# Critical two-point function for the $\varphi^4$ model in high dimensions

**@ Workshop on Stochastic Processes in Random Media**

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1. A pedagogical, yet nontrivial, model in lattice scaler-field theory.
2. The Hamiltonian for the spin configuration $\boldsymbol{\varphi} = \{\varphi_x\}_{x\in\Lambda}$, on $\Lambda \subset \mathbb{Z}^d$:

$$\mathscr{H}_\Lambda(\boldsymbol{\varphi}) = -\sum_{\{u,v\}\subset\Lambda} \mathscr{J}_{u,v}\,\varphi_u\varphi_v + \sum_{v\in\Lambda}\Big(\frac{\mu}{2}\varphi_v^2 + \frac{\lambda}{4!}\varphi_v^4\Big).$$

- $\mathscr{J}_{u,v} \equiv \mathscr{J}(u-v) \ge 0$ (ferro), $\mathscr{J}(o) = 0$, $\mathbb{Z}^d$-symmetric, all moments are finite. In particular, $\hat{\mathscr{J}} \equiv \sum_{x\in\mathbb{Z}^d} \mathscr{J}(x) < \infty$, $V \equiv \sum_{x\in\mathbb{Z}^d} |x|^2 \frac{\mathscr{J}(x)}{\hat{\mathscr{J}}} < \infty$.
- The "temperature" $\mu \in \mathbb{R}$ varies, while the "nonlineariry" $\lambda \ge 0$ is unchanged.

3. The two-point function: $\langle\varphi_o\varphi_x\rangle_{\mu,\Lambda} = \dfrac{\int_{\mathbb{R}^\Lambda} \varphi_o\varphi_x\, e^{-\mathscr{H}_\Lambda(\boldsymbol{\varphi})}\mathrm{d}\boldsymbol{\varphi}}{\int_{\mathbb{R}^\Lambda} e^{-\mathscr{H}_\Lambda(\boldsymbol{\varphi})}\mathrm{d}\boldsymbol{\varphi}} \underset{\Lambda\uparrow\mathbb{Z}^d}{\nearrow} \langle\varphi_o\varphi_x\rangle_\mu$

($\because$ the 2nd Griffiths inequality).

Since $\mathscr{H}_\Lambda(\boldsymbol{\varphi}) = \underbrace{\frac{\hat{\mathscr{J}}}{2}(\boldsymbol{\varphi}, -\Delta\boldsymbol{\varphi})}_{\text{Kinetic energy}} + \underbrace{\sum_{v\in\Lambda}\Big(\frac{\mu-\hat{\mathscr{J}}}{2}\varphi_v^2 + \frac{\lambda}{4!}\varphi_v^4\Big)}_{\text{Potential energy}}$,

- $\lambda = 0 \;\Rightarrow\;$ well-defined only when $\mu \ge \hat{\mathscr{J}}$: the Gaussian critical point.
- $\lambda > 0$, $\mu \begin{cases} \ge \hat{\mathscr{J}} & \Rightarrow \text{ single-well potential,} \\ < \hat{\mathscr{J}} & \Rightarrow \text{ double-well potential.} \end{cases}$

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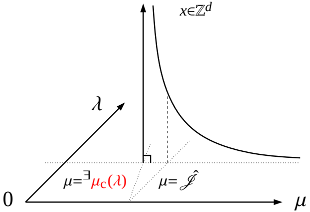
1. Lebowitz' inequality implies

$$\chi_\mu \equiv \sum_{x\in\mathbb{Z}^d} \langle\varphi_o\varphi_x\rangle_\mu$$




2. The bubble condition for the mean-field behavior [Aiz:1982]:

$$\sum_{x\in\mathbb{Z}^d} \langle\varphi_o\varphi_x\rangle_{\mu_c}^2 < \infty \quad\Rightarrow\quad \chi_\mu \underset{\mu\downarrow\mu_c}{\asymp} (\mu-\mu_c)^{-1}.$$

