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Fluctuation results in some positive temperature directed polymer models

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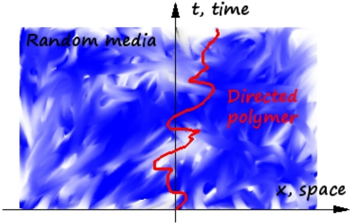
1. Directed polymer problems and universality claim

- Directed polymers: are paths π in space-time (t time, $x \in \mathbb{R}^d$ space) directed in the t-direction.
- Random media: $\omega(x,t)$
- Energy of a polymer: $-H(\pi) = \int_{(x,t)\in\pi} \omega(x,t)dxdt$



- Measure on π:

$$dP^{(\beta)}(\pi) := \frac{1}{Z^{(\beta)}} e^{-\beta H(\pi)} dP_0(\pi)$$

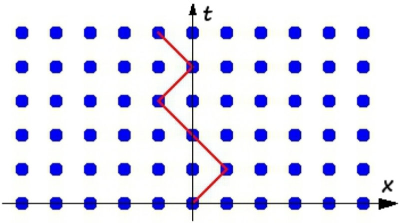
with P_0 a reference measure, $\beta > 0$ the inverse temperature, $e^{-\beta H(\pi)}$ the Boltzmann weight.

- Partition function (quenched: depends on the randomness):

$$Z^{(\beta)} := \int e^{-\beta H(\pi)} dP_0(\pi)$$

We can consider for example:

- P_0 the law of a simple random walk of length n: $\pi = (0, \pi(1), \pi(2), \ldots, \pi(n))$
- $\omega = \{\omega(i,j), i, j \in \mathbb{Z}\}$ with $\omega(i,j)$ iid random variables
- Measure proportional to $e^{-\beta H(\pi)} = e^{\beta \sum_{(x,t)\in\pi} \omega(x,t)}$



Extreme cases:

$\beta = 0$: H becomes irrelevant, one sees only simple random walks

$\beta = \infty$: The measure concentrates on paths π with minimal $H(\pi)$

Some questions on polymers in random media

Q1 How large are the transversal fluctuations? Determine ζ such that

$$|\pi(n)| \approx n^{\zeta}, \quad n \gg 1.$$

Q2 Consider the free energy: $F_\beta := \frac{1}{\beta} \ln(Z^{(\beta)})$. Determine χ such that

$$\mathrm{Var}(F_\beta) \approx n^{2\chi}, \quad n \gg 1.$$

Q3 Determine the asymptotic laws of fluctuations for $\pi(n)$ and F_β.

Q4 How do the results depend on β and the dimension d?

$d = 1$ and $d = 2$ are the physically relevant situations

What is it expected by universality?

From now on we focus on the $d = 1$ case only. Under weak assumptions on the law of the $\omega(i, j)$'s one expects the followings:

- Critical temperature with change of behavior is $\beta = \beta_c = 0$ (infinite temperature)
- For any $\beta > 0$, the scaling exponent are

  Foster,Nelson,Stephen'77;van Beijeren,Kutner,Spohn'85

$$\chi = 1/3, \quad \zeta = 2/3$$

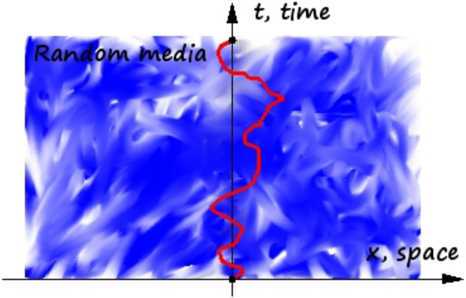
  which are the scaling exponents of the KPZ universality class

  Kardar,Parisi,Zhang'86

Now we consider the point-to-point geometry.

- Let P_0 be the law of a directed path pinned at (x, τ), $Z_{\mathrm{pp}}^{(\beta)}$, $F_{\mathrm{pp}}^{(\beta)}$ the corresponding partition functions / free energy.
- Let $f_{\mathrm{pp}}^{(\beta)}$ be the law of large number of $F_{\mathrm{pp}}^{(\beta)}$:

$$f_{\mathrm{pp}}^{(\beta)}(x, \tau) = \lim_{n\to\infty} \frac{1}{n} F_{\mathrm{pp}}^{(\beta)}(xn, \tau n).$$



