A determinantal structure for the O'Connell-Yor polymer model

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(Based on a collaboration with T. Imamura)

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0. The model and result

2001 O'Connell Yor

Semi-discrete directed polymer in random media

$B_i, 1 \le i \le N$: independent Brownian motions

Energy of the polymer π

$$E[\pi] = B_1(s_1) + B_2(s_1, s_2) + \cdots + B_N(s_{N-1}, t)$$

with $B_j(s,t) = B_j(t) - B_j(s),\ j = 2, \cdots, N$ for $s < t$

Partition function

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta E[\pi]} ds_1 \cdots ds_{N-1}$$

$\beta = 1/k_B T$: inverse temperature

Zero-temperature limit

In the $T \to 0$ (or $\beta \to \infty$) limit

$$f_N(t) := \lim_{\beta\to\infty} F_N(t) = \max_{0<s_1<\cdots<s_{N-1}<t} E[\pi]$$

2001 Baryshnikov Connection to random matrix theory

$$\text{Prob}\,(f_N(1) \le s) = \int_{(-\infty,s]^N} \prod_{j=1}^{N} dx_j \cdot P_{\text{GUE}}(x_1, \cdots, x_N),$$

$$P_{\text{GUE}}(x_1, \cdots, x_N) = \prod_{j=1}^{N} \frac{e^{-x_j^2/2}}{j!\sqrt{2\pi}} \cdot \prod_{1\le j<k\le N} (x_k - x_j)^2$$

where $P_{\text{GUE}}(x_1, \cdots, x_N)$ is the probability density function of the eigenvalues in the Gaussian Unitary Ensemble (GUE)

A generalization to finite β

Thm

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t)$$

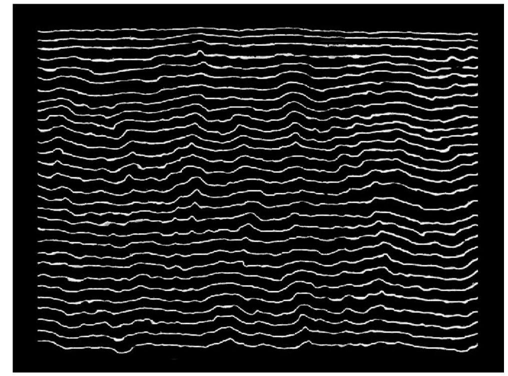
$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \prod_{1 \le j < k \le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j; t)\right)_{j,k=1}^{N}$$

where $f_F(x) = 1/(e^{\beta x} + 1)$ is the Fermi distribution function and

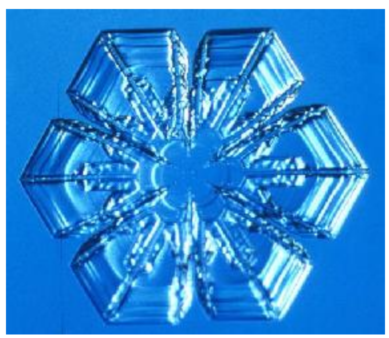
$$\psi_k(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx - w^2 t/2} \frac{(iw)^k}{\Gamma\left(1 + iw/\beta\right)^N}$$

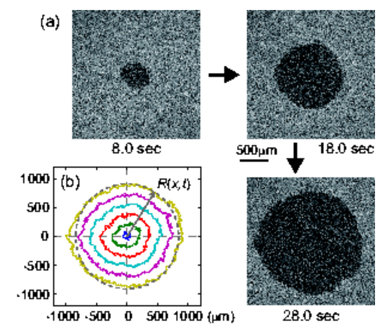
Proof by generalizing Warren's process on the Gelfand-Tsetlin cone

