Domino tilings and other games

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Joint work with Mark Adler, Sunil Chhita & Kurt Johansson

(1) M. Adler, K. Johansson, and P. van Moerbeke, *Double Aztec Diamonds and the Tacnode Process*, Adv in Math, **252**, 518-571 (2014). (arXiv:1112.5532).

(2) M. Adler, Sunil Chhita, Kurt Johansson and Pierre van Moerbeke, *Tacnode GUE-minor processes and double Aztec Diamonds*, Prob. Theory Relat. Fields, **160**, 1-51 (2014)

(3) M. Adler and P. van Moerbeke, *Coupled GUE-minor Processes*, Intern. Math. Research Notices, 1-58 (February 2015) (doi:10.1093/imrn/rnu280). (arXiv:1312.3859).

1. Introduction: Domino tilings of an Aztec diamond

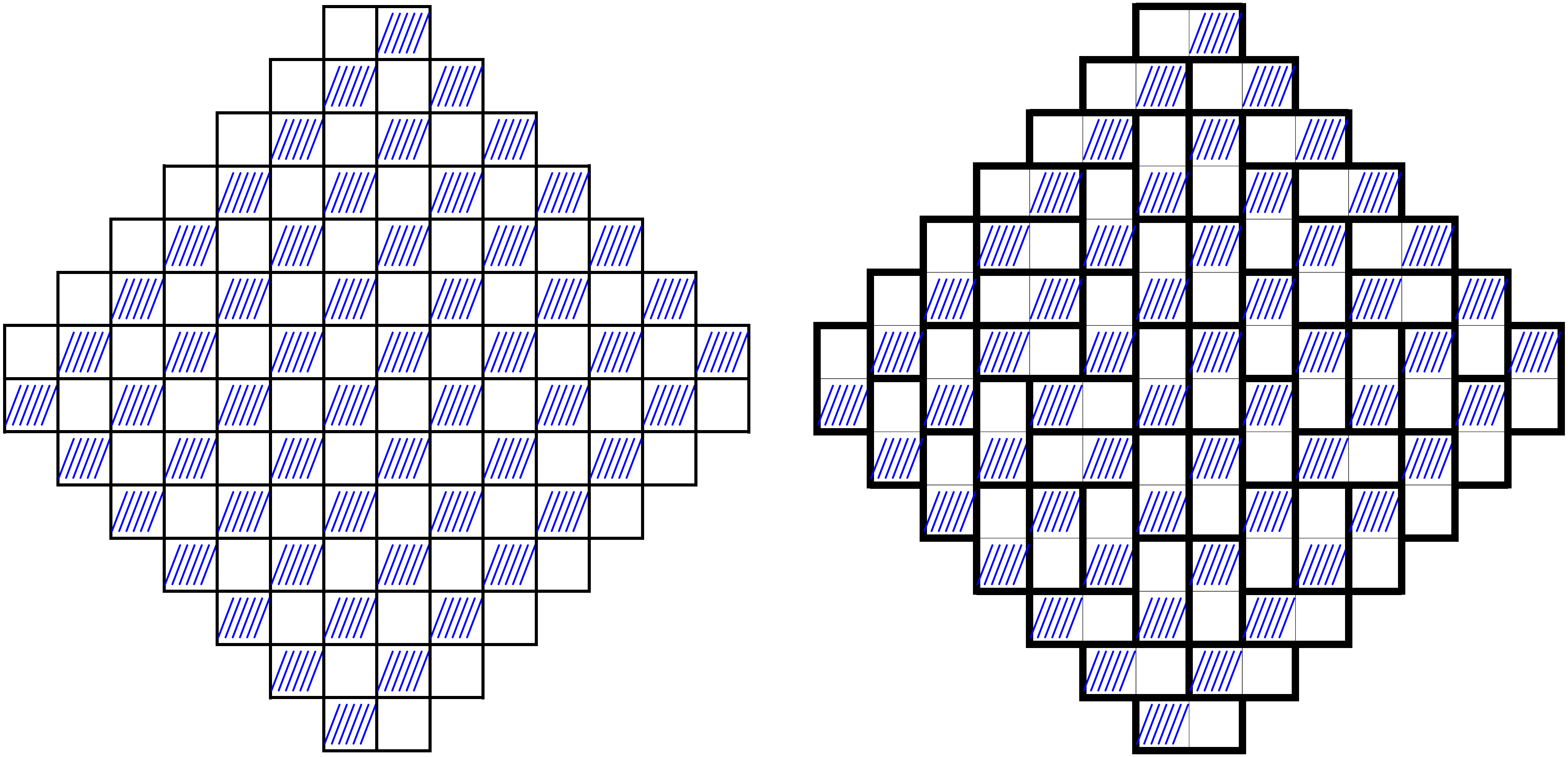
2. Domino tiling of a double Aztec diamond: $\mathbb{K}-$ and $\mathbb{L}$-processes

3. Tacnode Process ($n \to \infty$)

4. The Edge-Tacnode ($n \to \infty$)

5. The Edge-Tacnode kernel and coupled GUE-matrices

1. Introduction: Domino tilings of an Aztec diamond :



4 types of coverings by domino's!

Kasteleyn 1961, Elkies, Kuperberg, Larsen, Propp '92
Cohn, Elkies, Propp '96, Jockusch, Propp, Shor '98,
Johansson '00, '03, '05
Johansson-Nordenstam '06

A weight on domino's and a probability on domino tilings:

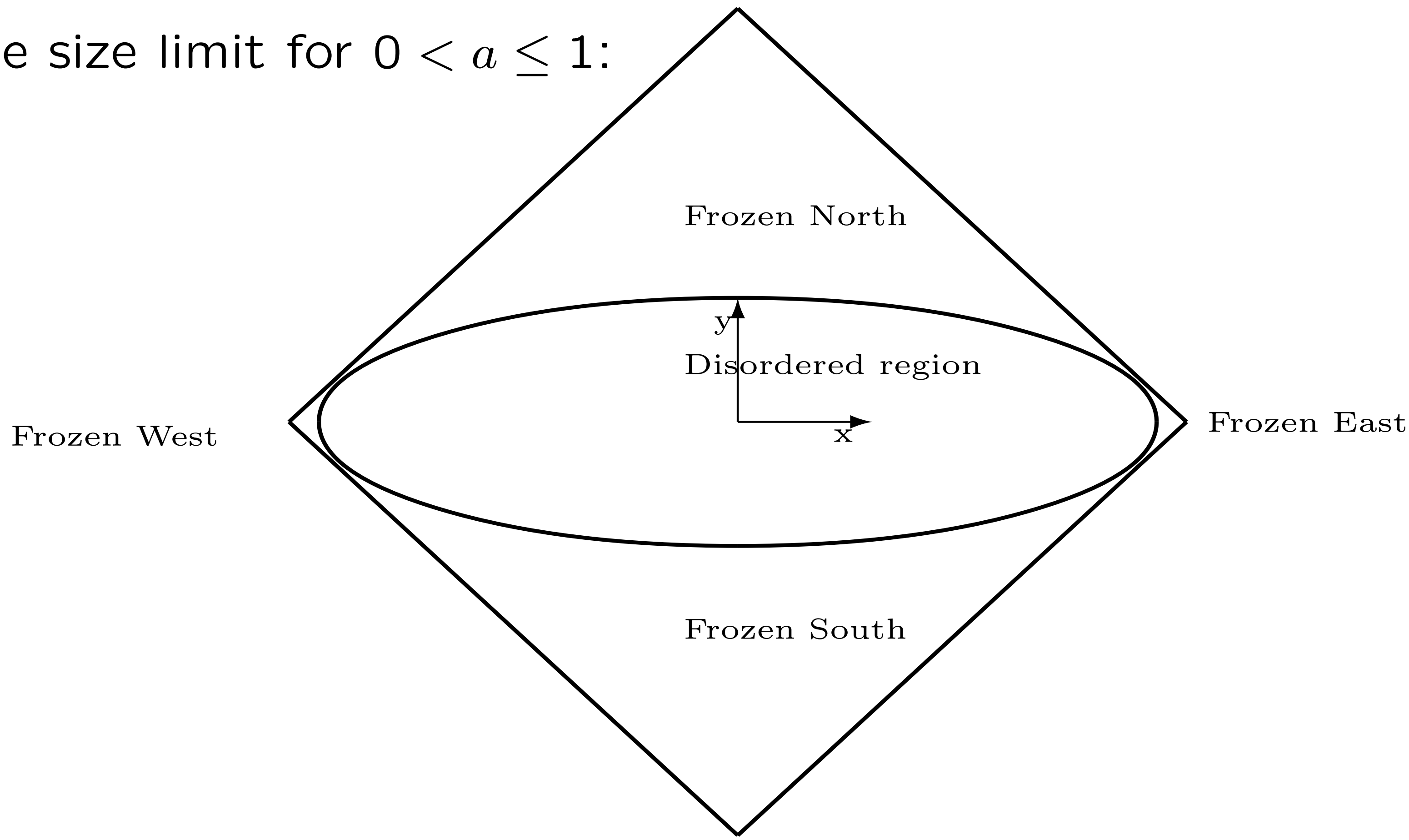
- put the weight $0 < a \leq 1$ on vertical dominoes

- put the weight 1 on horizontal dominoes,

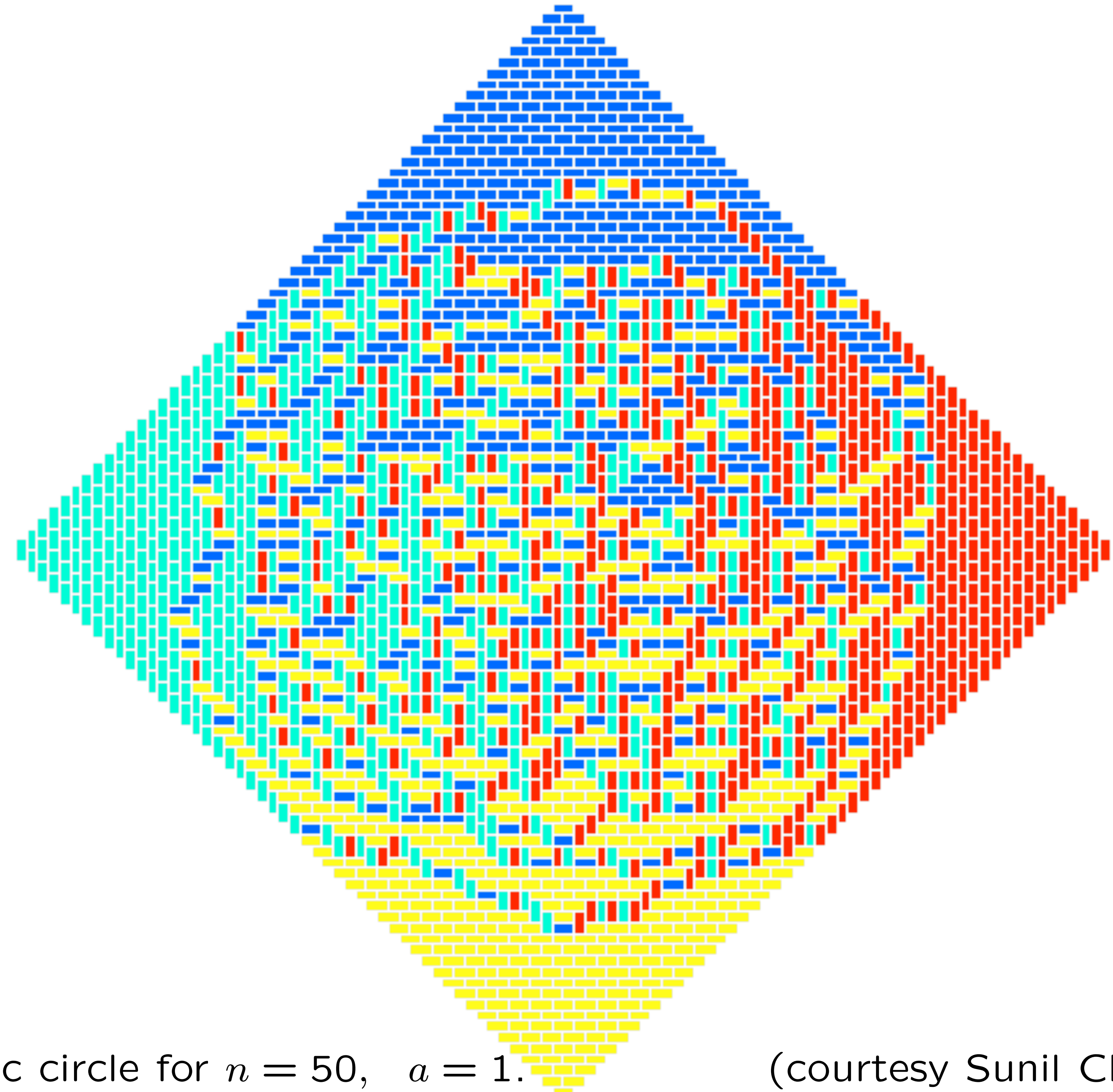
Define:

$$\mathbb{P}(\text{domino tiling } T) = \frac{a^{\#\text{vertical domino's in } T}}{\sum_{\text{all possible tilings } T} a^{\#\text{vertical domino's in } T}}$$

In the large size limit for $0 < a \leq 1$:

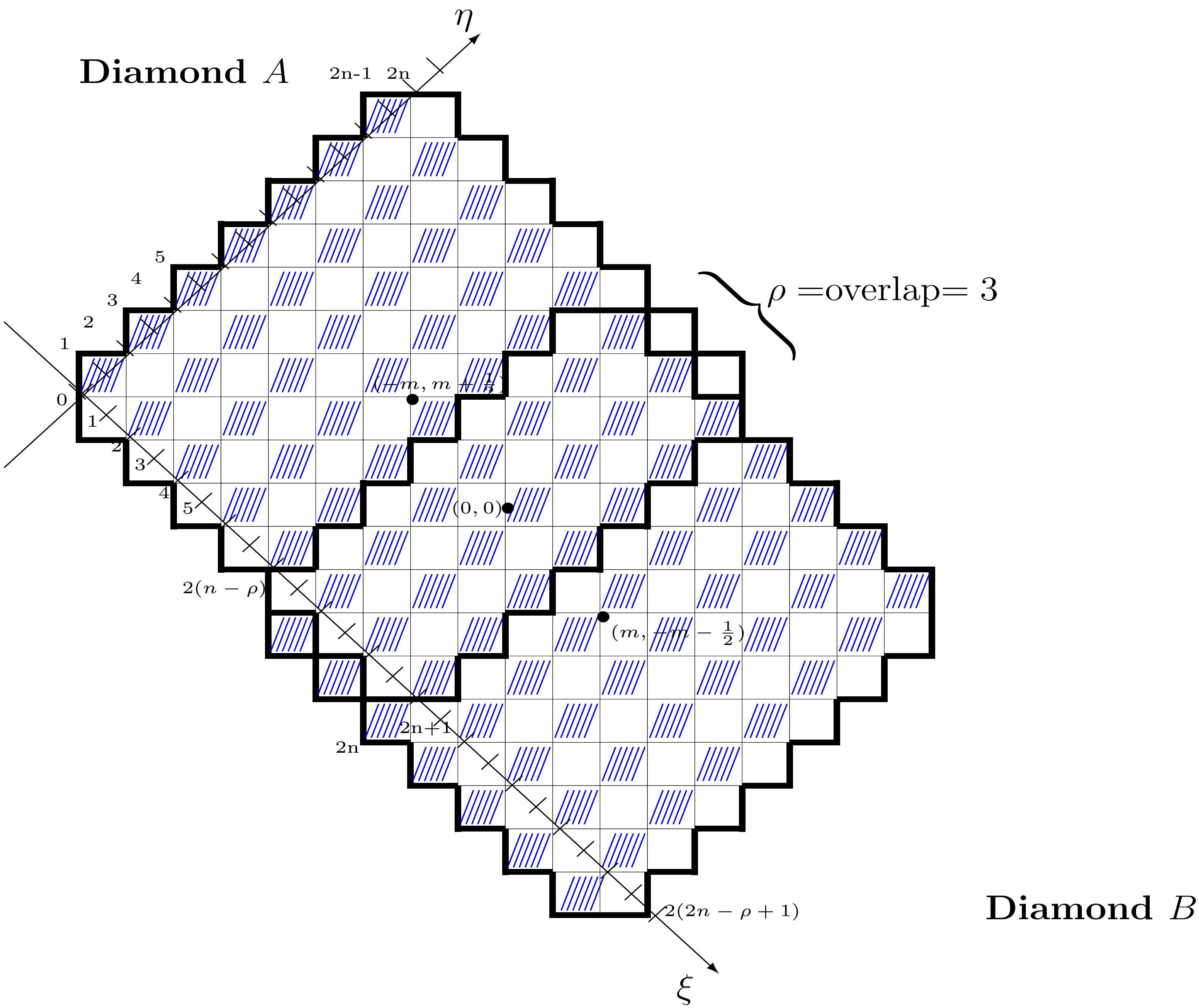


(horizontal domino's are more likely than vertical ones for $0 < a < 1$)



Arctic circle for $n = 50$, $a = 1$. (courtesy Sunil Chhita)

2. Domino tiling of a double Aztec diamond



Double Aztec diamond of type $n = 7$, with overlap $=\rho = 3$

Coordinate system: (ξ, η)

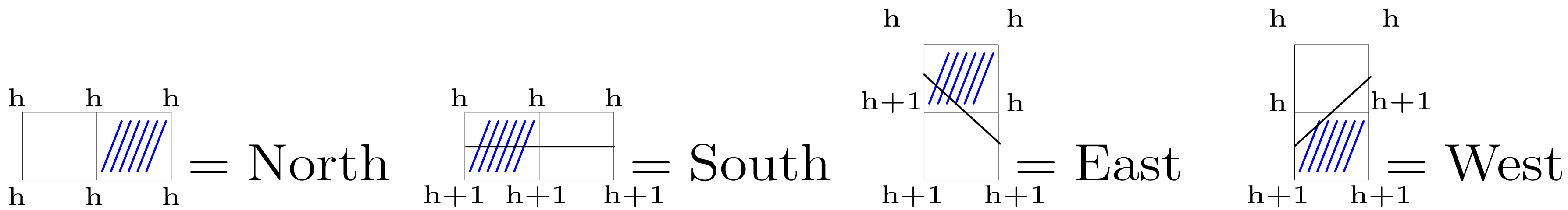
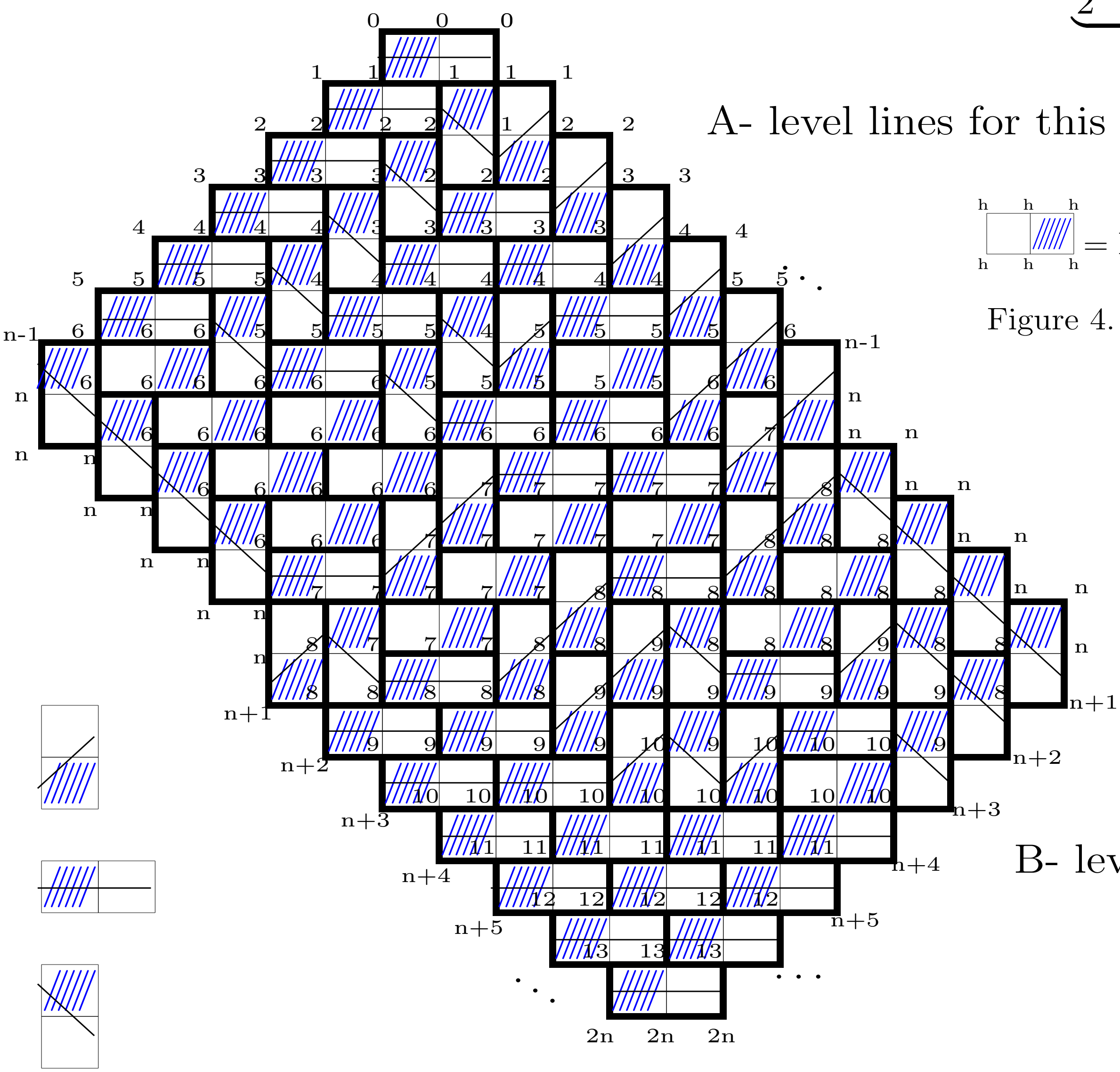


Figure 4. Height function on domino's and level-lines.