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- Universality claim: there exists a universal limit process $\mathcal{A}$ such that, for any $\beta > 0$, there are model-dependent constants c_1, c_2, c_3 s.t.

$$\lim_{n\to\infty} \frac{1}{n^{\chi}} \left[F^{(\beta)}_{\rm pp}(xn^{\zeta}, \tau n) - n f^{(\beta)}_{\rm pp}(xn^{\zeta-1}, \tau) \right] = c_1 \mathcal{A}(c_2 x, c_3 \tau).$$

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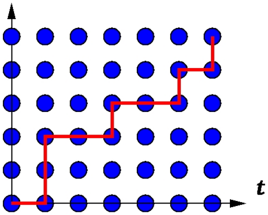
- At zero temperature $\beta = \infty$, for special models proven that $\chi = 1/3, \zeta = 2/3$; $x \mapsto \mathcal{A}(x, 1)$ is the Airy$_2$ process, with one-point distribution the GUE Tracy-Widom F_{GUE}
  Baik, Deift, Johansson, Prähofer, Spohn, Sasamoto, Borodin, Ferrari, Péché, Corwin, Veto, Seppäläinen, Valko, ...

2. Semi-discrete directed polymer:

a solvable model at positive temperature

The polymer model we consider is a limit of the following generic situation:



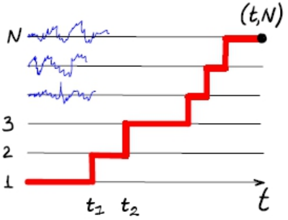
- i.i.d. random variables (finite second moment)
- reference measure: one-sided discrete time simple random walk

Taking the limit to continuous time one gets the semi-discrete directed polymer model (a.k.a. O'Connell-Yor directed polymer model)

Semi-discrete directed polymer model at positive temperature

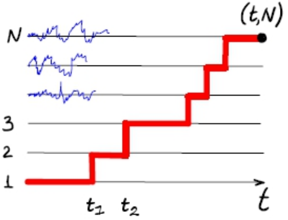
O'Connell-Yor'01

- Path measure P_0: Continuous time one-sided simple random walk from $(0,1)$ to (t,N). The directed polymer is parameterized by the jump times $0 < t_1 < t_2 < \ldots < t_{N-1} < t$.



- Random media: for each $k \in \mathbb{N}$, let B_k an independent standard Brownian motion. The energy is given by

$$-H(\pi) = B_1(t_1)+(B_2(t_2)-B_2(t_1))+\ldots+(B_N(t)-B_N(t_{N-1}))$$



- By Brownian scaling wlog $\beta = 1$
- Partition function:

$$Z_N(t) := \int_{0<t_1<t_2<\ldots<t_{N-1}<t} e^{B_1(t_1)+(B_2(t_2)-B_2(t_1))+\ldots+(B_N(t)-B_N(t_{N-1}))} dt_1 \ldots dt_{N-1}.$$

- Law of large numbers: for any $\kappa > 0$,

$$f(\kappa) := \lim_{N\to\infty} \frac{1}{N} F_N(\kappa N) \equiv \lim_{N\to\infty} \frac{1}{N} \ln Z_N(\kappa N) = \inf_{t>0} (\kappa t - (\ln\Gamma)'(t)).$$

O'Connell-Yor'01;Moriarty,O'Connell'07

What about fluctuations of $F_N(\kappa N)$?

Theorem (Borodin,Corwin,Ferrari'12)

For any $\kappa > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{F_N(\kappa N) - N f(\kappa)}{c(\kappa) N^{1/3}} \leq r\right) = F_{\mathrm{GUE}}(r)$$

where $c(\kappa) = (-\frac{1}{2}\Psi''([\Psi']^{-1}(\kappa)))^{1/3}$ *(*Ψ *is the digamma function), and* F_{GUE} *the GUE Tracy-Widom distribution function.*

(Further results available for Brownian motions with drifts where the distribution is the Baik-Ben Arous-Péché distribution)

3. Continuous Directed Random Polymer (CDRP)

The continuous directed random polymer (CDRP) is the natural fully-continuous scaling limit of discrete models

Alberts,Khanin,Quastel'12

- the background noise is white noise $\dot{W}$,
- the reference measure P_0 is the law of a Brownian motions.

The partition function of a CDRP is given by

$$\mathcal{Z}(T,X) = \mathbb{E}_{T,X}\left(\mathcal{Z}_0(\pi(0)) : \exp : \left\{-\int_0^T ds \dot{W}(\pi(s),s)\right\}\right)$$

where the expectation is with respect Brownian paths, π, backwards in time with $\pi(T) = X$.