1. Universal distribution in surface growth



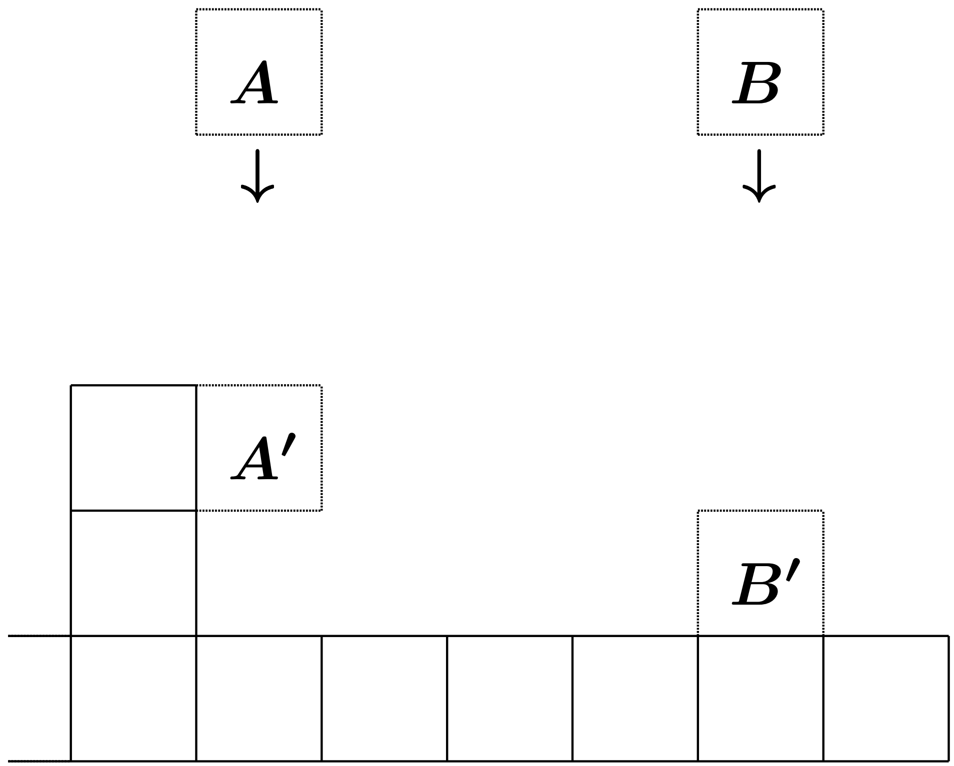
- Paper combustion, bacteria colony, crystal growth, etc
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems
- Connections to integrable systems, representation theory, etc





