffusion is due to Bass.
  - Bass divided the adopters into two groups- 'Innovators' and 'Imitators'
  - Innovators buy a new product by the external influence.
  - The imitators purchase a new product from internal influences.
  - The two forces are assumed to operate simultaneously.
- Several models capturing the variability in the adopter's behavior and market conditions are proposed (Easingwood , Mahajan and Peterson , Kalish , Jain and Rao , Jones and Ritz , Horsky, etc.).

The Bass Model

The differential equation describing **Bass model** can be written as

$$\frac{dN(t)}{dt} = \left[p + q\frac{N(t)}{\bar{N}} \right] \left(\bar{N} - N(t) \right) = r(t)[\bar{N} - N(t)] \qquad (1)$$

solution of the model with *N(0)=0* is

$$N(t) = \bar{N}\frac{1 - \exp^{-(p+q)t}}{1 + (q/p)\exp^{-(p+q)t}}$$

The Bass model is flexible in the sense it can **capture exponential as well as logistic curve**.

The model initiated extensive research in innovation diffusion modeling.

Bass Model Revisited(2004)

Bass model is derived alternatively by changing rate function *r(t)* and avoiding the distinction between innovators and imitators, given as

$$r(t) = \frac{b}{1+\beta \exp^{-bt}}$$

Solving above equation under the initial condition $N(0)=0$, we get

$$N(t) = \bar{N}\frac{1-\exp^{-bt}}{1+\beta \exp^{-bt}}$$

Substituting $b = p+q$ and $\beta = (p/q)$ above model is equivalent to Bass Model.

Change-Point Problem in Product life cycle:

Under Bass model, it is assumed that each purchase decision is made randomly in time with same distribution. But the adoption process may get affected by many factors such as

- Promotional effort
- Intensity of advertisement campaign,
- Expectations and satisfaction level of the purchasers, after sales customer care etc.
- Change in market conditions such as entry/exit of competitors from the market
- launch of better quality/technology product

There are many more factors which can cause change points in sales

The adoption rate may not be smooth and can be changed at some time moment τ *known as change-point.*

Stochastic Differential Equation(Oksendal B.)

A **stochastic differential equation (SDE)** is a differential equation in which one or more of the terms is a stochastic process, thus resulting in a solution which is itself a stochastic process

Stochastic Analog of Classical Differential Equation:

If we allow for some randomness in some of the coefficients of a differential equation, we often obtain a more realistic mathematical model of the situation.

Consider the simple population growth model

$$\frac{dN(t)}{dt} = a(t)N(t) \qquad , N(0) = N_0 \text{ (constant)} \tag{1.1}$$

Where $N(t)$ is the size of the population at time t and $a(t)$ is the relative rate of growth at time t. It might happen that $a(t)$ is not completely known , but subject to some random environment effects, so that we have

$$a(t) = r(t) + \text{"}noise\text{"}$$

Where we do not know the exact behaviour of the noise term, only its probability distribution. The function $r(t)$ is assumed to be a non random.

Definition: A real valued stochastic process W(.) is called a Brownian or Wiener process if

1). W(0)=0

2). W(t)-W(s) is N(0,t-s) for $t \geq s \geq 0$

3). For all times $0 < t_1 < t_2 \_\_\_\_ < t_{n,}$,the random variables $W(t_1), W(t_2)-W(t_1), \_\_\_\_\_ W(t_n)-W(t_{n-1})$ are independent .

In particular

$E[W(t)] = 0, E[W^2(t)] = t$ for real time $t \geq 0$

$Var(W(t)) = t$

As W(t) follows normal distribution with Mean 0 & Variance t, so for all t>0 & $a \leq b$

$$p[a \leq W(t) \leq b] = \frac{1}{\sqrt{2\prod t}} \int_a^b e^{-\frac{1}{2t}W^2(t)} dw(t)$$

$It\ \hat{O}$ **Integral:-**

We now turn to the question of finding a reasonable mathematical interpretation of the "noise" term in the equation (1.1)

$$\frac{dN(t)}{dt} = \left(r(t) + \text{"}noise\text{"}\right) N(t)$$

or more generally in equation of the form

$$\frac{dN(t)}{dt} = b\left(t, N(t)\right) + \sigma\left(t, N(t)\right).noise$$

where b and σ are some given functions.