- One can recover the partition function of the CDRP from the semi-discrete by taking $t = \sqrt{TN} + X$ and $N \to \infty$, i.e., for a function $C(N,X,T)$ Moreno Flores,Quastel,Remenik

$$\frac{Z(\sqrt{TN}+X,N)}{C(N,X,T)} \Rightarrow \mathcal{Z}(T,X)$$

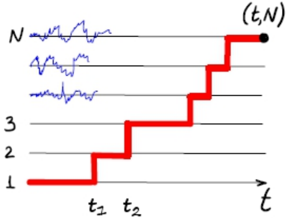
Recall that

$$Z_N(t) := \int_{0<t_1<t_2<\ldots<t_{N-1}<t} e^{B_1(t_1)+(B_2(t_2)-B_2(t_1))+\ldots+(B_N(t)-B_N(t_{N-1}))} dt_1 \ldots dt_{N-1}.$$

- The quantity $u(t,N) := e^{-t} Z_N(t)$ satisfies

$$\partial_t u(t,N) = (u(t,N-1) - u(t,N)) + u(t,N)\dot{B}_N(t)$$

with initial condition $u(0,N) = \delta_{1,N}$.



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$$\partial_t u(t,N) = (u(t,N-1) - u(t,N)) + u(t,N)\dot{B}_N(t)$$

with initial condition $u(0,N) = \delta_{1,N}$.

- The partition function of the CDRP $\mathcal{Z}(T,X)$ satisfies

$$\partial_T \mathcal{Z} = \frac{1}{2}\partial_X^2 \mathcal{Z} + \mathcal{Z}\dot{W}$$

with initial conditions $\mathcal{Z}(0,X) = \delta_0(X)$.

Theorem (Borodin,Corwin,Ferrari'12)

For any $T>0$ and S with $\Re(S)>0$ it holds

$$\mathbb{E}\left(e^{-S\mathcal{Z}(T,0)e^{T/4!}}\right)=\det(\mathbb{1}-K)_{L^2(\mathbb{R}_+)}$$

where, with $\sigma=(2/T)^{1/3}$,

$$K(\eta,\eta')=\frac{1}{(2\pi \mathrm{i})^2}\int_{-\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}dw\int_{\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}dz\,\frac{\sigma\pi S^{(z-w)\sigma}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-z\eta'}}{e^{w^3/3-w\eta}}.$$

Our result is actually more general. This formula obtained also from the weakly ASEP to approach KPZ was obtained by

Sasamoto,Spohn'10;Amir,Corwin,Quastel'10

- In arXiv:1407.6977 with Borodin, Corwin, and Vető we obtain the analogue result for the stationary case, i.e., the solution of

$$\partial_T Z = \frac{1}{2}\partial_X^2 Z + Z\dot{W}$$

with initial conditions $Z(0,X)$ being any two-sided standard Brownian motion with fixed drift.

- The stationary case was considered also using replica approach Sasamoto,Imamura'13

4. Continuous Directed Random Polymer and KPZ equation

The KPZ equation

- The Kardar-Parisi-Zhang (KPZ) equation is one of the models in the KPZ universality class, class of irreversible stochastic random growth models. Kardar,Parisi,Zhang'86

- The KPZ equation writes (by a choice of parameters) in one-dimension is

$$\partial_T h = \frac{1}{2}\partial_X^2 h + \frac{1}{2}(\partial_X h)^2 + \dot{W}$$

where $\dot{W}$ is the space-time white noise

- Stationary initial conditions are two-sided Brownian motions with fixed drift.

- KPZ equation

$$\partial_T h = \frac{1}{2}\partial_X^2 h + \frac{1}{2}(\partial_X h)^2 + \dot{W}$$

⇒ Problem in defining the object $(\partial_X h)^2$.
For a way of doing it, see Hairer's work (Fields Medal 2014)

Hairer'11

- Setting $h = \ln \mathcal{Z}$ (and ignoring the Itô-correction term) one gets the (well-defined) Stochastic Heat Equation (SHE):

$$\partial_T \mathcal{Z} = \frac{1}{2}\partial_T^2 \mathcal{Z} + \mathcal{Z}\dot{W}$$

- Given the solution of the SHE with initial condition $\mathcal{Z}(0,X) := e^{h(0,X)}$, one calls

$$h(T,X) = \ln(\mathcal{Z}(T,X))$$

the Cole-Hopf solution of the KPZ equation.