3. For reflection-positive $\mathscr{J}$ in dimensions $d>2$ [Frö-Sim-Spe:1976],

$$0 \le \sum_{x\in\mathbb{Z}^d} e^{ik\cdot x}\langle\varphi_o\varphi_x\rangle_\mu \le O(|k|^{-2}) \qquad {}^\forall k\in[-\pi,\pi]^d, \text{ uniformly in } \mu>\mu_c.$$

In particular, for the nearest-neighbor model [Sok:1982],

$$\langle\varphi_o\varphi_x\rangle_\mu \le \frac{O(1)}{(|x|\vee 1)^{d-2}} \qquad {}^\forall x\in\mathbb{Z}^d, \text{ uniformly in } \mu>\mu_c.$$

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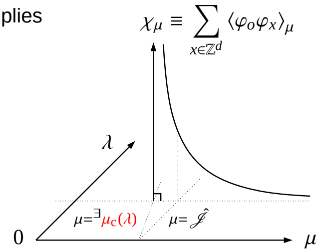


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4 Renormalization-group results for the nearest-neighbor model in 4 dimensions:

$$\lambda \ll 1 \quad \Rightarrow \quad \begin{cases} \langle \varphi_o \varphi_x \rangle_{\mu_c} \underset{|x|\uparrow\infty}{\sim} {}^{\exists}C_1 |x|^{-2} & \text{[Gaw-Kup:1984]}, \\ \chi_\mu \underset{\mu\downarrow\mu_c}{\sim} {}^{\exists}C_2 (\mu-\mu_c)^{-1} \big|\log(\mu-\mu_c)\big|^{1/3} & \text{[Har-Tas:1987]}. \end{cases}$$

For the $n$-component $|\boldsymbol{\varphi}|^4$ model, $1/3$ is replaced by $\frac{n+2}{n+8}$ [Bau-Bry-Sla:2014].

5 Simulation for the nearest-neighbor Ising model in 3 dimensions [Pel-Vic:2002]:

$$\chi_\mu \underset{\mu\downarrow\mu_c}{\approx} (\mu-\mu_c)^{-1.23725\cdots}.$$

6 By Onsager's exact solution for the nearest-neighbor Ising model in 2 dimensions,

$$\chi_\mu \underset{\mu\downarrow\mu_c}{\approx} (\mu-\mu_c)^{-7/4}.$$

The Schwinger-Dyson equation: by integration-by-parts $\left\langle \frac{\partial \mathscr{H}_\Lambda(\varphi)}{\partial \varphi_o} \varphi_x \right\rangle_{\mu,\Lambda} = \delta_{o,x}$,

$$\underset{\Lambda\uparrow\mathbb{Z}^d}{\rightarrow} \quad -\sum_{v\in\mathbb{Z}^d} \mathscr{J}(v)\,\langle\varphi_v\varphi_x\rangle_\mu + \mu\,\langle\varphi_o\varphi_x\rangle_\mu + \frac{\lambda}{3!}\left\langle\varphi_o^3\varphi_x\right\rangle_\mu = \delta_{o,x}.$$

$$\updownarrow$$

$$-\sum_{v\in\mathbb{Z}^d} \mathscr{J}(v)\,\langle\varphi_v\varphi_x\rangle_\mu^{\lambda=0} + \mu\,\langle\varphi_o\varphi_x\rangle_\mu^{\lambda=0} = \delta_{o,x} \quad \overset{d>2}{\Longrightarrow} \quad \langle\varphi_o\varphi_x\rangle_{\hat{\mathscr{J}}}^{\lambda=0} \underset{|x|\uparrow\infty}{\sim} \frac{\frac{d}{2}\pi^{-d/2}\Gamma(\frac{d-2}{2})}{\hat{\mathscr{J}}\,V\,|x|^{d-2}}.$$