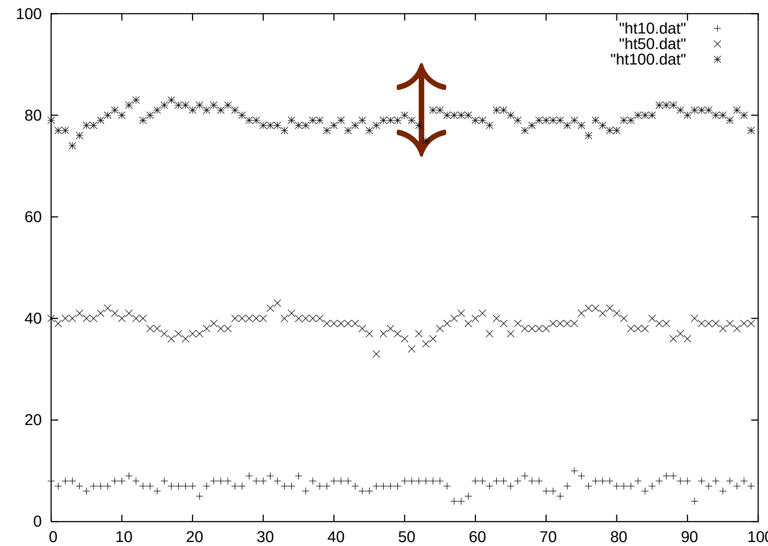
Simulation models

Ex: ballistic deposition



Height fluctuation

$$O(t^{\beta}), \beta = 1/3$$

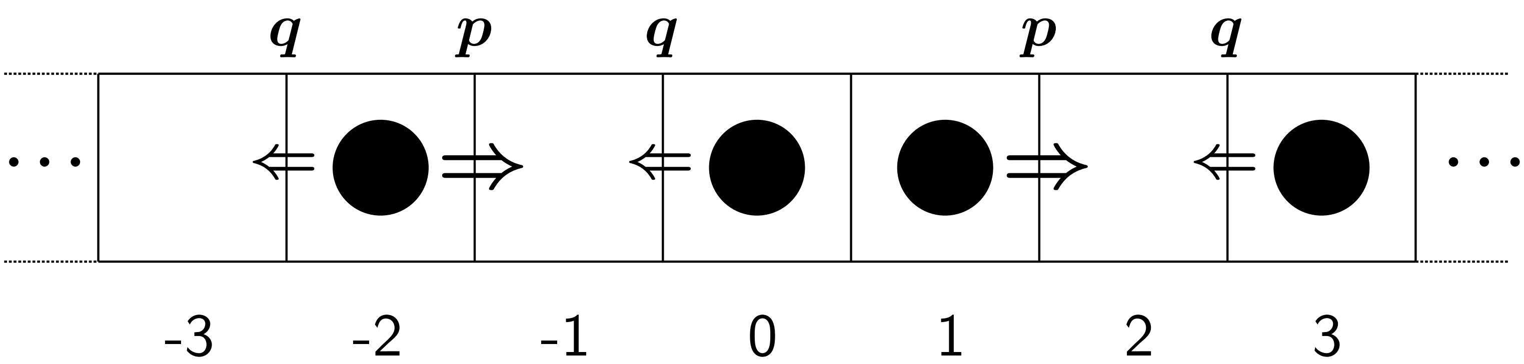


Flat

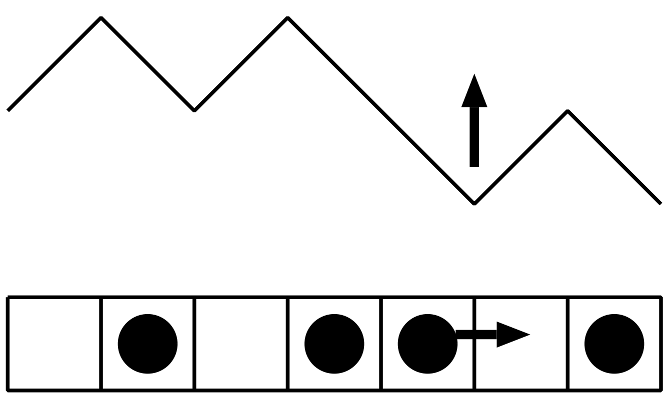
Universality: exponent and height distribution

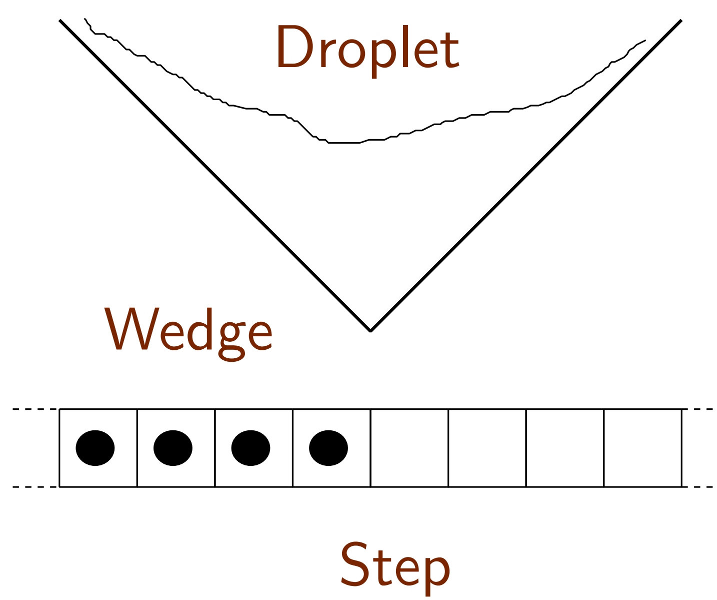
Totally ASEP ($q = 0$)

ASEP (asymmetric simple exclusion process)



Mapping to a surface growth model (single step model)





TASEP with step i.c.