Cont.

Let us first concentrate on the case where the noise is 1-dimensional. It is reasonable to look for some stochastic process $\gamma(t)$ to represent the noise term, so that

$$\frac{dN(t)}{dt} = b\left(t, N(t)\right) + \sigma\left(t, N(t)\right)\gamma(t) \qquad (1.2)$$

Nevertheless it is possible to represent $\gamma(t)$ as a generalized stochastic process called the white noise process.

The time derivative of the Wiener process (or Brownian motion) is White noise i.e. $\frac{dW(t)}{dt} = \gamma(t)$. So equation (2) can be rewritten as

$$dN(t) = b\left(t, N(t)\right)dt + \sigma\left(t, N(t)\right)dW(t)$$

This is called a Stochastic differential equation of $It\hat{O}$ type.

Result: The 1-dimensional formula

Let X_t be an $It\hat{O}$ process given by

$$dx_t = udt + vdW(t)$$

let $g(t,x) \in C^2([0,\infty) \times R)$ (i.e. g is twice continuously differentiable on $([0,.\infty) \times R)$),

then

$Y_t = g(t, X_t)$ is again an $It\hat{O}$ process , and

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x_t} dx_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2{}_t} (dx_t)^2 \qquad (1.3)$$

Where $(dx_t)^2 = (dx_t . dx_t)$ is computed according to the rules

$$dt.dt = dt.dW(t) = dW(t).dt = 0 \quad \& \quad dW(t).dW(t) = dt$$

Solution of Equation (1.2):

$$\frac{dN(t)}{dt} = a(t)N(t) \qquad , N(0) = N_0$$

where $a(t) = b(t) + \sigma\gamma(t)$, σ is a constant representing the magnitude of the irregular fluctuations & $\gamma(t)$ is a standardized Gaussian White noise.

Let $b(t) =$b (constant)

$$\frac{dN(t)}{dt} = bN(t) + \sigma\gamma(t)N(t)$$

$$dN(t) = bN(t)dt + \sigma N(t)dW(t)$$

Solving above, we have

$$N(t) = N_0 e^{\left(b - \frac{1}{2}\right)t + \sigma W(t)}$$

Stochastic Differential Equation Based Model Development :

- If the product under consideration has a considerably large life cycles and caters to needs of a large section of society, then as the time progresses, the adoption process has smaller increment in total adoption as compared to the potential adopter population. In such cases, adoption process defined by $\{N(t), t \geq 0\}$ behaves as a stochastic process with continuous state space
- Modeling of time evolution of a process with Stochastic Differential Equation is not new. They have been successfully used in financial engineering, Portfolio Management, Physics and life sciences to name a few [2].

Proposed Modelling framework

- Consider equation (1)

$$\frac{dN(t)}{dt} = r(t)\left[\overline{N} - N(t)\right]$$

Where $r(t)$ is a Product adoption rate per remaining potential adopter at time t. However, the behavior of is not completely known since it is subject to random changes due to a large number of factors at play - these factors can be internal to the company such as the promotional effort expenditure, marketing strategy, product quality or can be external such as change in customer preferences or change in competitors strategy and so on and thus might have irregular fluctuation. Thus, we have:

$$r(t) = b(t) + noise$$

Cont.

- Let $\gamma(t)$ be a standard Gaussian white noise and σ a positive constant representing a magnitude of the irregular fluctuations. So can be written as:

$$r(t) = b(t) + \sigma\gamma(t) \qquad (2)$$

Hence, equation (1) becomes

$$\frac{dN(t)}{dt} = \left[b(t) + \sigma\gamma(t)\right]\left[\overline{N} - N(t)\right] \qquad (3)$$

Cont.