**Theorem [Sak:2015]**

Let $d > 4$ and $\lambda \ll 1$ (depending on $d$ and $\mathscr{J}$). Then,

$$^\exists\Phi_\mu(x) = \left\langle\varphi_o^2\right\rangle_\mu \delta_{o,x} + \frac{O(\lambda)}{(|x|\vee 1)^{3(d-2)}} \qquad \text{uniformly in } \mu > \mu_c,$$

such that the following linearized Schwinger-Dyson equation holds:

$$-\sum_{v\in\mathbb{Z}^d} \mathscr{J}(v)\,\langle\varphi_v\varphi_x\rangle_\mu + \mu\,\langle\varphi_o\varphi_x\rangle_\mu + \frac{\lambda}{2}\sum_{v\in\mathbb{Z}^d} \Phi_\mu(v)\,\langle\varphi_v\varphi_x\rangle_\mu = \delta_{o,x}.$$

As a result, $\mu_c = \hat{\mathscr{J}} - \frac{\lambda}{2}\hat{\Phi}_{\mu_c} \equiv \hat{\mathscr{J}} - \frac{\lambda}{2}\sum_{v\in\mathbb{Z}^d}\Phi_{\mu_c}(v)$, $\langle\varphi_o\varphi_x\rangle_{\mu_c} \underset{|x|\uparrow\infty}{\sim} \frac{\frac{d}{2}\pi^{-d/2}\Gamma(\frac{d-2}{2})}{\hat{\mathscr{J}}\,V + O(\lambda^2)}\,|x|^{2-d}$.

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$$\underset{\Lambda \uparrow \mathbb{Z}^d}{\rightarrow} \quad -\sum_{v \in \mathbb{Z}^d} \mathscr{J}(v) \, \langle \varphi_v \varphi_x \rangle_\mu + \mu \, \langle \varphi_o \varphi_x \rangle_\mu + \frac{\lambda}{3!} \left\langle \varphi_o^3 \varphi_x \right\rangle_\mu = \delta_{o,x}.$$

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## Remark

1. Similar results hold for large $\lambda$ if $\hat{\mathscr{J}} \gg 1$ (e.g., $d \gg 4$).
2. If $\mathscr{J}(x) \propto |x|^{-d-\alpha}$ for some $\alpha > 0$, then

$$1-\frac{\hat{\mathscr{J}}(k)}{\hat{\mathscr{J}}} \equiv 1-\sum_{x\in\mathbb{Z}^d} e^{ik\cdot x}\frac{\mathscr{J}(x)}{\hat{\mathscr{J}}} \underset{|k|\to 0}{\sim} \frac{{}^{\exists}\hat{V}_\alpha}{2d}|k|^{\alpha\wedge 2}\times\begin{cases}1 & [\alpha\neq 2],\\ \log\frac{1}{|k|} & [\alpha=2].\end{cases}$$

For $d > 2(\alpha\wedge 2)$, or for $d \geq 4$ when $\alpha = 2$,

$$\Big|\sum_{x\in\mathbb{Z}^d} e^{ik\cdot x}\langle\varphi_o\varphi_x\rangle_\mu\Big| \leq \frac{O(1)}{\hat{\mathscr{J}}-\hat{\mathscr{J}}(k)}$$

uniformly in $\mu > \mu_c$, hence $\chi_\mu \asymp (\mu-\mu_c)^{-1}$ ($\because$ extension of [Hey-Hof-Sak:2008]).