For the step (wedge for surface) initial condition for TASEP
2000 Johansson

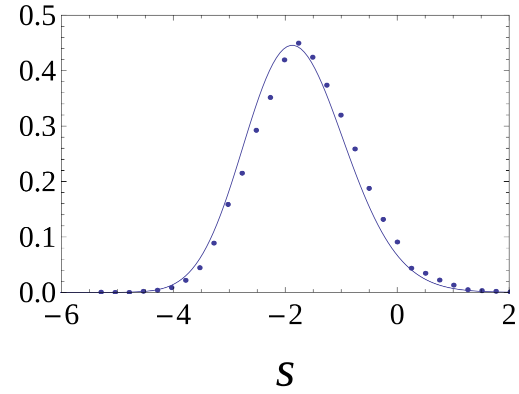
$$\lim_{t\to\infty} \mathbb{P}\left[\frac{h(0,t)-t/4}{-2^{-4/3}t^{1/3}} \le s\right] = F_2(s)$$

where $F_2(s)$ is the GUE Tracy-Widom distribution

$$F_2(s) = \det(1 - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R})}$$

where P_s: projection onto the interval $[s,\infty)$
and K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$



KPZ equation

$h(x,t)$: height at position $x \in \mathbb{R}$ and at time $t \geq 0$

1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu \partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with mean 0 and covariance

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

By a simple scaling we can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

The KPZ equation now looks like

$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

Cole-Hopf transformation

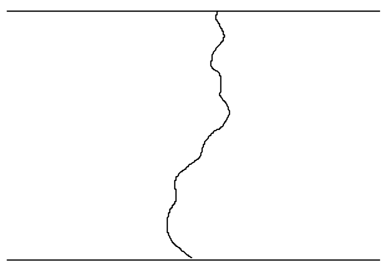
If we set

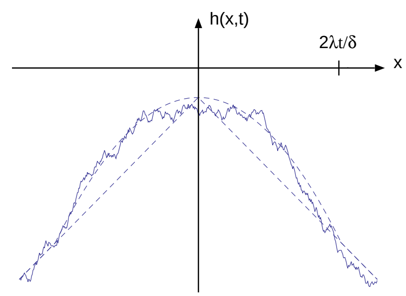
$$Z(x,t) = \exp\left(h(x,t)\right)$$

this quantity (formally) satisfies

$$\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t) Z(x,t)$$

This can be interpreted as a (random) partition function for a directed polymer in random environment η.





The polymer from the origin: $Z(x,0) = \delta(x) = \lim_{\delta \to 0} c_\delta e^{-|x|/\delta}$ corresponds to narrow wedge for KPZ.

KPZ equation for sharp wedge i.c.

For the initial condition $Z(x,0) = \delta(x)$ (narrow wedge for KPZ)

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{h(0,t)+\frac{t}{24}}{(t/2)^{1/3}} \le s\right] = F_2(s)$$

- The Tracy-Widom distribution appears universally in various surface growth models in the KPZ class.
- Experiment
- Technically there is a big difference between TASEP and KPZ equation. The structure for TASEP is well-understood but for KPZ equation, not really yet.

2. "Determinantal"s

Random matrix theory

GUE (Gaussian unitary ensemble): For a matrix H: $N \times N$ hermitian matrix

$$P(H)dH \propto e^{-\mathrm{Tr}H^2} dH$$

Each independent matrix element is independent Gaussian.

Joint eigenvalue density

$$\frac{1}{Z} \prod_{i<j} (x_j - x_i)^2 \prod_i e^{-x_i^2}$$

This is written in the form of a product of two determinants using

$$\prod_{i<j} (x_j - x_i) = \det(x_i^{j-1})_{i,j=1}^N$$

From this follows

- All m point correlation functions can be written as determinants using the "correlation kernel" $K(x, y)$.
- The largest eigenvalue distribution

$$\mathbb{P}\left[x_{\max} \le s\right] = \frac{1}{Z} \int_{[-\infty, s]^N} \prod_{i<j} (x_j - x_i)^2 \prod_i e^{-x_i^2} \prod_i dx_i$$

can be written as a Fredholm determinant using the same kernel $K(x, y)$.