- Equation (3) can be extended to the following stochastic differential equation of an type $\hat{Ito}$ [2].

$$d\,N(t) = [b(t) - \frac{1}{2}\sigma^2][\overline{N} - N(t)]dt + \sigma[\overline{N} - N(t)]dW(t) \qquad (4)$$

Where , $W(t)$ is a one-dimensional Wiener process which is formally defined as an integration of the white noise $\gamma(t)$ with respect to time *t*.
Using formula $\hat{Ito}$, solution to equation (4) with initial condition $N(0)=0$, we get *N(t)* as follows:

$$N(t) = \overline{N}\left[1 - \exp\left\{-\int_0^t b(x)\,dx - \sigma W(t)\right\}\right] \qquad (5)$$

Stochastic Differential Equation based Bass Model

In this proposed model, it is assumed that the product adoption rate r(t) may change at any time moment and it can be defined as

$$r(t) = \begin{cases} \dfrac{b_1}{1+\beta_1 \exp(-b_1 t)} + \sigma\gamma(t) & \text{for } 0 \le t \le \tau \\ \dfrac{b_2}{1+\beta_2 \exp(-b_2 t)} + \sigma\gamma(t) & \text{for } t > \tau \end{cases} \tag{6}$$

Here we consider the case when the irregular fluctuations in adoption rate is same before and after the change-point.

Cont.

- The corresponding stochastic differential equation for product adoption process can be written as

$$\frac{dN(t)}{dt} = \begin{cases} \left[\frac{b_1}{1+\beta_1 \exp(-b_1 t)} + \sigma\gamma(t)\right]\left[\overline{N} - N(t)\right] & for \quad 0 \le t \le \tau \\ \left[\frac{b_2}{1+\beta_2 \exp(-b_2 t)} + \sigma\gamma(t)\right]\left[\overline{N} - N(t)\right] & for \quad t > \tau \end{cases} \qquad (7)$$

- Therefore, the transition probability distribution of the above is obtained as follows:

$$N(t) = \begin{cases} \overline{N}\left[1 - \frac{(1+\beta_1)}{(1+\beta_1 \exp(-b_1 t))} \exp(-b_1 t - \sigma W(t))\right] & for \quad 0 \le t \le \tau \\ \overline{N}\left[1 - \frac{(1+\beta_1)(1+\beta_2 \exp(-b_2 \tau))}{(1+\beta_1 \exp(-b_1 \tau))(1+\beta_2 \exp(-b_2 t))} \exp(-b_1 \tau - b_2(t-\tau) - \sigma W(t))\right] & for \quad t > \tau \end{cases} \qquad (8)$$

Cont.

- We consider the mean number adopters of product up to time t. As we know that the Brownian motion or Weiner Process follows normal distribution, thus the mean number of adopters is given as:

$$E\left[N(t)\right]=\begin{cases}\bar{N}\left[1-\dfrac{(1+\beta_1)}{(1+\beta_1\exp(-b_1 t))}\exp\left(-b_1 t+\dfrac{\sigma^2 t}{2}\right)\right] & for\quad 0\le t\le\tau \\[2ex] \bar{N}\left[1-\dfrac{(1+\beta_1)(1+\beta_2\exp(-b_2\tau))}{(1+\beta_1\exp(-b_1\tau))(1+\beta_2\exp(-b_2 t))}\exp\left(-b_1\tau-b_2(t-\tau)+\dfrac{\sigma^2 t}{2}\right)\right] \\ \qquad for\quad t>\tau\end{cases} \tag{9}$$

Data Description and Estimation Results

- The proposed model has been validated on IBM Systems-in-use Generation-I (USA) [Phister,14] and sales data of three products namely air conditioners, telephone answering machines and colour television receivers [Bass et al., 6]. We have estimated the parameters of the proposed model using SPSS tool based on nonlinear least square method. The estimated parameter values of the SDE based proposed model given by equation (9) as well as the Bass model, given by equation (1), for four datasets are given in Table 2. R^2 (Coefficient of determination) and MSE (Mean Squared Error) have been used as comparison criteria for model validation.

Cont.

- For the four datasets under consideration, the position of change-point τ has been fixed as follows:

S.No.	Product	Data Set Number	Position of τ
1	IBM Systems-in-use Generation-I	(DS-I)	8^{th} Year
2	Room air conditioner	(DS-II)	7^{th} Year
3	Colour television	(DS-III)	6^{th} Year
4	Telephone answering machine	(DS-IV)	8^{th} Year

Cont.

- Table 1 gives the description of all these four datasets.

- **Table-2** shows the estimated values of the parameters for Bass model, the proposed model and two comparison criteria: R^2 and MSE. Overall, the proposed model has the best-estimated value for all the four data sets under consideration.