3. Under an additional condition on the "derivative" of $\mathscr{J}^{*n}(x)$ [Che-Sak:2015, 201?],

$$\langle\varphi_o\varphi_x\rangle_{\mu_c} \underset{|x|\uparrow\infty}{\sim} \frac{\frac{2d}{2^{\alpha\wedge 2}}\pi^{-d/2}\Gamma(\frac{d-\alpha\wedge 2}{2})\big/\Gamma(\frac{\alpha\wedge 2}{2})}{\hat{\mathscr{J}}\hat{V}_\alpha+O(\lambda^2)\mathbb{1}_{\{\alpha>2\}}}|x|^{\alpha\wedge 2-d}\times\begin{cases}1 & [\alpha\neq 2,\ d>2(\alpha\wedge 2)],\\ \frac{1}{\log|x|} & [\alpha=2,\ d\geq 4],\end{cases}$$

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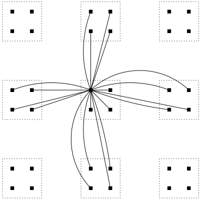
**1** Ising spins on $\tilde{\Lambda}_N \equiv \Lambda \times \{1, \dots, N\}$ $(\Lambda \subset \mathbb{Z}^d,\ N \in \mathbb{N})$.



Nearest-neighbor bonds from a single site on $\tilde{\mathbb{Z}}^d_4 \equiv \mathbb{Z}^d \times \{1, \dots, 4\}$.

**2** The Hamiltonian for the Ising-spin configuration $\boldsymbol{\sigma} = \{\sigma_{(x,j)}\} \in \{\pm 1\}^{\tilde{\Lambda}_N}$, on $\tilde{\Lambda}_N$:

$$H_{\tilde{\Lambda}_N}(\boldsymbol{\sigma}) = -\frac{1}{2} \sum_{u \neq v \in \Lambda} J_{u,v} \tilde{\sigma}_u \tilde{\sigma}_v - \frac{I}{2} \sum_{v \in \Lambda} \tilde{\sigma}_v^2,$$

where $\tilde{\sigma}_x \equiv \sum_{j=1}^N \sigma_{(x,j)}$ is the block spin at $x \in \Lambda$.

**The Griffiths-Simon construction [Sim-Gri:1973]**

Let $\epsilon_N = (\frac{\lambda}{2} N^3)^{-1/4}$, $J_{u,v} = \mathscr{J}_{u,v} \epsilon_N^2$, $I = \frac{1}{N} - \mu \epsilon_N^2$. Then,

$$\epsilon_N^2 \langle\!\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\!\rangle_{\tilde{\Lambda}_N} \equiv \epsilon_N^2 \frac{\sum_{\boldsymbol{\sigma}} \tilde{\sigma}_o \tilde{\sigma}_x \, e^{-H_{\tilde{\Lambda}_N}(\boldsymbol{\sigma})}}{\sum_{\boldsymbol{\sigma}} e^{-H_{\tilde{\Lambda}_N}(\boldsymbol{\sigma})}} \underset{N \uparrow \infty}{\longrightarrow} \langle \varphi_o \varphi_x \rangle_\Lambda .$$

## Idea of the Griffiths-Simon construction

1. The marginal distribution given $\tilde{\boldsymbol{\sigma}} = \{\tilde{\sigma}_x\}_{x\in\Lambda}$:

$$(\bigstar) \quad \Big(\frac{1}{2}\Big)^{|\tilde{\Lambda}_N|} \sum_{\substack{\boldsymbol{\sigma}\in\{\pm1\}^{\tilde{\Lambda}_N} \\ (\tilde{\boldsymbol{\sigma}} \text{ fixed})}} e^{-H_{\tilde{\Lambda}_N}(\boldsymbol{\sigma})} = \exp\bigg(\frac{1}{2}\sum_{u\neq v\in\Lambda} J_{u,v}\tilde{\sigma}_x\tilde{\sigma}_y + \frac{I}{2}\sum_{v\in\Lambda}\tilde{\sigma}_v^2\bigg)\prod_{x\in\mathbb{Z}^d}\binom{N}{\frac{N+\tilde{\sigma}_x}{2}}\Big(\frac{1}{2}\Big)^N.$$