Random cover with domino's: two groups of non-intersecting paths

level lines corresponding to height $\underbrace{\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ldots,n-\frac{1}{2},}_{A-\text{level lines}}\ \underbrace{n+\frac{1}{2},\ldots,\ 2n-\frac{1}{2}}_{B-\text{level lines}}$

A- level lines for this random surface



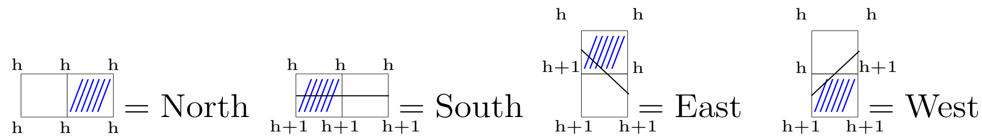
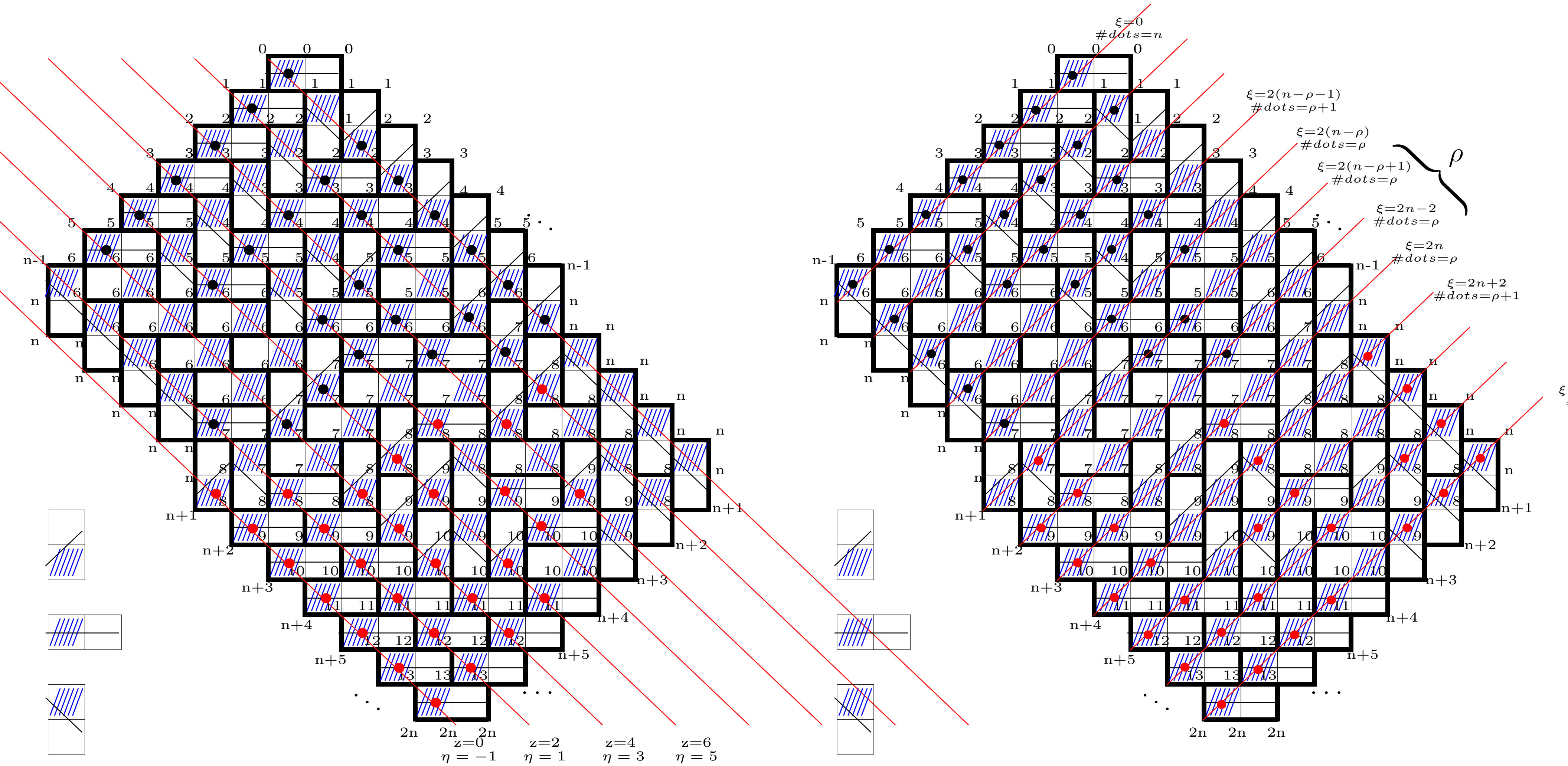


Figure 4. Height function on domino's and level-lines.

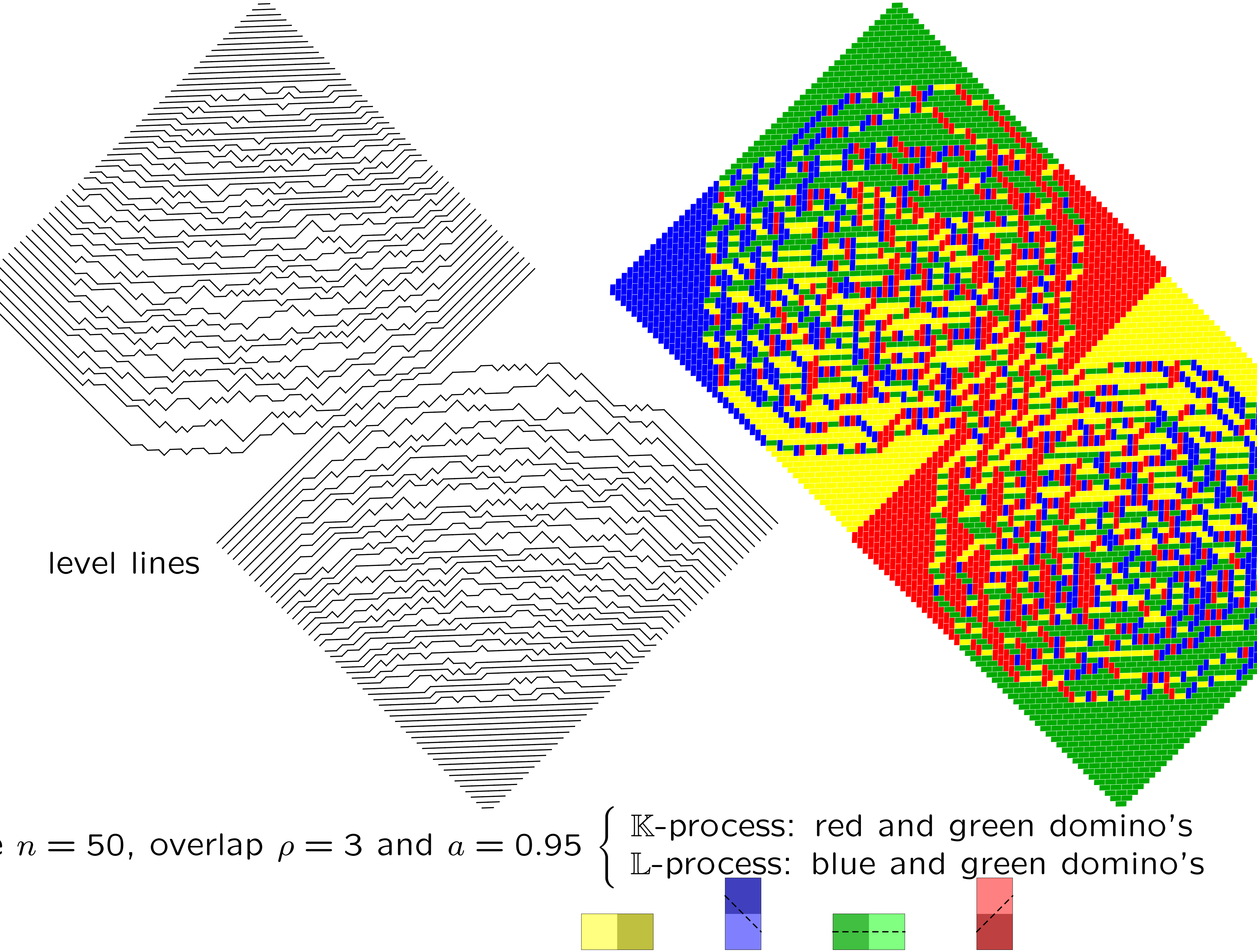
B- level lines for this random surface

Two processes:



$\mathbb{K}$-process
dots =(lines$\{z = 2k\}\cap$level curves)
# dots per line = n

$\mathbb{L}$-process
dots = (lines$\{\xi = 2k\}\cap$level curves)
# dots per line varies from n to ρ

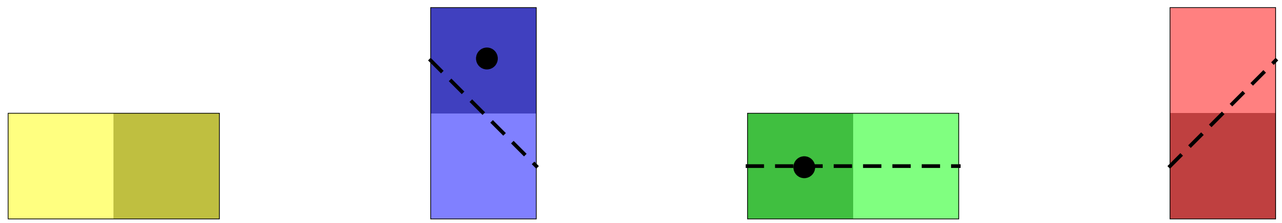

level lines
e $n = 50$, overlap $\rho = 3$ and $a = 0.95$
$\mathbb{K}$-process: red and green domino's
$\mathbb{L}$-process: blue and green domino's

For the $\mathbb{L}$-process: Which probability are we computing?

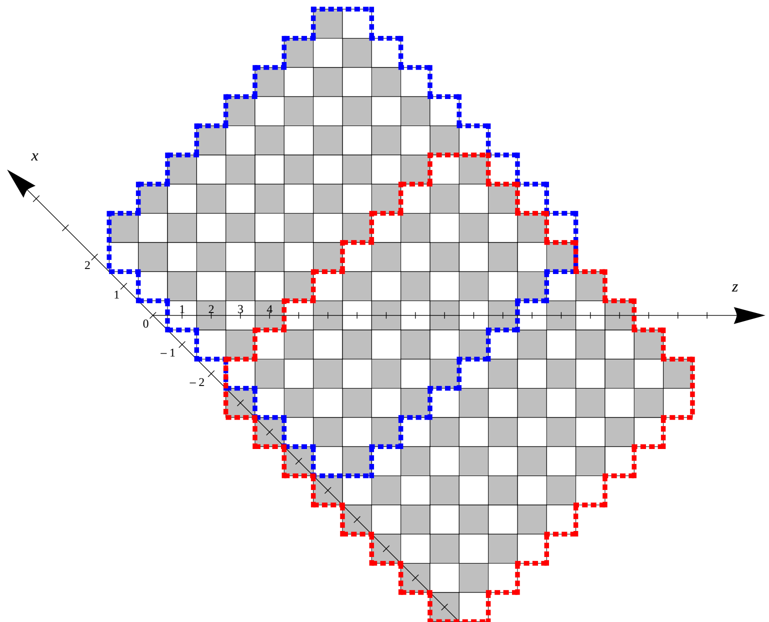
$$\mathbb{P}\{\text{The interval } [k,\ell] \subset \{\xi = 2s\} \text{ contains no dot-particles}\}$$