Table-I: Data Description

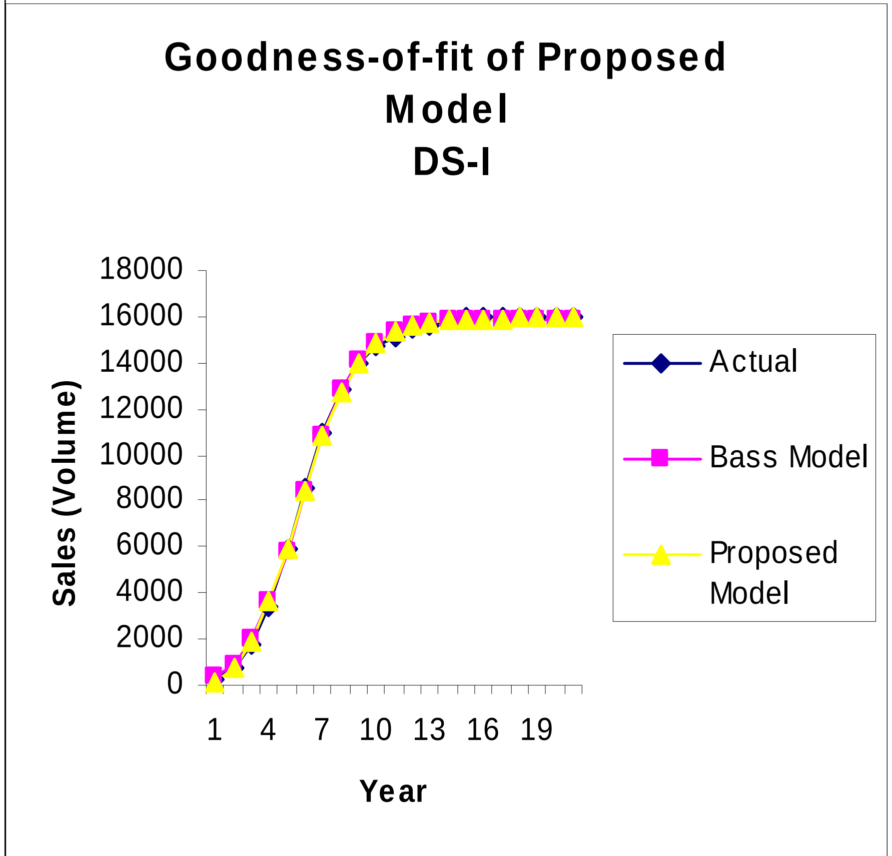
DS-I		DS-II		DS-III		DS-IV	
Year	IBM-Gen-1	Year	Sales	Year	Sales	Year	Sales
1955	190	1	96	1	147	1	400
1956	750	2	291	2	585	2	895
1957	1750	3	529	3	1332	3	1474
1958	3430	4	909	4	2795	4	2171
1959	5972	5	1954	5	5441	5	3039
1960	8612	6	3184	6	10559	6	5133
1961	10962	7	4451	7	16336	7	7761
1962	12782	8	6279	8	22318	8	11067
1963	13952	9	7865	9	28280	9	15877
1964	14702	10	9538	10	32911	10	21432
1965	15157	11	11338				
1966	15460	12	12918				
1967	15663	13	14418				
1968	15833						
1969	15882						
1970	15911						
1971	15925						
1972	15931						
1973	15935						
1974	15939						
1975	15942						