- KPZ equation

$$\partial_T h = \frac{1}{2}\partial_X^2 h + \frac{1}{2}[(\partial_X h)^2 - \infty] + \dot{W}$$

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The KPZ class

The KPZ universality class of stochastic growth models contains for example:

- directed polymers (and last passage percolation as a zero-temperature limit)
- exclusion processes (partially asymmetric)
- growth models like polynuclear growth model or eden model
- some random tilings (dynamics given by the shuffling algorithms)

Universality:

- Universality of fluctuations is expected in the large time limit
- In some models with tunable parameter, there is universality of the KPZ equation under appropriate "weak asymmetry" scaling limit (remember Tuesday's talk by Caravenna).
- However, the second universality is not required for the first one

Nematic liquid crystals: stable (black) vs metastable (gray) cluster

Takeuchi,Sano'10: PRL 104, 230601 (2010)

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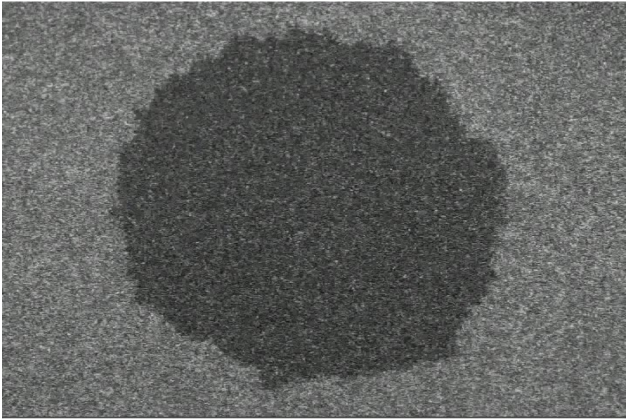

5. Some hints on the methods

Consider again the O'Connell-Yor semi-discrete directed polymer model.

- Starting point: a formula for the Laplace transform

$$\mathbb{E}\left(e^{-uZ_N(t)}\right).$$

Q How to recover the distribution function?

- One expects: $\ln(Z_N(\kappa N)) \approx Nf(\kappa) - c(\kappa)N^{1/3}\xi$, $c(\kappa) > 0$, and ξ a random variable GUE Tracy-Widom distributed.
- Set: $u = e^{-Nf(\kappa)+c(\kappa)N^{1/3}r}$. Then

$$\mathbb{E}\left(e^{-uZ_N(t)}\right) = \mathbb{E}\left(e^{-e^{c(\kappa)N^{1/3}(\xi-r)}}\right) \to \mathbb{E}(\mathbb{1}_{\xi\leq r}) = \mathbb{P}(\xi \leq r)$$

as $N \to \infty$.

- One could try to proceed using moments:

$$\mathbb{E}\left(e^{-uZ_N(\kappa N)}\right) \text{"}=\text{"} \sum_{\ell\geq 0} \frac{(-u)^\ell}{\ell!}\mathbb{E}\left((Z_N(\kappa N))^\ell\right)$$

  Problem: $\mathbb{E}\left((Z_N(\kappa N))^\ell\right) \simeq e^{c\ell^2}$: so RHS not convergent :-(

  The exponential moments do not determine the distribution function!

- Replica trick uses formally the above "equality" and sums up the terms canceling infinities to get the result

- Explicit Fredholm determinant expression for

$$\mathbb{E}\left(e^{-uZ_N(t)}\right) = \det(\mathbb{1} + K_u)_{L^2(\mathcal{C}_0)}$$

where $\mathcal{C}_0$ a small contour around 0 and K_u is the kernel

$$K_u(v, v') = \frac{\mathrm{i}}{2} \int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \frac{ds}{\sin(\pi s)} \left(\frac{\Gamma(v-1)}{\Gamma(s+v-1)}\right)^N \frac{u^s e^{vts+ts^2/2}}{s+v-v'}$$

Borodin,Corwin'11

- Problem: Asymptotics analysis restricted to $\kappa > \kappa^* > 0$, so the SHE limit is not reachable $\kappa \sim 1/\sqrt{N}$.