In the limit of large matrix dimension, we get

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_{\max} - \sqrt{2N}}{2^{-1/2}N^{-1/6}} \leq s\right] = F_2(s) = \det(1 - P_s K_2 P_s)_{L^2(\mathbb{R})}$$

where P_s: projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x, y) = \int_0^\infty d\lambda \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda)$$

$F_2(s)$ is known as the **GUE Tracy-Widom distribution**

Determinantal process

- The point process whose correlation functions are written in the form of determinants are called a determinantal process.
- Eigenvalues of the GUE is determinantal.
- This is based on the fact that the joint eigenvalue density can be written as a product of two determinants. The Fredholm determinant expression for the largest eigenvalue comes also from this.
- Once we have a measure in the form of a product of two determinants, there is an associated determinantal process and the Fredholm determinant appears naturally.

"TASEP is determinantal": Schur measure

- Finite t formula

$$\mathbb{P}\left[\frac{h(0,t)-t/4}{-2^{-4/3}t^{1/3}} \leq s\right] = \frac{1}{Z}\int_{[0,s]^N} \prod_{i<j}(x_j - x_i)^2 \prod_i e^{-x_i} \prod_i dx_i$$

As $t \to \infty$ we get $F_2(s)$.

- The proof is based on Robinson-Schensted-Knuth (RSK) correspondence. For a discrete TASEP with parameters $a = (a_1, \cdots, a_N), b = (b_1, \cdots, b_M)$ associated with the Schur measure for a partition λ

$$\frac{1}{Z} s_\lambda(a) s_\lambda(b)$$

The schur function s_λ can be written as a single determinant (Jacobi-Trudi identity).

Dyson's Brownian motion

In GUE, one can replace the Gaussian random variables by Brownian motions. The eigenvalues are now stochastic process, satisfying SDE

$$\boldsymbol{dX_i = dB_i + \sum_{j \neq i} \frac{dt}{X_i - X_j}}$$

known as the Dyson's Brownian motion.

Warren's Brownian motion in Gelfand-Tsetlin cone

Let $Y(t)$ be the Dyson's BM with m particles starting from the origin and let $X(t)$ be a process with $(m+1)$ components which are interlaced with those of Y, i.e.,