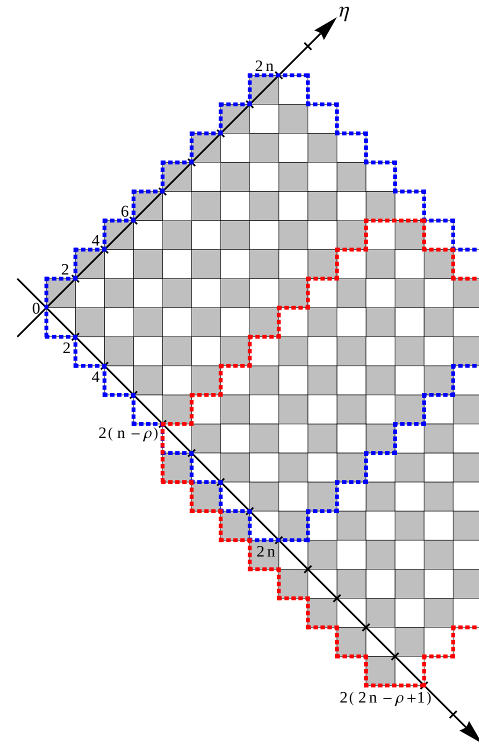
$$= \mathbb{P}\{\text{The random surface is flat along the interval } [k,\ell] \subset \{\xi = 2s\}\}$$

$$= \mathbb{P}\{\text{Dominos covering interval } [k,\ell] \subset \{\xi = 2s\} \text{ are red or yellow }\}$$



Determinantal point processes $\mathbb{K}$ and $\mathbb{L}$





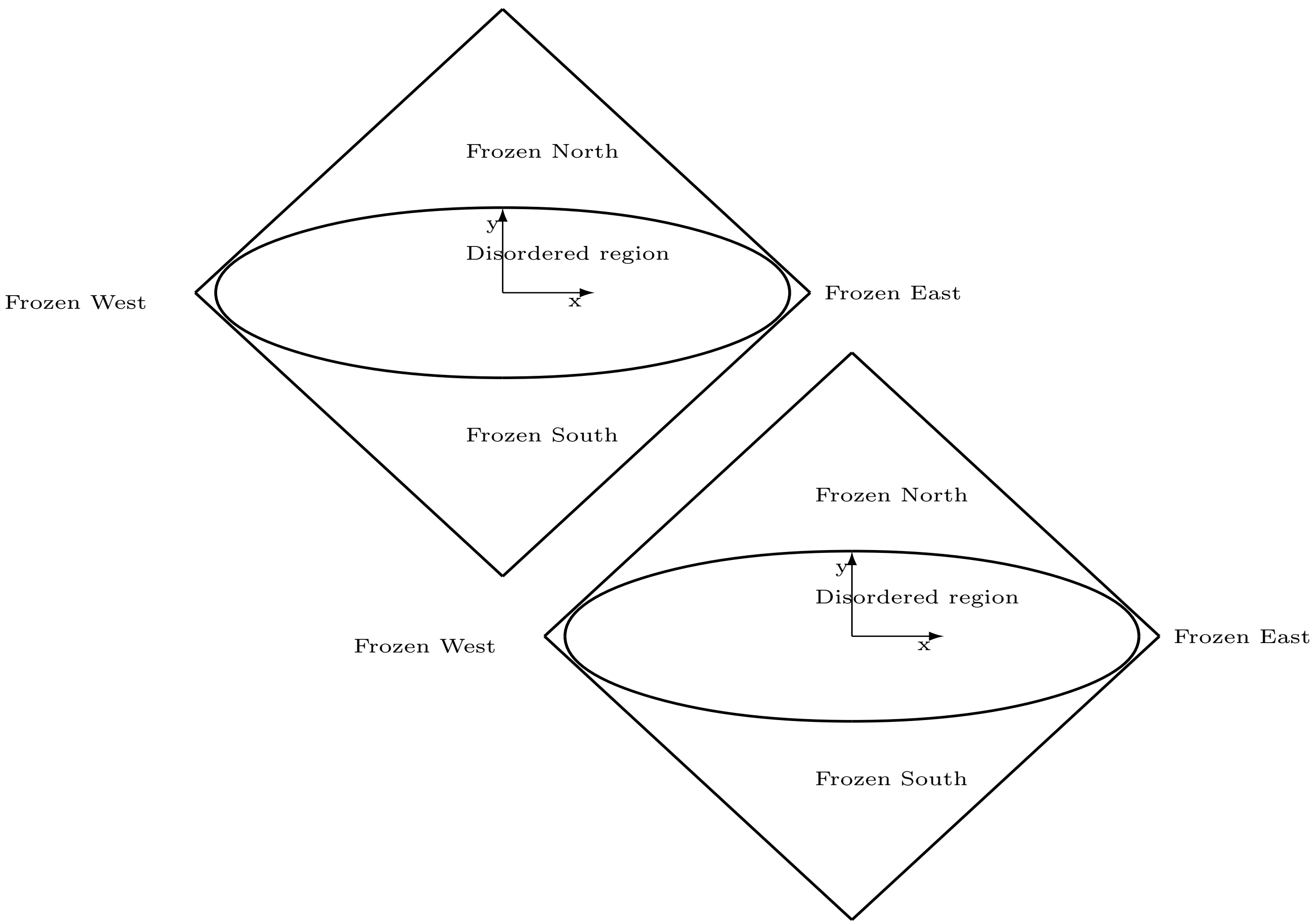
$$(-1)^{x-y}\mathbb{K}_{n,\rho}(z,x;z',y) = \tilde{\mathbb{K}}_n^{\text{oneAzt}}\Big(z,x;z',y\Big) - \left\langle ((\mathbb{1}-K_n)^{-1}\mathcal{A}_{-y,\frac{z'}{2}})(k), \mathcal{B}_{-x,\frac{z}{2}}(k) \right\rangle_{\geq n-\rho+1}.$$

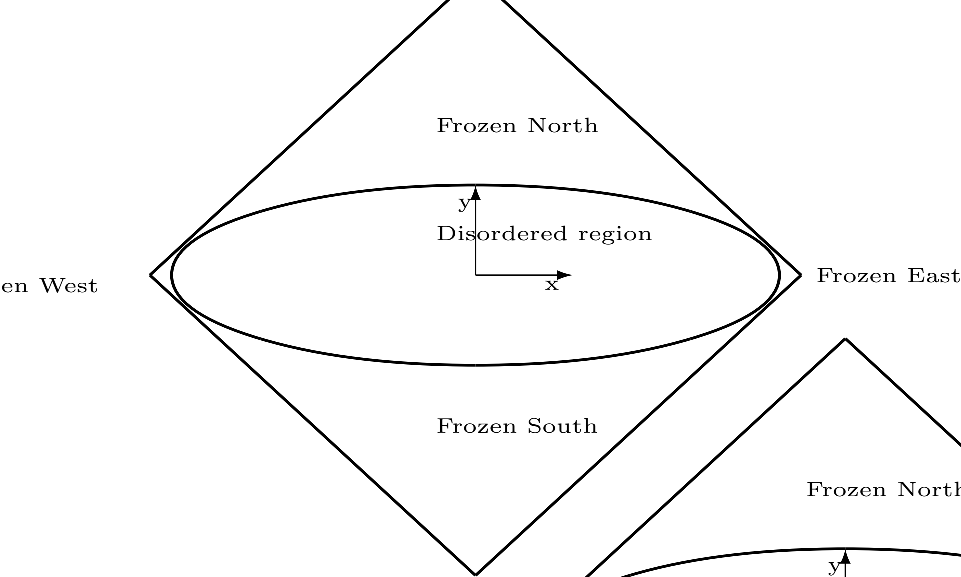
$$\frac{1}{1+a^2}\mathbb{L}_{n,\rho}(\xi_1,\eta_1;\xi_2,\eta_2) = \tilde{\tilde{\mathbb{K}}}_n^{\text{oneAzt}}(\xi_1,\eta_1;\xi_2,\eta_2) - \left\langle ((I-K_n)^{-1}A_{\xi_1,\eta_1})(k), B_{\xi_2,\eta_2}(k) \right\rangle_{\geq n-\rho+1},$$

Reminder:

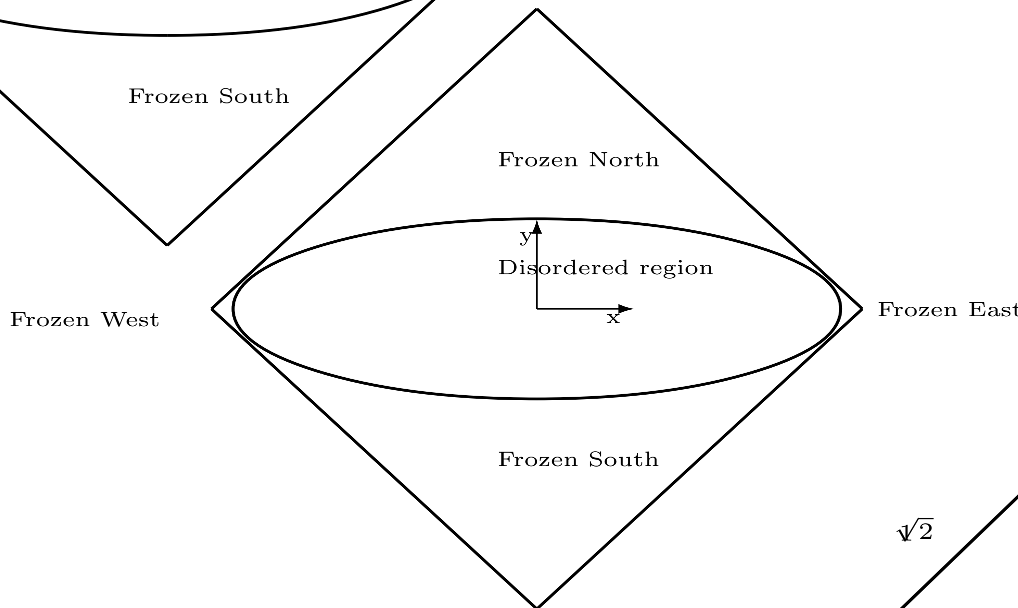
$$\mathbb{K}_n^{\textbf{OneAzt}}(2r,x;2s,y)$$
$$=\frac{(-1)^{x-y}}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}du\oint_{\Gamma_{0,u,a}}\frac{dv}{v-u}\frac{v^{-x}}{u^{1-y}}\frac{(1+au)^{n-s}(1-\frac{a}{u})^s}{(1+av)^{n-r}(1-\frac{a}{v})^r}$$
$$-\mathbb{1}_{s>r}\oint_{\Gamma_{0,a}}\frac{dz}{2\pi iz}z^{x-y}\left(\frac{1+az}{1-\frac{a}{z}}\right)^{s-r},$$