2. $\frac{I}{2}\tilde{\sigma}_x^2 + \log\Big(\binom{N}{\frac{N+\tilde{\sigma}_x}{2}}\Big(\frac{1}{2}\Big)^N\Big)$

$$\underset{\varphi_x\equiv\epsilon_N\tilde{\sigma}_x}{\overset{\text{Stirling}}{\sim}} \underbrace{\frac{\epsilon_N^{-2}}{2}\Big(I - \frac{1}{N}}_{-\mu/2} + O(N^{-2})\Big)\varphi_x^2 - \underbrace{\frac{\epsilon_N^{-4}}{12N^3}}_{\lambda/4!}\varphi_x^4 + o_\varphi(N^{-1/4}) + O(\log N).$$

3. $$(\bigstar) \underset{J_{u,v}=\mathscr{J}_{u,v}\epsilon_N^2}{\propto} \exp\bigg(\frac{1}{2}\sum_{u\neq v\in\Lambda}\mathscr{J}_{u,v}\varphi_u\varphi_v - \sum_{v\in\Lambda}\Big(\frac{\mu + O(N^{-1/2})}{2}\varphi_v^2 + \frac{\lambda}{4!}\varphi_v^4\Big) + o_\varphi(N^{-1/4})\bigg)$$

$$\sim e^{-\mathscr{H}_\Lambda(\boldsymbol{\varphi})}.$$

1 To make life easier, let $N = 1$ and $I = 0$:

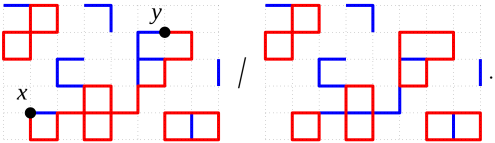
$$\langle\!\langle \tilde{\sigma}_x \tilde{\sigma}_y \rangle\!\rangle_{\tilde{\Lambda}_N} = \frac{\sum_{\boldsymbol{\sigma}} \sigma_x \sigma_y \, e^{-H_\Lambda(\boldsymbol{\sigma})}}{\sum_{\boldsymbol{\sigma}} e^{-H_\Lambda(\boldsymbol{\sigma})}}, \qquad H_\Lambda(\boldsymbol{\sigma}) = -\sum_{\{u,v\}\subset\Lambda} J_{u,v}\sigma_u\sigma_v.$$

2
$$\left(\frac{1}{2}\right)^{|\Lambda|} \sum_{\boldsymbol{\sigma}} e^{-H_\Lambda(\boldsymbol{\sigma})} = \sum_{\boldsymbol{\sigma}} \sum_{\mathbf{n}=\{n_b\}} \prod_b \frac{J_b^{n_b}}{n_b!} \prod_v \frac{\sigma_v^{\sum_{b\ni v} n_b}}{2} = \sum_{\mathbf{n}=\{n_b\}} \prod_b \frac{J_b^{n_b}}{n_b!} \prod_v \mathbb{1}_{\{\sum_{b\ni v} n_b \text{ is even}\}},$$

$$\left(\frac{1}{2}\right)^{|\Lambda|} \sum_{\boldsymbol{\sigma}} \sigma_x \sigma_y \, e^{-H_\Lambda(\boldsymbol{\sigma})} = \sum_{\mathbf{n}=\{n_b\}} \prod_b \frac{J_b^{n_b}}{n_b!} \prod_v \mathbb{1}_{\{\delta_{v,x}+\delta_{v,y}+\sum_{b\ni v} n_b \text{ is even}\}}.$$

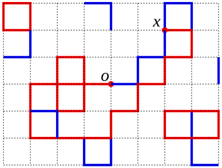
**The random-current representation (for $N = 1$, $I = 0$) [Gri-Hur-She:1970]**

$\langle\!\langle \tilde{\sigma}_x \tilde{\sigma}_y \rangle\!\rangle_{\tilde{\Lambda}_N} =$