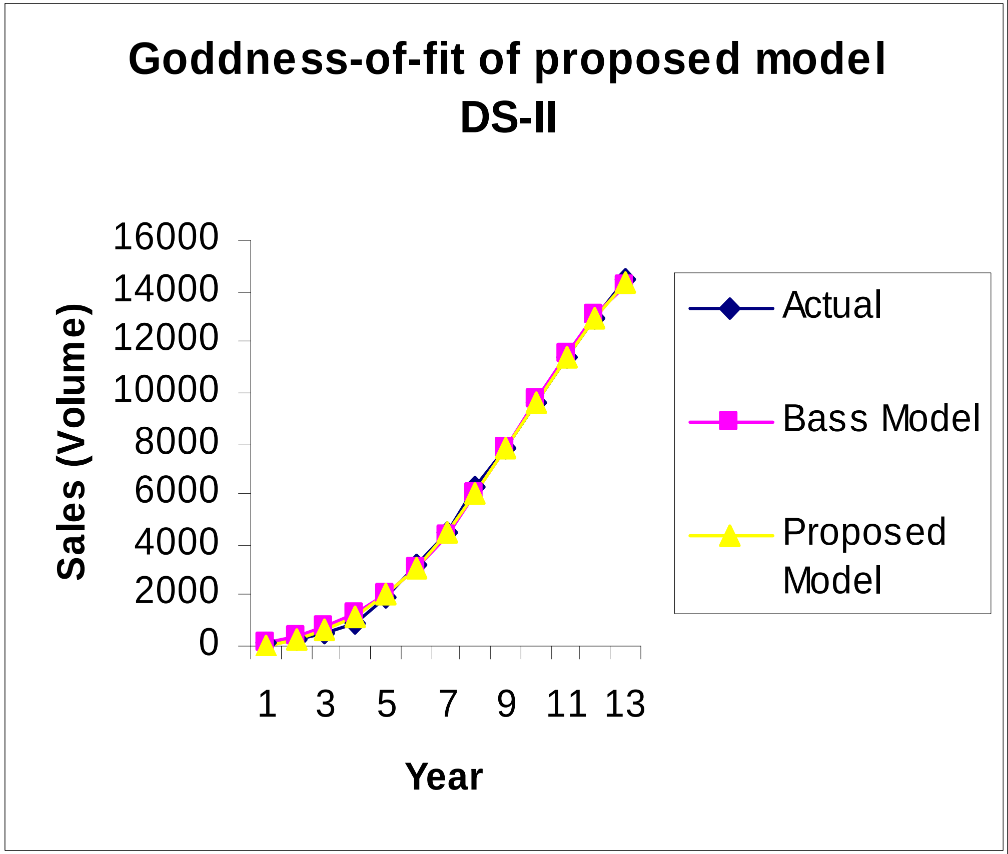
Table-2:Parameter Estimates and Comparison Criteria

Model Under Comparison	Parameter Estimation				
		DS-I	DS-II	DS-III	DS-IV
Bass Model	$\overline{N}$	15861	17173	38464	73693
	b	.644912	.434422	.688099	.406281
	β	41	57	170	137
	R^2	.99947	.99868	.99951	.99942
	MSE	16615.66	31622.17	66262.39	26706.55
Proposed Model	$\overline{N}$	15933	18433	39787	54494
	b_1	.582565031	.232691	.580456	.024698
	b_2	.607863053	.371327	.631201	.459274
	β_1	23	14.8352	81.7412	1.04024
	β_2	27	30.6128	107.825	151.902
	σ	.132312	.113935	.111836	.000051
	R^2	.99974	.99934	.99970	.99969
	MSE	9154.32	10214.04	21021.02	7536.23

Goodness of Fit Curves

- The following figures show the goodness of fit of Bass model and the proposed model graphically.