$$X_1(t) \leq Y_1(t) \leq X_2(t) \leq \ldots \leq Y_m(t) \leq X_{m+1}(t)$$

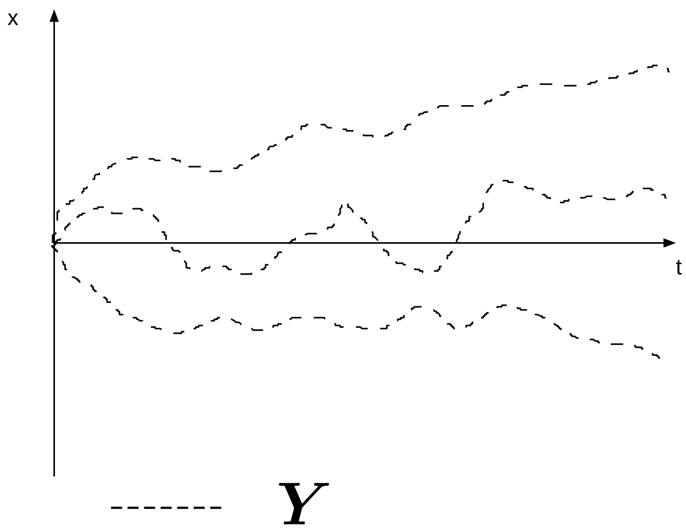
and satisfies

$$X_i(t) = x_i + \gamma_i(t) + \{L_i^-(t) - L_i^+(t)\}.$$

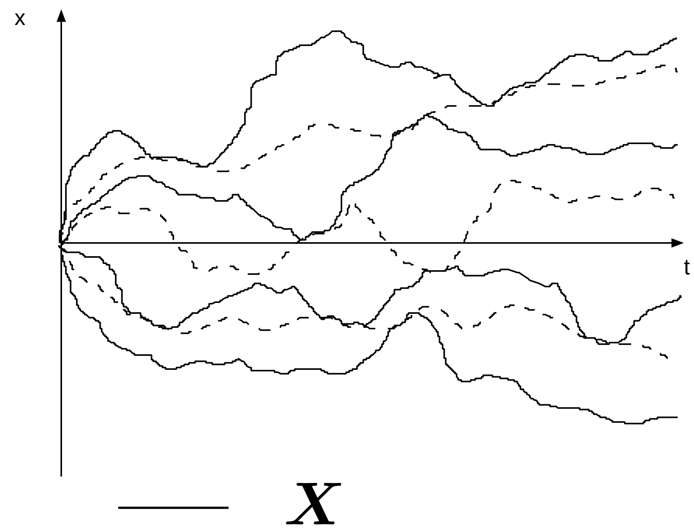
Here $\gamma_i, 1 \leq i \leq m$ are indep. BM and $L_i^{\pm}$ are local times.

Warren showed that the process X is distributed as a Dyson's BM with $(m+1)$ particles.

$m = 3$ Dyson BM



$m = 3, 4$ Dyson BM



Warren's Brownian motion in Gelfand-Tsetlin cone

- Repeating the same procedure for $m = 1, 2, \ldots, n-1$, one can construct a process $X_i^j, 1 \le j \le n, 1 \le i \le j$ in Gelfand-Tsetlin cone
- The marginal $X_i^i, 1 \le i \le n$ is the diffusion limit of TASEP (reflective BMs). One can understand how the random matrix expression for TASEP appears.

$$
\begin{array}{ccccccccc}
 & & & & x_1^1 & & & & \\
 & & & x_1^2 & & x_2^2 & & & \\
 & & x_1^3 & & x_2^3 & & x_3^3 & & \\
 & \iddots & & & \vdots & & & \ddots & \\
x_1^n & x_2^n & x_3^n & & \cdots & & & x_{n-1}^n & x_n^n
\end{array}
$$

The formala for KPZ equation

Thm (2010 TS Spohn, Amir Corwin Quastel)

For the initial condition $Z(x, 0) = \delta(x)$ (narrow wedge for KPZ)

$$\mathbb{E}\left[e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}}\right] = \det(1 - K_{s,t})_{L^2(\mathbb{R}_+)}$$

where $\gamma_t = (t/2)^{1/3}$ and $K_{s,t}$ is

$$K_{s,t}(x, y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1}$$

The final result is written as a Fredholm determinant, but this was obtained without using a measure in the form of a product of two determinants (Bethe ansatz, Macdonald measure, replica, δ-Bose gas).

3 O'Connell-Yor polymer

2001 O'Connell Yor

Semi-discrete directed polymer in random media

$B_i, 1 \leq i \leq N$: independent Brownian motions

Energy of the polymer π

$$E[\pi] = B_1(s_1) + B_2(s_1, s_2) + \cdots + B_N(s_{N-1}, t)$$

Partition function

$$Z_N(t) = \int_{0<s_1<\cdots<s_{N-1}<t} e^{\beta E[\pi]} ds_1 \cdots ds_{N-1}$$

$\beta = 1/k_B T$: inverse temperature

In a limit, this becomes the polymer related to KPZ equation.

Whittaker measure: non-determinantal

O'Connell discovered that the OY polymer is related to the quantum version of the Toda lattice, with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N-1} e^{x_i - x_{i-1}}$$

and as a generalization of Schur measure appears a measure written as a product of the two Whittaker functions (which is the eigenfunction of the Toda Hamiltonian):

$$\frac{1}{Z}\Psi_0(\beta x_1, \cdots, \beta x_N)\Psi_\mu(\beta x_1, \cdots, \beta x_N)$$

A determinant formula for Ψ is not known.