3. Tacnode Process ($n \to \infty$).

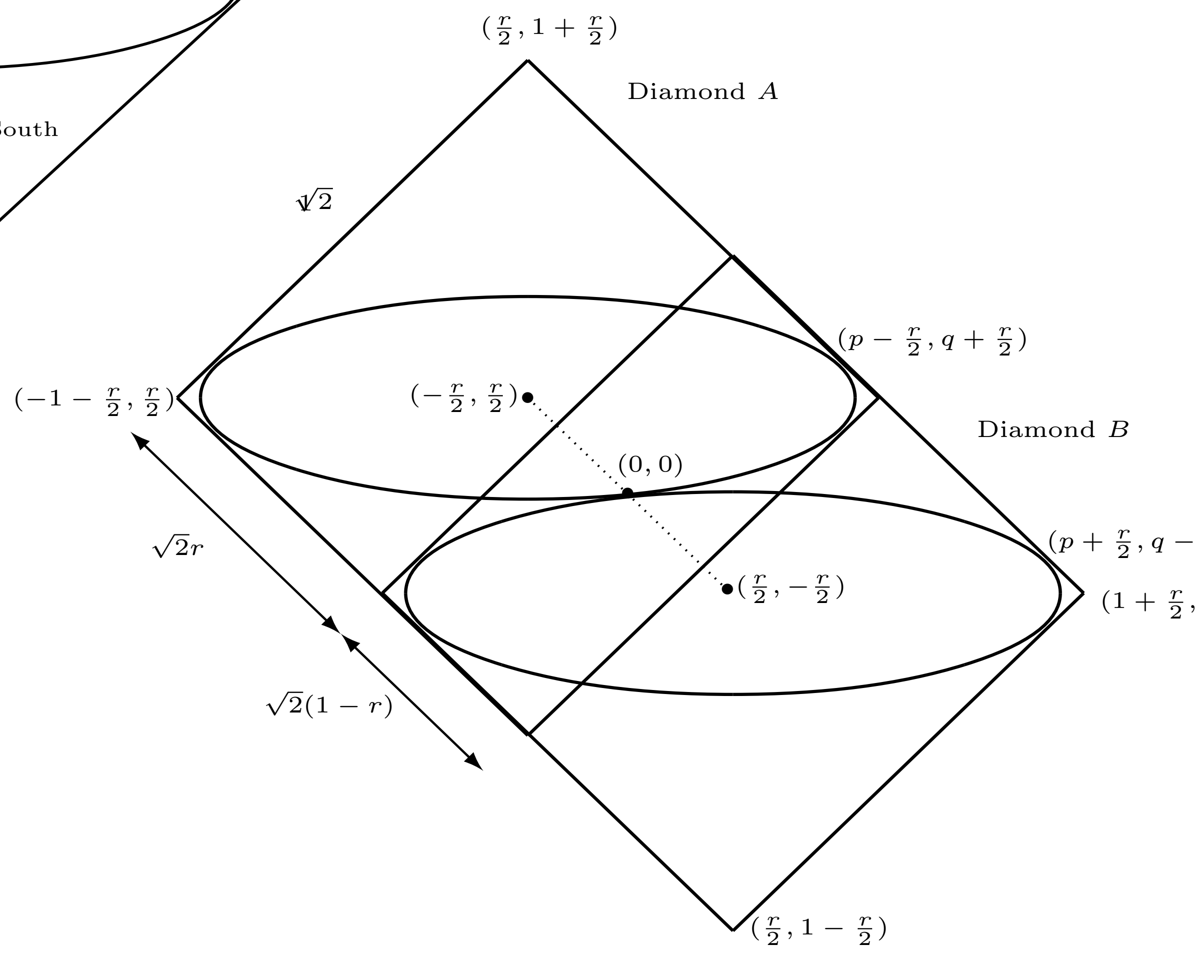




Frozen North
y
Disordered region
en West
x
Frozen East
Frozen South



Frozen North
y
Disordered region
Frozen West
x
Frozen East
Frozen South



$(\frac{r}{2}, 1+\frac{r}{2})$
Diamond A
$\sqrt{2}$
$(p-\frac{r}{2}, q+\frac{r}{2})$
$(-1-\frac{r}{2}, \frac{r}{2})$
$(-\frac{r}{2}, \frac{r}{2})$
Diamond B
$(0,0)$
$\sqrt{2}r$
$(p+\frac{r}{2}, q-$
$(\frac{r}{2}, -\frac{r}{2})$
$(1+\frac{r}{2},$
$\sqrt{2}(1-r)$
$(\frac{r}{2}, 1-\frac{r}{2})$

New statistics near the point of tangency of the two ellipses, when $n \to \infty$

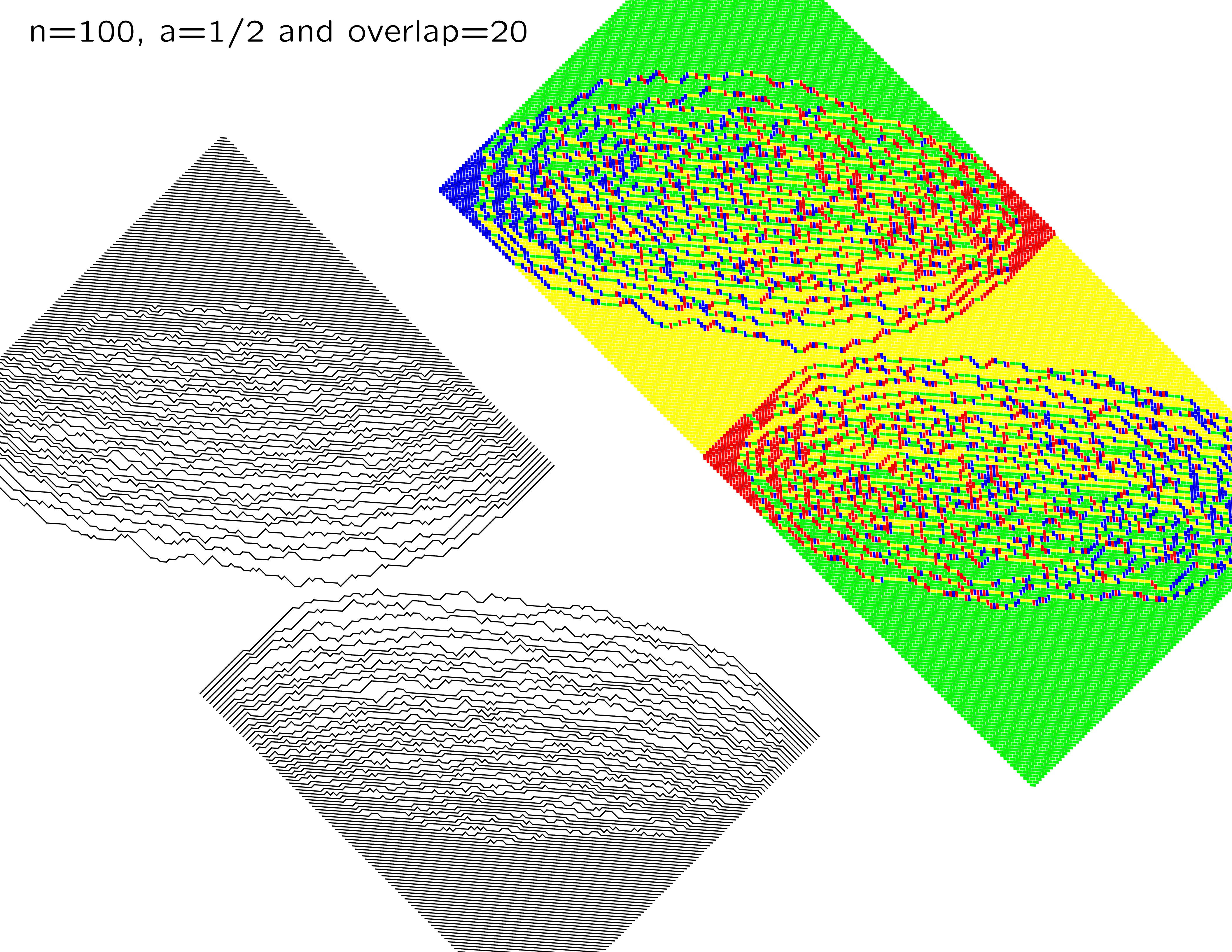
- **Overlap of the two diamonds** $= \rho = n(1 - \frac{2}{a+a^{-1}})$

- **Scaling** $z \sim \tau n^{-1/3}$ **in the tangential direction, giving new time** τ
- **Scaling** $x \sim \xi n^{-2/3}$ **in the oblique direction, giving new space** ξ

$$\mathbb{K}^{\mathsf{tac}}(\tau_1, \xi_1; \tau_2, \xi_2) = \frac{q_\sigma(\tau_1, \xi_1)}{q_\sigma(\tau_2, \xi_2)} \mathbb{K}^{\mathsf{AiryProcess}}(\tau_2, \sigma - \xi_2 + \tau_2^2; \tau_1, \sigma - \xi_1 + \tau_1^2) + 2^{1/3} \int_{\tilde{\sigma}}^{\infty} \left((\mathbb{1} - K_{\mathsf{Ai}})_{\tilde{\sigma}}^{-1} \mathcal{A}_{\xi_1 - \sigma}^{\tau_1} \right)(\lambda) \mathcal{A}_{\xi_2 - \sigma}^{-\tau_2}(\lambda) d\lambda.$$

(Adler-Johansson-PvM 2011)

Related work by: Ferrari-Vetö (2012), Delvaux-Kuijlaars-Zhang (2011), Adler-Ferrari-PvM (2013), Johansson (2012) n=100, a=1/2 and overlap=20



4. The Edge Tacnode (a limit of the $\mathbb{L}$-process!) : Two Aztec diamonds overlapping.

Letting the size $n \to \infty$ and keeping the **overlap fixed**.