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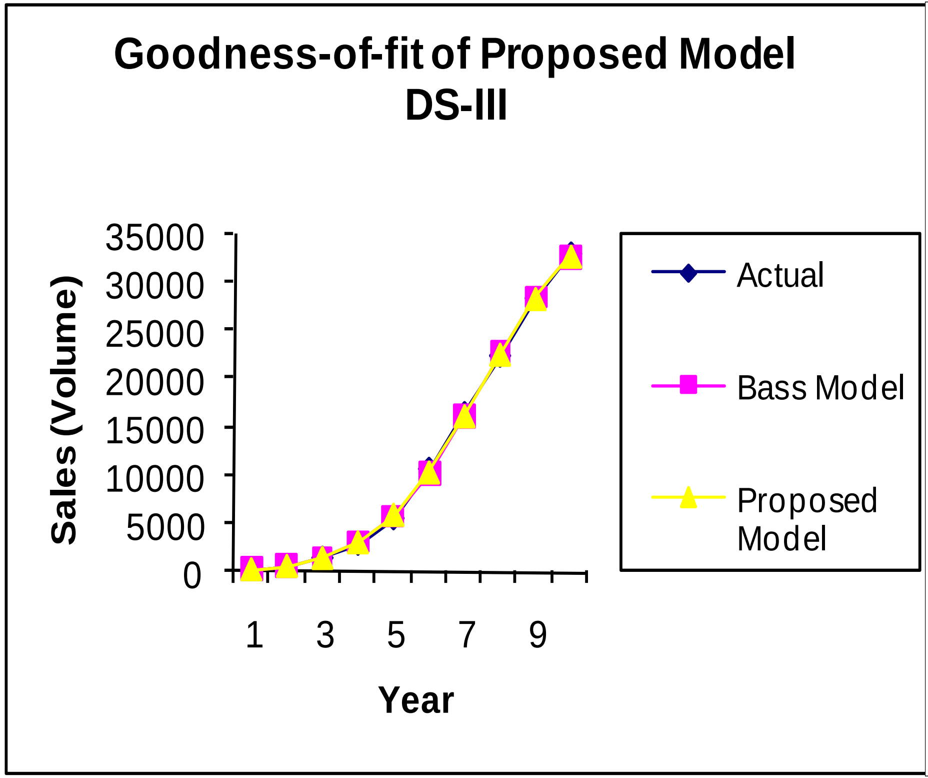
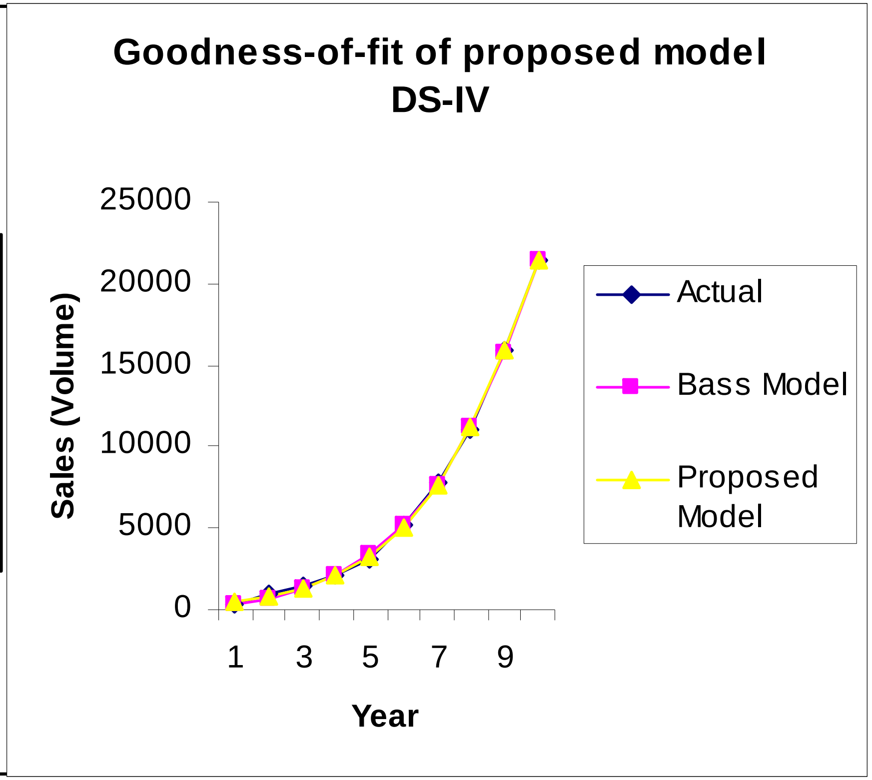

Goodness-of-fit of Proposed Model
DS-III
Sales (Volume)
35000
30000
25000
20000
15000
10000
5000
0
1
3
5
7
9
Year
Actual
Bass Model
Proposed Model


Goodness-of-fit of proposed model
DS-IV
Sales (Volume)
25000
20000
15000
10000
5000
0
1
3
5
7
9
Year
Actual
Bass Model
Proposed Model

Conclusion:

- Time Evolution of adoption process as a continuous state space stochastic process is the modeling framework proposed here.
- Incorporate change in the adoption rate called change-point due to changes in marketing strategies.
- Use of type Stochastic differential equation for innovation-diffusion process puts the modeling on the larger canvas.
- The applicability and accuracy of the proposed model has been checked for product sales data.
- The model can be extended for the possibility of more than one change-points. SDE based model, on one hand capture irregular fluctuation in adoption and on the other hand, concept of change point can assist in identifying future generation of adoption because of change in technology/marketing strategies. This aspect makes an interesting area of study for our future research efforts.

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Thank You