From this connection one can find a formala

$$\text{Prob}\,(\boldsymbol{F_N(t)} \le \boldsymbol{s}) = \int_{(-\infty,s]^N} \prod_{j=1}^{N} dx_j \cdot m_t(x_1, \cdots, x_N)$$

where $m_t(x_1, \cdots, x_N) \prod_{j=1}^{N} dx_j$ is given by

$$m_t(x_1, \cdots, x_N) = \Psi_0(\beta x_1, \cdots, \beta x_N)$$
$$\times \int_{(i\mathbb{R})^N} d\boldsymbol{\lambda} \cdot \Psi_{-\boldsymbol{\lambda}}(\beta x_1, \cdots, \beta x_N) e^{\sum_{j=1}^{N} \lambda_j^2 t/2} s_N(\boldsymbol{\lambda})$$

where $s_N(\boldsymbol{\lambda})$ is the Sklyanin measure

$$s_N(\boldsymbol{\lambda}) = \frac{1}{(2\pi i)^N N!} \prod_{i<j} \Gamma(\lambda_i - \lambda_j)$$

Doing asymptotics using this expression has not been possible.

Macdonald measure and Fredholm determinant formula

Borodin, Corwin (2011) introduced the Macdonald measure

$$\frac{1}{Z} P_\lambda(a) Q_\lambda(b)$$

Here $P_\lambda(a), Q_\lambda(b)$ are the Macdonald polynomials, which are also not known to be a determinant.

By using this, they found a formula for OY polymer

$$\mathbb{E}[e^{-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}}] = \det(1+L)_{L^2(C_0)}$$

where the kernel $L(v, v'; t)$ is written as

$$\frac{1}{2\pi i} \int_{i\mathbb{R}+\delta} dw \frac{\pi/\beta}{\sin(v'-w)/\beta} \frac{w^N e^{w(t^2/2-u)}}{v'^N e^{v'(t^2/2-u)}} \frac{1}{w-v} \frac{\Gamma(1+v'/\beta)^N}{\Gamma(1+w/\beta)^N}$$

By using this expression, one can study asymptotics.

Our formula for finite β

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t)$$

$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \prod_{1 \le j < k \le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j; t)\right)$$

where $f_F(x) = 1/(e^{\beta x} + 1)$ is Fermi distribution function and

$$\psi_k(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx - w^2 t/2} \frac{(iw)^k}{\Gamma\left(1 + iw/\beta\right)^N}$$

A formula in terms of a determinantal measure W for finite temperature polymer.

From this one gets the Fredholm determinant by using standard techniques of random matrix theory and does asymptotics.

Proof of the formula

We start from a formula by O'Connell

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right)=\int_{(i\mathbb{R}-\epsilon)^N}\prod_{j=1}^{N}\frac{d\lambda_j}{\beta}e^{-u\lambda_j+\lambda_j^2t/2}\Gamma\left(-\frac{\lambda_j}{\beta}\right)^N s_N\left(\frac{\lambda}{\beta}\right)$$

where $\epsilon > 0$.

This is a formula which is obtained by using Whittaker measure.

In this sense, we have not really found a determinant structure for the OY polymer itself.

An intermediate formula

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} dx_\ell f_F(x_\ell - u) \cdot \det\left(F_{jk}(x_j;t)\right)_{j,k=1}^{N}$$

with $(0 < \epsilon < \beta)$

$$F_{jk}(x;t) = \int_{i\mathbb{R}-\epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x + \lambda^2 t/2}}{\Gamma\left(\frac{\lambda}{\beta}+1\right)^N} \left(\frac{\pi}{\beta} \cot \frac{\pi\lambda}{\beta}\right)^{j-1} \lambda^{k-1}$$

Now it is sufficient to prove the relation

$$\int_{\mathbb{R}^N} \prod_{\ell=1}^{N} dt_\ell f_F(t_\ell - u) \cdot \det\left(F_{jk}(t_j;t)\right)_{j,k=1}^{N}$$