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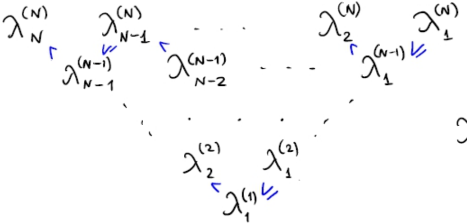
Borodin,Corwin'11

- Problem: Asymptotics analysis restricted to $\kappa > \kappa^* > 0$, so the SHE limit is not reachable $\kappa \sim 1/\sqrt{N}$.

- Alternative Fredholm determinant expression, from $\mathcal{C}_0$ to $\mathcal{C}_{\mathbb{R}_-}$, which is good for asymptotic analysis for any $\kappa > 0$ and even when $\kappa \to 0$ as $N \to \infty$. So-far obtainable only through q-Whittaker process (see next) Borodin,Corwin,Ferrari'12

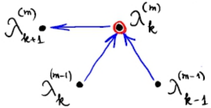
- The configurations are elements on



$$\lambda_j^{(m)} \in \mathbb{Z}$$

- Let $q \in (0,1)$ be fixed. Particle $\lambda_k^{(m)}$ jumps to the right with rate

$$\mathrm{rate}(\lambda_k^{(m)}) = \frac{\left(1-q^{\lambda_{k-1}^{(m-1)}-\lambda_k^{(m)}-1}\right)\left(1-q^{\lambda_k^{(m)}-\lambda_{k+1}^{(m)}}\right)}{\left(1-q^{\lambda_k^{(m)}-\lambda_k^{(m-1)}+1}\right)}$$

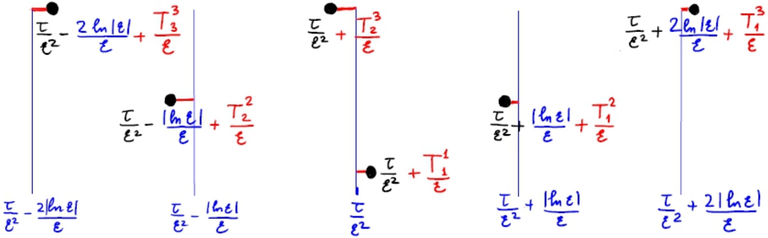


- The set of coordinates $\lambda_k^{(k)}$, $k \geq 1$, forms what we call q-TASEP. Have jump rate given by $1 - q^{\text{gap to next particle}}$.

The relation between q-TASEP and the O'Connell-Yor model

Borodin,Corwin'11

- Set $q = e^{-\varepsilon}$ and look at time $t = \tau/\varepsilon^2$ with "packed" initial condition.
- As $\varepsilon \to 0$,



- In particular, $-T_N^N = \ln Z_N(\tau)$ in distribution.

- Using Macdonald polynomials (generically)

  Borodin,Corwin'11

  or duality (for some initial conditions)

  Borodin,Corwin,Sasamoto'13

  one obtains expressions for

$$\mathbb{E}\left(q^{\ell\lambda_N^{(N)}(t)}\right)$$

- From this algebraic manipulations give (perfectly legal q-version of the replica trick)

$$\mathbb{E}\left(\frac{1}{(\zeta q^{\lambda_N^{(N)}(t)};q)_\infty}\right) = \mathbb{E}\left(\sum_{\ell=0}^{\infty} \frac{q^{\ell\lambda_N^{(N)}(t)}\zeta^\ell}{(1-q)\cdots(1-q^\ell)}\right)$$

$$= \sum_{\ell=0}^{\infty} \frac{\mathbb{E}\left(q^{\ell\lambda_N^{(N)}(t)}\right)\zeta^\ell}{(1-q)\cdots(1-q^\ell)} = \det(1+\widetilde{K})$$

$$\mathbb{E}\left(\frac{1}{(\zeta q^{\lambda_N^{(N)}(t)};q)_\infty}\right) = \det(1+\widetilde{K})$$

- Then take $q \to 1$ limit of both sides:
- LHS $\to$ Laplace transform of semi-discrete directed polymer
- RHS $\to$ Fredholm determinant with kernel

$$K_u(v,v') = \frac{\mathrm{i}}{2}\int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \frac{ds}{\sin(\pi s)}\left(\frac{\Gamma(v-1)}{\Gamma(s+v-1)}\right)^N \frac{u^s e^{vts+ts^2/2}}{s+v-v'}$$

- Take $N \to \infty$ limit: difficulties in the asymptotics comes from the poles of $1/\sin(\pi s)$ in the kernel since no steep descent path exists ...