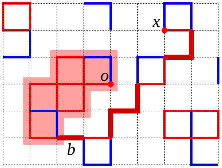



3 Next, extract a recursive structure out of the connection between the two points.

1. Split the event depending on whether $^{\exists}$a pivotal bond for the connection:




$$\equiv \pi^{(0)}_{\tilde{\Lambda}_N}(o,x)$$



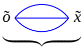

$$\simeq \sum_{b=(\underline{b},\overline{b})} \pi^{(0)}_{\tilde{\Lambda}_N}(o,\underline{b})\,(\tanh J_b)\,\langle\!\langle \sigma_{\overline{b}}\sigma_x \rangle\!\rangle_{\tilde{\Lambda}_N}.$$

2. Further investigation of the difference in "$\simeq$" yields

**The lace expansion for the Ising model [Sak:2007]**

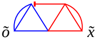
For $\tilde{u}=(u,i)$, $\tilde{v}=(v,j)\in\tilde{\Lambda}_N$, let $\tilde{J}_{\tilde{u},\tilde{v}}=\delta_{u,v}(1-\delta_{i,j})I+(1-\delta_{u,v})J_{u,v}$. Then,

$$^{\exists}\pi_{\tilde{\Lambda}_N}(\tilde{o},\tilde{x}) \equiv \sum_{n=0}^{\infty}(-1)^n\pi^{(n)}_{\tilde{\Lambda}_N}(\tilde{o},\tilde{x}) =$$




$$\le \langle\!\langle \sigma_{\tilde{o}}\sigma_{\tilde{x}}\rangle\!\rangle^3_{\tilde{\Lambda}_N}$$

$-$

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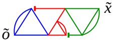



$-\cdots$

such that $\langle\!\langle \sigma_{\tilde{o}}\sigma_{\tilde{x}}\rangle\!\rangle_{\tilde{\Lambda}_N} = \pi_{\tilde{\Lambda}_N}(\tilde{o},\tilde{x}) + \sum_{\tilde{u},\tilde{v}\in\tilde{\Lambda}_N}\pi_{\tilde{\Lambda}_N}(\tilde{o},\tilde{u})\,(\tanh\tilde{J}_{\tilde{u},\tilde{v}})\,\langle\!\langle\sigma_{\tilde{v}}\sigma_{\tilde{x}}\rangle\!\rangle_{\tilde{\Lambda}_N}.$

**Bound on $\pi_{\tilde{\Lambda}_N}$ [Sak:2015]**

Let $\tilde{\lambda} = \frac{1}{\mu}\sqrt{\frac{\lambda}{2N}}$ and set $\lambda \ll 1$ and $N \gg 1$. Then, for $d > 4$,

$$\left|\pi_{\tilde{\Lambda}_N}(\tilde{o},\tilde{x}) - \delta_{\tilde{o},\tilde{x}}\Big(1 - \sum_{\tilde{v}} (\tanh \tilde{J}_{\tilde{o},\tilde{v}}) \langle\!\langle \sigma_{\tilde{v}} \sigma_{\tilde{o}} \rangle\!\rangle_{\tilde{\Lambda}_N}\Big)\right| \le O(\tilde{\lambda}^2)\left(\delta_{\tilde{o},\tilde{x}} + \frac{O(\tilde{\lambda})}{(|x| \vee 1)^{3(d-2)}}\right).$$

$$\Rightarrow \quad \Pi_N(x) \equiv \lim_{\Lambda \uparrow \mathbb{Z}^d} \sum_{\substack{\tilde{o}=(o,\cdot) \\ \tilde{x}=(x,\cdot)}} \pi_{\tilde{\Lambda}_N}(\tilde{o},\tilde{x}) = N\delta_{o,x}\Big(1 -$$

$$\Big) + \frac{O(\tilde{\lambda}^3 N^2)}{(|x| \vee 1)^{3(d-2)}}.$$

$$\Rightarrow \quad \Phi_N(x) \equiv -\epsilon_N^2 N\big(\Pi_N(x) - N\delta_{o,x}\big) \underset{N\uparrow\infty}{\sim} \langle \varphi_o^2 \rangle_\mu \delta_{o,x} + \frac{O(\lambda/\mu^3)}{(|x| \vee 1)^{3(d-2)}}.$$