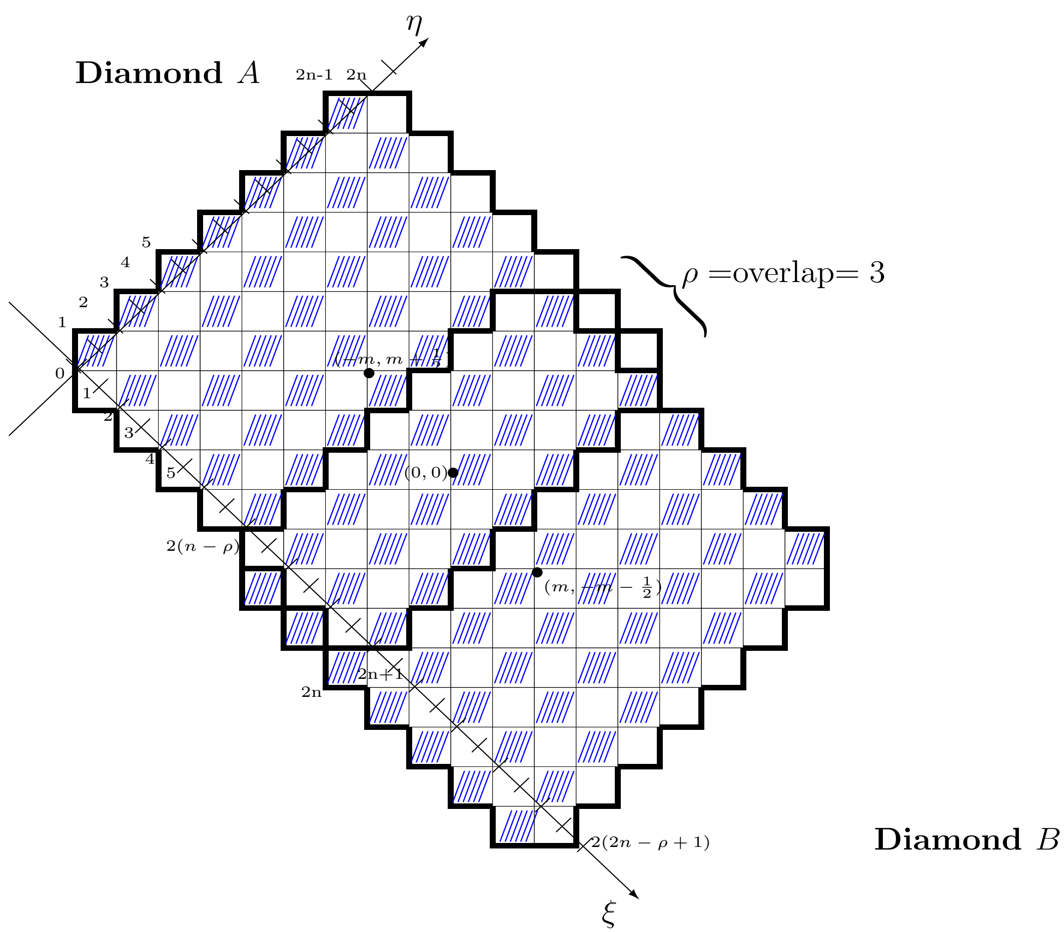
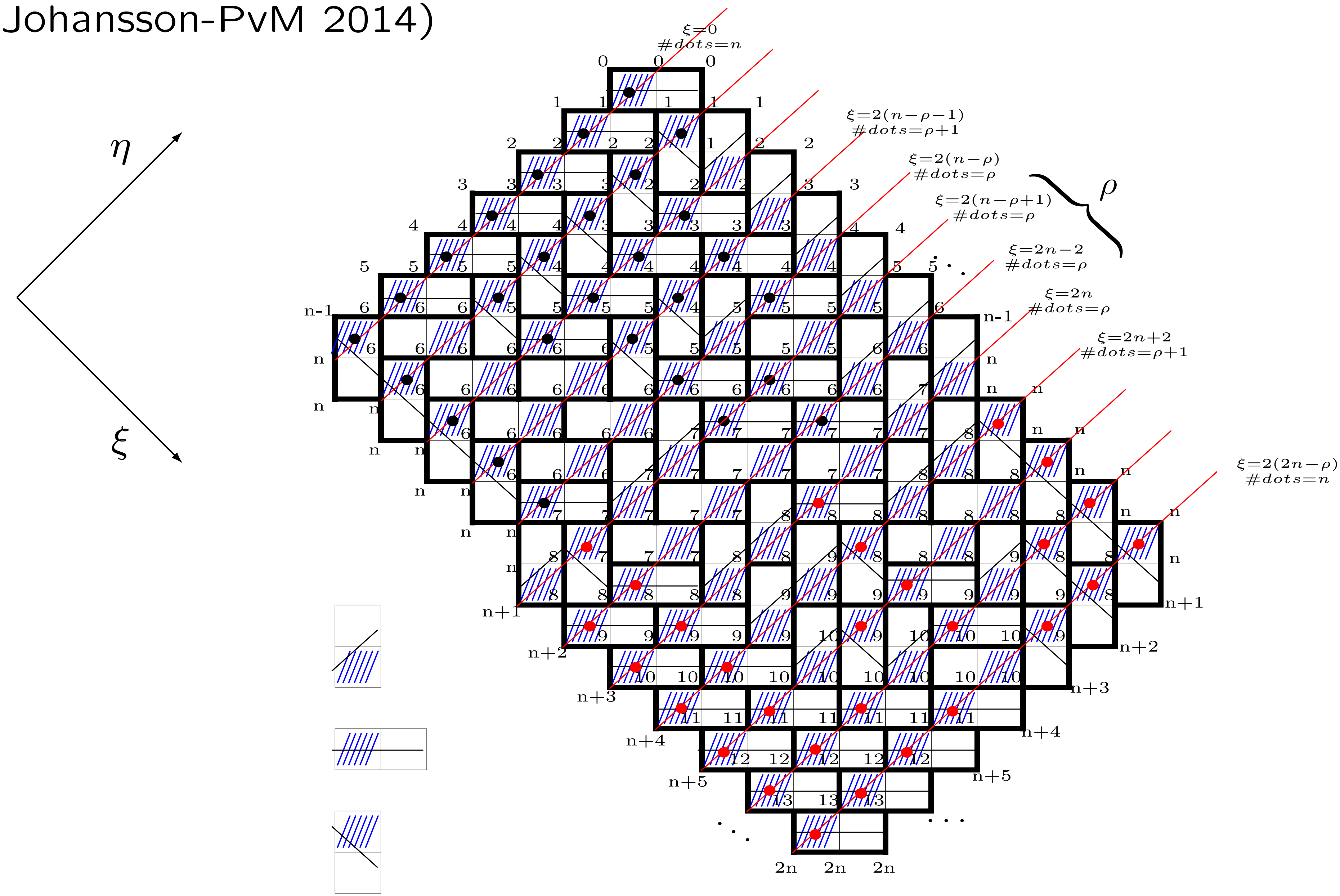


Figure 1. Double Aztec diamond of size $n = 7$ and overlap $\rho = 3$.

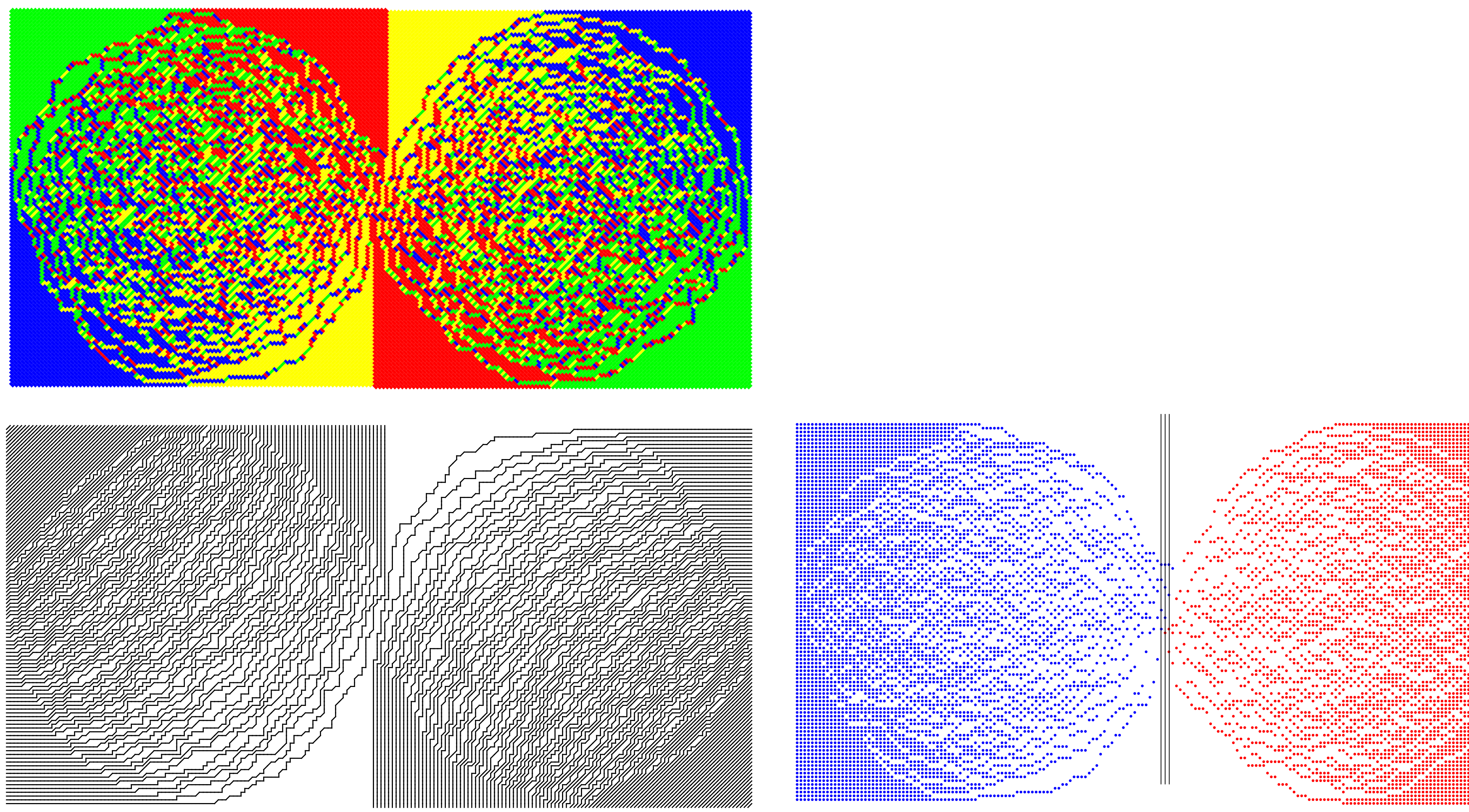
Reminder: The $\mathbb{L}$-determinantal Processes (Adler-Chhita-Johansson-PvM 2014)



Each time a line $\xi = 2k$ intersects an

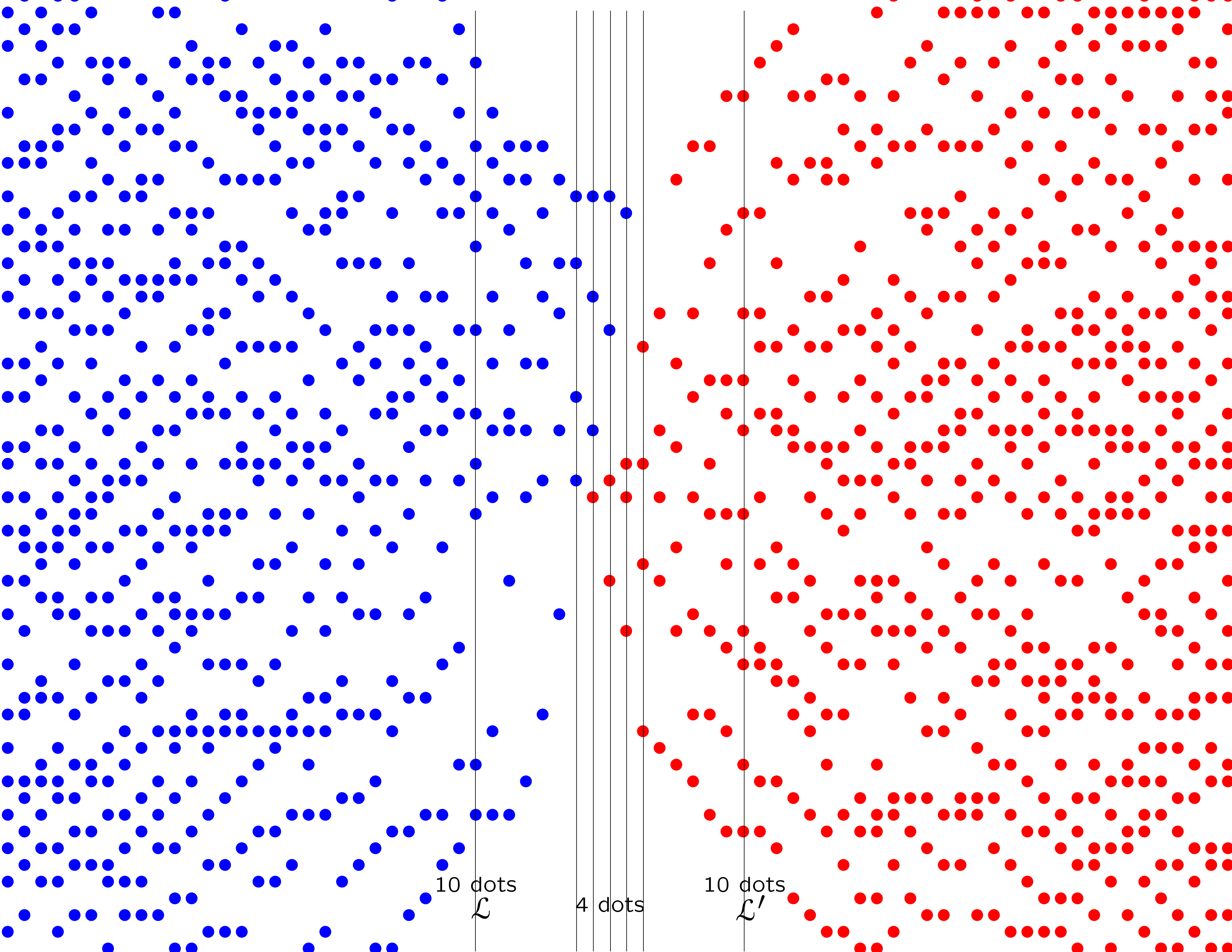
(1) A-level line: put a black dot in that square
(2) B-level line: put a red dot in that square

} *interlacing*!



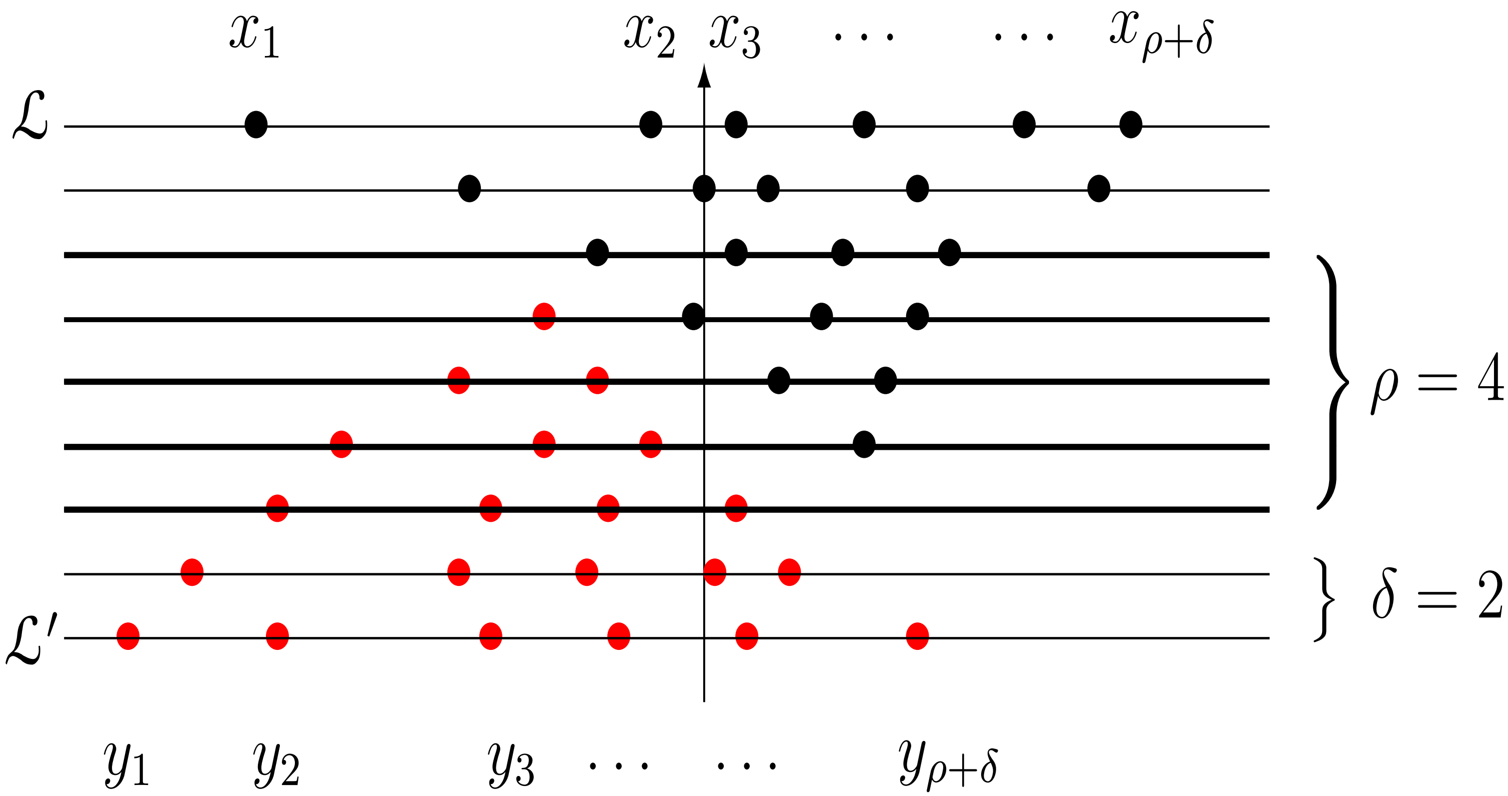
Diamond size $n = 100$ and overlap $\rho = 4$ with weight $a = 1$

(Courtesy Sunil Chhita)



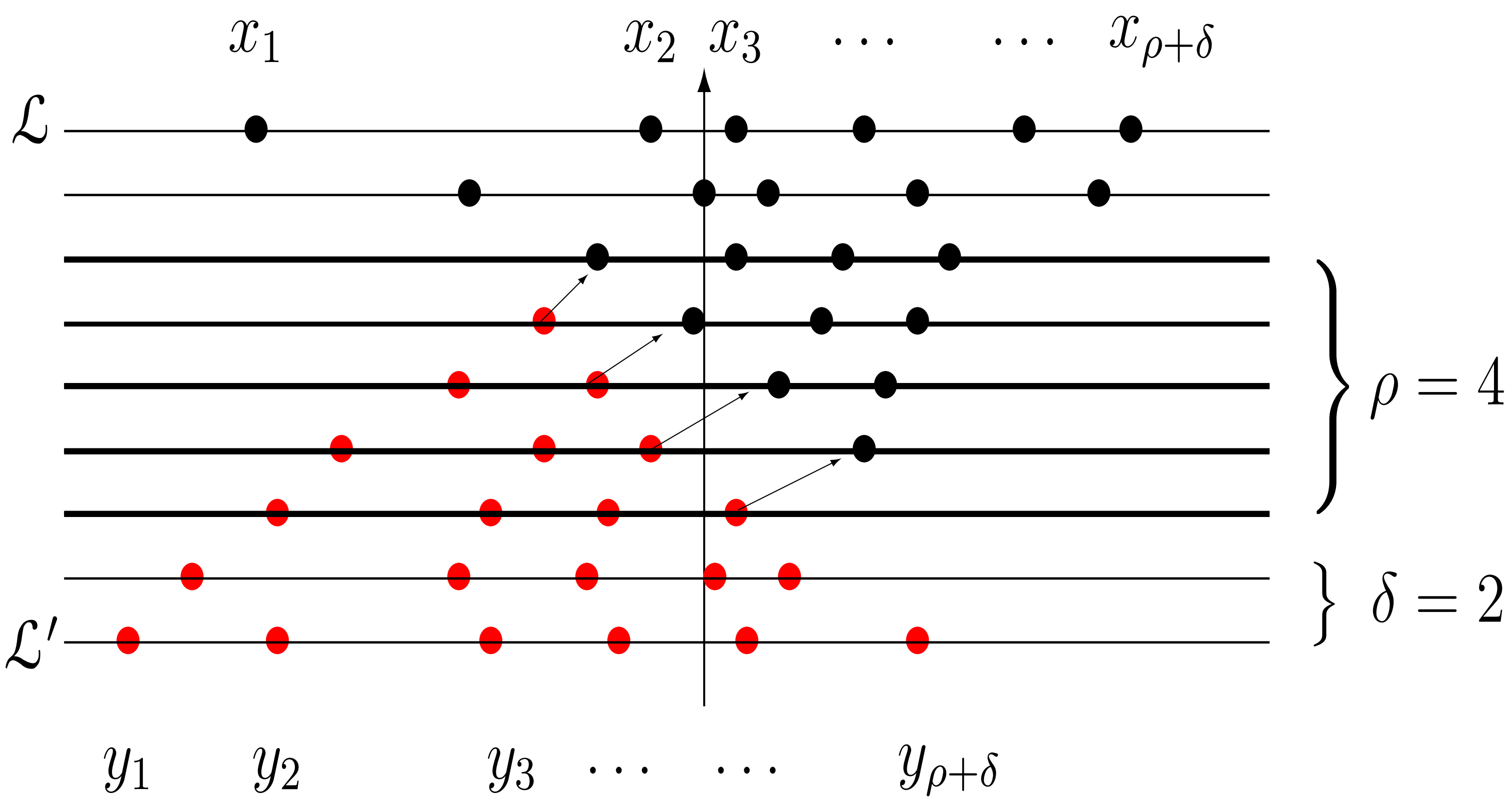
10 dots
$\mathcal{L}$
4 dots
10 dots
$\mathcal{L}'$

• **(Discrete) interlacing:**



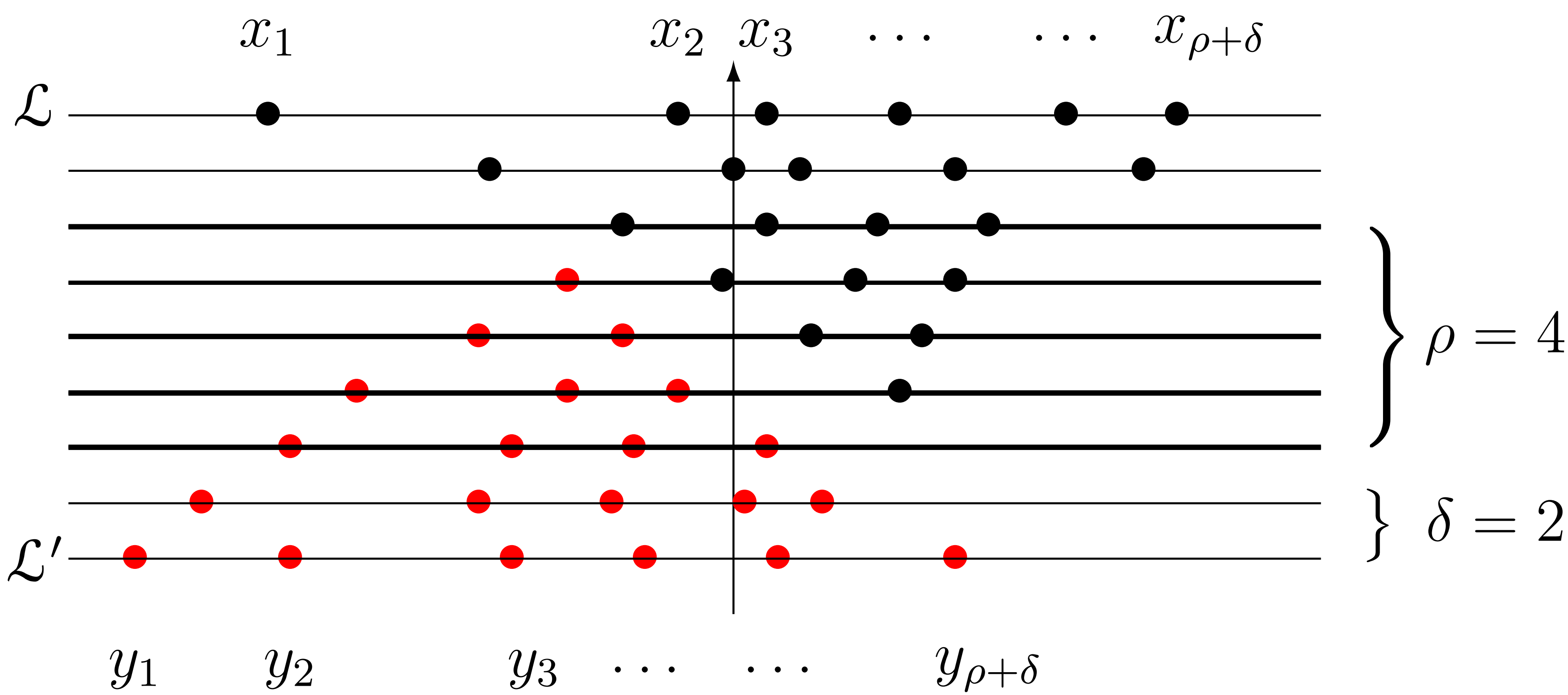
Any such set of (discrete) blue and red dots, respecting the interlacing corresponds to a tiling with domino's !