**Derivation of the linearized Schwinger-Dyson equation from the lace expansion**

$$\begin{aligned} \langle\!\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\!\rangle_N &= \Pi_N(x) + \sum_{\tilde{u},\tilde{v}} \frac{\Pi_N(u)}{N} (\tanh \tilde{J}_{\tilde{u},\tilde{v}}) \frac{\langle\!\langle \sigma_{\tilde{v}} \tilde{\sigma}_x \rangle\!\rangle_N}{N} \\ &= N\delta_{o,x} - \frac{\Phi_N(x)}{\epsilon_N^2 N} + (N-1)(\tanh I)\,\langle\!\langle \tilde{\sigma}_o \tilde{\sigma}_x \rangle\!\rangle_N + N\sum_v (\tanh J_{o,v})\,\langle\!\langle \tilde{\sigma}_v \tilde{\sigma}_x \rangle\!\rangle_N \\ &\quad - \sum_{u,v} \frac{\Phi_N(u)}{\epsilon_N^2 N^2}\Big((N-1)(\tanh I)\delta_{u,v} + N(\tanh J_{u,v})\Big)\langle\!\langle \tilde{\sigma}_v \tilde{\sigma}_x \rangle\!\rangle_N . \end{aligned}$$

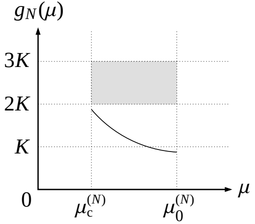
## Derivation of the linearized Schwinger-Dyson equation from the lace expansion (cont.)

$$\Rightarrow \underbrace{\frac{1-(N-1)\tanh I}{N\epsilon_N^2}}_{\sim\,\mu} \underbrace{\epsilon_N^2 \langle\!\langle \tilde\sigma_o \tilde\sigma_x \rangle\!\rangle_N}_{\sim\,\langle\varphi_o\varphi_x\rangle_\mu} = \delta_{o,x} - \underbrace{\frac{\Phi_N(x)}{\epsilon_N^2 N^2}}_{O(N^{-1/2})} + \sum_v \underbrace{\frac{\tanh J_{o,v}}{\epsilon_N^2}}_{\sim\,\mathscr{J}_{o,v}} \underbrace{\epsilon_N^2 \langle\!\langle \tilde\sigma_v \tilde\sigma_x \rangle\!\rangle_N}_{\sim\,\langle\varphi_v\varphi_x\rangle_\mu}$$

$$- \sum_{u,v} \Phi_N(u) \underbrace{\frac{(N-1)(\tanh I)\delta_{u,v} + N(\tanh J_{u,v})}{\epsilon_N^4 N^3}}_{\sim\,\frac{\lambda}{2}\delta_{u,v}} \underbrace{\epsilon_N^2 \langle\!\langle \tilde\sigma_v \tilde\sigma_x \rangle\!\rangle_N}_{\sim\,\langle\varphi_v\varphi_x\rangle_\mu}$$

$$\underset{N\uparrow\infty}{\longrightarrow} \quad \mu \langle\varphi_o\varphi_x\rangle_\mu = \delta_{o,x} + \sum_v \left( \mathscr{J}_{o,v} - \frac{\lambda}{2}\Phi_\mu(v) \right) \langle\varphi_v\varphi_x\rangle_\mu .$$

## Verification of the uniform boundedness of $g_N(\mu) \equiv \sup_x (|x| \vee 1)^{d-2} \epsilon_N^2 \langle\langle \tilde\sigma_o \tilde\sigma_x \rangle\rangle_N$



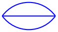

(i) $g_N(\mu)$ is continuous in $(\mu_c^{(N)}, \mu_0^{(N)}]$ $\big(\sim (\mu_c, \hat{\mathscr{J}}]\big)$.