• **(Discrete) interlacing:**



Any such set of (discrete) blue and red dots, respecting the interlacing corresponds to a tiling with domino's !

• **Given the position of the dots along the lines $\mathcal{L}$ and $\mathcal{L}'$, the blue and red dots in between are uniformly distributed, respecting the numbers and the interlacing.**



• Edge-Tacnode kernel, as a scaling limit of the $\mathbb{L}$-kernel

Take scaling limit:

$$\begin{cases} \text{size } n \to \infty, \\ \rho = \text{fixed overlap of two diamonds}, \\ a = 1 - \beta\sqrt{\frac{2}{n}} = \text{weight of vertical domino's}, \quad \text{with } \beta \geq 0. \end{cases}$$

Scaling of the coordinates (ξ, η), $\Longrightarrow$ coordinates $(u, y) \in \mathbb{Z} \times \mathbb{R}$:

$$\xi = 2n - 2u, \qquad \eta = n + [y\sqrt{2n}] - 1.$$

LIMITING KERNEL: Edge-Tacnode Kernel

$$\lim_{n\to\infty} (-a)^{\frac{1}{2}(\eta_1-\eta_2)}(-\sqrt{n/2})^{\frac{1}{2}(\xi_1-\xi_2)}\mathbb{L}_{n,\rho}(\xi_1, \eta_1; \xi_2, \eta_2)\frac{\Delta\eta_2}{2}$$
$$= \mathbb{K}^{\text{EdgeTac}}_{\beta,\rho}(u_1, y_1; u_2, y_2)dy_2.$$

Edge-Tacnode kernel, two parameters: $\begin{cases} \beta = \text{speed at which } \ a \to 1 \\ \rho = \text{overlap of two diamonds} \end{cases}$

$$\begin{aligned} \tfrac{1}{2}\mathbb{K}^{\text{EdgeTac}}_{\beta,\rho}(u_1, y_1; u_2, y_2) = {} & \tfrac{1}{2}\mathbb{K}^{\text{minor}}(u_1, \beta - y_1;\ u_2, \beta - y_2) \\ & + \left\langle (\mathbb{1} - \mathcal{K}^{\beta}(\lambda,\kappa))^{-1}\ \mathcal{A}^{\beta, y_1-\beta}_{u_1}(\kappa), \mathcal{B}^{\beta, y_2-\beta}_{u_2}(\lambda) \right\rangle_{\geq -\rho} \end{aligned}$$

where

$$\begin{aligned} \tfrac{1}{2}\text{GUE-minor kernel} = \tfrac{1}{2}\mathbb{K}^{\text{minor}}(n, x; n', x') := {} & -\mathbb{1}_{n>n'} \frac{(2(x-x'))^{n-n'-1}}{(n-n'-1)!} \mathbb{1}_{x \geq x'} \\ & + \oint_{\Gamma} \frac{dz}{(2\pi \mathrm{i})^2} \oint_L \frac{dw}{w-z} \frac{e^{-z^2+2zx}}{e^{-w^2+2wx'}} \frac{w^{n'}}{z^n}, \end{aligned}$$

where, given a GUE-matrix A,

$$\det(\mathbb{K}^{\text{minor}}(n_i, x_i; n_j, x_j))_{1 \leq i,j \leq 2} dx_1 dx_2$$

$$= \mathbb{P} \begin{pmatrix} \text{an eigenvalue of the } n_1\text{th minor } A^{(n_1)} \in dx_1 \\ \text{an eigenvalue of the } n_2\text{th minor } A^{(n_2)} \in dx_2 \end{pmatrix}$$

5. The Edge-Tacnode kernel and coupled GUE-matrices

The main message is the following equivalence:

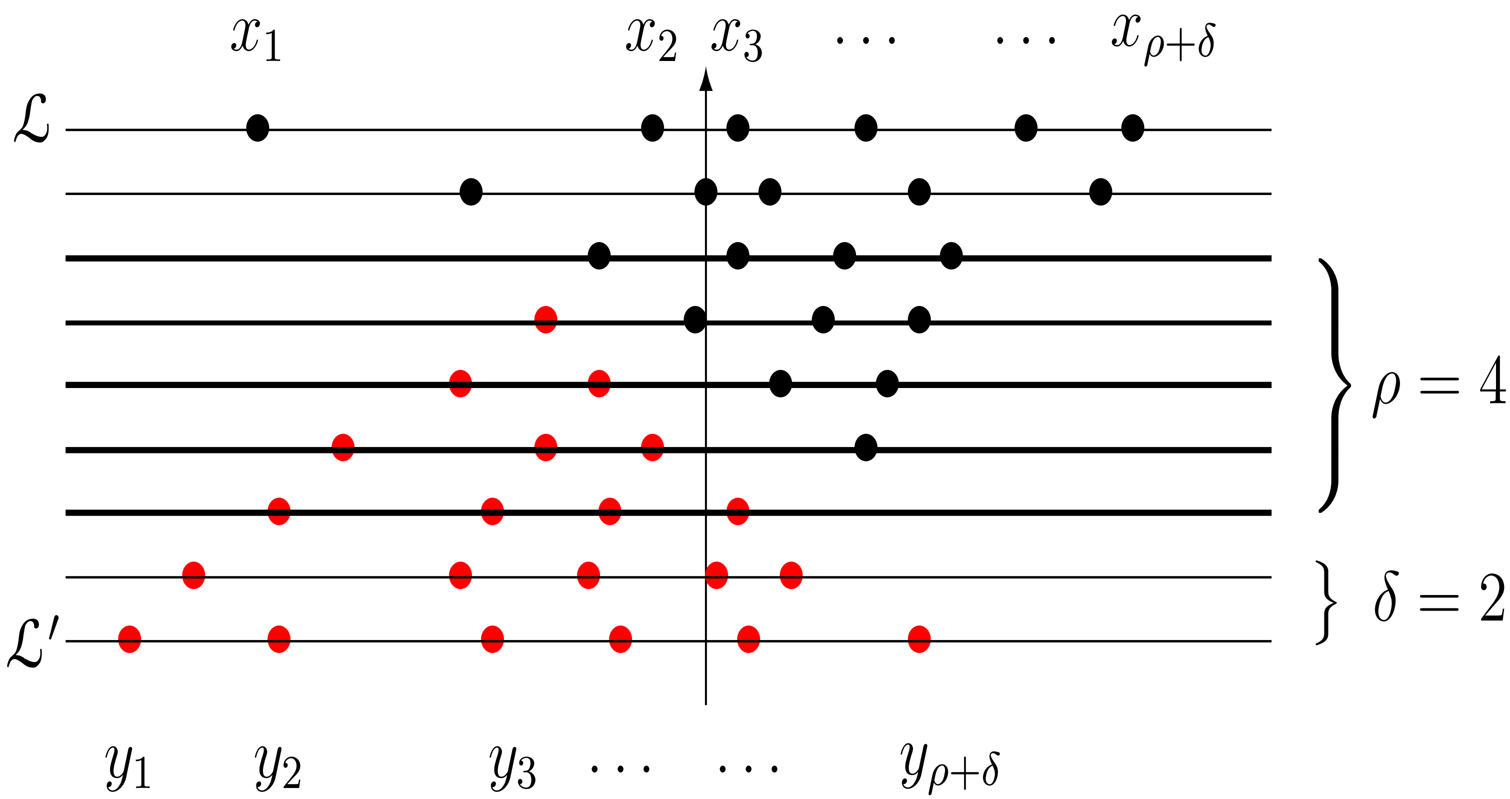
$$\left\{\begin{array}{l}\textit{The statistics of the eigenvalues of}\\ \textit{the consecutive minors } A^{(i)} \textit{ and } B^{(i)}\\ \textit{of the } \textbf{coupled} \textit{ GUE-matrices } A \textit{ and } B\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\textit{The statistics of the dot-}\\ \textit{particles in the rescaled}\\ \textit{double-Aztec diamonds}\\ \textit{when its size } n \to \infty\\ \textit{(Edge-tacnode Process)}\end{array}\right\}$$

What coupling of GUE-matrices do we refer to? (Cond. probability)

$$\mathbb{P}(E) := \mathbb{P}^{\mathrm{GUE}}\left(E \;\middle|\; \bigcap_{1}^{\rho}\left\{\max \mathit{spec}(B^{(i)}) \leq \min \mathit{spec}(A^{(\rho-i+1)}) \text{ , for } 1 \leq i \leq \rho\right\}\right)$$

Example: Two GUE-matrices A and B of size 6, conditioned by

$$\rho = 4 \quad \text{conditions} \begin{cases} \text{spectrum } (A^{(1)}) < \text{spectrum } (B^{(4)}) \\ \text{spectrum } (A^{(2)}) < \text{spectrum } (B^{(3)}) \\ \text{spectrum } (A^{(3)}) < \text{spectrum } (B^{(2)}) \\ \text{spectrum } (A^{(4)}) < \text{spectrum } (B^{(1)}) \end{cases}$$



- Extension of Baryshnikov result:

$$\mathbb{P}\left(\bigcap_1^{n-1}\{\mathbf{x}^{(k)} \in d\mathbf{x}^{(k)},\ \mathbf{y}^{(k)} \in d\mathbf{y}^{(k)}\}\Big|\ \mathbf{x}^{(n)} = \mathbf{x},\ \mathbf{y}^{(n)} = \mathbf{y}\right) = \frac{d\mu_{\mathbf{x},\mathbf{y}}}{\mathrm{Vol}(\mathcal{C}_{\mathbf{xy}}^{(n)})},$$

• In the limit $n \to \infty$, for $u = -\delta$ (at a distance δ outside the overlap)
$\mathbb{P}$(particles in $dy_1, \ldots, dy_{\rho+\delta}$)

$$= C_{\rho,\delta} \Delta_{\rho+\delta}(y) \widetilde{\Delta}^{\beta}_{\rho+\delta}(y) \prod_{1}^{\rho+\delta} \frac{e^{-(y_i-\beta)^2}}{\sqrt{\pi}} dy_1 \ldots dy_{\rho+\delta}$$

where

$$\widetilde{\Delta}^{\beta}_{\rho+\delta}(y) :=$$

$$\det \begin{pmatrix} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ y_1 & y_2 & \ldots & y_{\rho+\delta} \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{\delta-1} & y_2^{\delta-1} & \ldots & y_{\rho+\delta}^{\delta-1} \end{pmatrix} \\ \\ \Phi_\delta(\beta - y_k) \\ \vdots \\ \Phi_{\delta+\rho-1}(\beta - y_k) \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\} \delta = \left\{ \begin{matrix}\text{distance} \\ \text{to overlap}\end{matrix}\right. \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \rho = \text{overlap} \end{matrix}$$

where

$$\Phi_n(\eta) := \frac{2^n}{\sqrt{\pi} n!} \int_0^\infty \xi^n e^{-(\xi-\eta)^2} d\xi, \qquad n \geq 0,$$

Thank you!