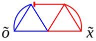
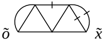
(ii) $g_N(\mu_0^{(N)}) \le {}^\exists K < \infty$.

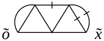
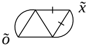
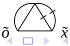
(iii) $g_N(\mu) \notin (2K, 3K]$ for every $\mu \in (\mu_c^{(N)}, \mu_0^{(N)})$.

## Bounding $\pi_{\tilde\Lambda_N}$ under the assumption $g_N(\mu) \le 3K$

1. By definition, $\langle\langle \sigma_{\tilde o} \sigma_{\tilde x} \rangle\rangle_{\tilde\Lambda_N} \le \delta_{\tilde o, \tilde x} + (1 - \delta_{\tilde o, \tilde x})\, O(\tilde\lambda)\, \epsilon_N^2 \langle\langle \tilde\sigma_o \tilde\sigma_x \rangle\rangle_N$.
2. By a BK-type inequality,

$\delta_{\tilde o, \tilde x} \le \pi^{(0)}_{\tilde\Lambda_N}(\tilde o, \tilde x) = \tilde o$  $\tilde x \le \langle\langle \sigma_{\tilde o} \sigma_{\tilde x} \rangle\rangle^3_{\tilde\Lambda_N} \le \delta_{\tilde o, \tilde x} + \dfrac{O(\tilde\lambda)^3}{(|x| \vee 1)^{3(d-2)}}$,

$\delta_{\tilde o, \tilde x}$  $\lesssim \pi^{(1)}_{\tilde\Lambda_N}(\tilde o, \tilde x) =$  $\lesssim \delta_{\tilde o, \tilde x}$  $+$  + other combinations,

etc., where  $\le {}^\exists \tilde C \tilde\lambda^2 N$  $\le \tilde C^2 \tilde\lambda^2 N$  $\le \tilde C^3 \tilde\lambda^2 N$ $\tilde o$  $\tilde x$.

**Theorem [Sak:2015], the finite-range case**

Let $\mathscr{J} \geq 0$ be translation-invariant, $\mathbb{Z}^d$-symmetric, supported on a finite set $\subset \mathbb{Z}^d \setminus \{o\}$, but not necessarily reflection-positive. If $d > 4$ and $\lambda \ll 1$ (depending on $d$ and $\mathscr{J}$), then

$$^{\exists}\Phi_\mu(x) = \left\langle \varphi_o^2 \right\rangle_\mu \delta_{o,x} + \frac{O(\lambda)}{(|x| \vee 1)^{3(d-2)}} \qquad \text{uniformly in } \mu > \mu_c,$$

such that

$$\mu_c = \hat{\mathscr{J}} - \frac{\lambda}{2}\hat{\Phi}_{\mu_c} \equiv \hat{\mathscr{J}} - \frac{\lambda}{2}\sum_{v \in \mathbb{Z}^d} \Phi_{\mu_c}(v), \qquad \langle \varphi_o \varphi_x \rangle_{\mu_c} \underset{|x|\uparrow\infty}{\sim} \frac{\frac{d}{2}\Gamma(\frac{d-2}{2})/\pi^{d/2}}{\hat{\mathscr{J}} V + O(\lambda^2)} |x|^{2-d}.$$

## Three key steps for the proof

1. The Griffiths-Simon construction of the $\varphi^4$ model from the Ising model.
2. The random-current representation for the Ising model.
3. The lace expansion for the Ising model.

## Future problems on the lace expansion

1. Relax the condition on the "derivative" of $\mathscr{J}^{*n}(x)$ for the power-law decaying case.
2. Application to other models, such as the FK random-cluster model, quantum Ising ferromagnets, self-avoiding walk on random conductances, etc.