$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t).$$

A determinantal measure on $\mathbb{R}^{N(N+1)/2}$

For $\underline{x}_k := (x_i^{(j)}, 1 \le i \le j \le k) \in \mathbb{R}^{k(k+1)/2}$, we define a measure $R_u(\underline{x}_N; t) d\underline{x}_N$ with R_u given by

$$\prod_{\ell=1}^{N} \frac{1}{\ell!} \det\left(f_i(x_j^{(\ell)} - x_{i-1}^{(\ell-1)}) \right)_{i,j=1}^{\ell} \cdot \det\left(F_{1i}(x_j^{(N)}; t) \right)_{i,j=1}^{N}$$

where $x_0^{(\ell-1)} = u$, $\underline{x}_N = \prod_{j=1}^{N} \prod_{i=1}^{j} dx_i^{(j)}$,

$$f_i(x) = \begin{cases} f_F(x) := 1/(e^{\beta x} + 1) & i = 1, \\ f_B(x) := 1/(e^{\beta x} - 1) & i \ge 2. \end{cases}$$

and $F_{1i}(x; t)$ is given by $F_{ji}(x; t)$ with $j = 1$ in the previous slide.

Two ways of integrations

$$\int_{\mathbb{R}^{N(N+1)/2}} d\underline{x}_N R_u(\underline{x}_N; t)$$

$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_1^{(j)} f_F\left(x_1^{(j)} - u\right) \cdot \det\left(F_{jk}\left(x_1^{(N-j+1)}; t\right)\right)_{j,k=1}^{N}$$

$$\int_{\mathbb{R}^{N(N+1)/2}} d\underline{x}_N R_u(\underline{x}_N; t)$$

$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j^{(N)} f_F\left(x_j^{(N)} - u\right) \cdot W\left(x_1^{(N)}, \cdots, x_N^{(N)}; t\right)$$

Lemma

1. For $\beta > 0$ and $a \in \mathbb{C}$ with $-\beta < \mathrm{Re}\ a < 0$, we have
$$\int_{-\infty}^{\infty} e^{-ax} f_B(x) dx = \frac{\pi}{\beta} \cot \frac{\pi}{\beta} a.$$

2. Let $G_0(x) = f_F(x)$ and
$G_j(x) = \int_{-\infty}^{\infty} dy f_B(x-y) G_{j-1}(y),\ j = 1, 2, \cdots$.
Then we have for $m = 0, 1, 2, \cdots$
$$G_m(x) = f_F(x) \left(\frac{x^m}{m!} + p_{m-1}(x) \right),$$
where $p_{-1}(x) = 0$ and $p_k(x)(k = 0, 1, 2, \cdots)$ is some kth order polynomial.

Dynamics of X_i^N

The density for the positions of $X_i^N, 1 \le i \le N$ satisfies

$$\frac{\partial}{\partial t} W(x_1, \cdots, x_N; t)$$
$$= \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} W(x_1, \cdots, x_N; t)$$
$$- \sum_{i=1}^{N} \left(\sum_{j \neq i} \frac{1}{x_i - x_j} \right) \frac{\partial}{\partial x_i} W(x_1, \cdots, x_N; t)$$

which is the equation for the Dyson's Brownian motion.

Dynamics of X_i^i's

The transition density of X_i^i's

$$R(x_1, \cdots, x_N; t) = \det \left(F_{jk}\left(x_k; t\right)\right)_{j,k=1}^{N}$$

satisfy

$$\frac{\partial}{\partial t} R(x_1, \cdots, x_N; t) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \cdot R(x_1, \cdots, x_N; t)$$

$$-\frac{\beta^2}{\pi^2} \int_{-\infty}^{\infty} dx_{j+1} \frac{e^{-\frac{\beta}{2}(x_{j+1}-x_j)}}{e^{\beta(x_{j+1}-x_j)} - 1} R(x_1, \cdots, x_N; t) = 0$$

As $\beta \to \infty$, the latter becomes

$$\partial_{x_i} R(x_1, \cdots, x_N; t)|_{x_{i+1}=x_i+0} = 0$$

which represents reflective interaction like TASEP.

Summary

- A determinantal formula for finite temperature O'Connell-Yor polymer
- Techniques from random matrix theory can readily be applied. Asymptotics possible.
- We started from a formula which is obtained from Whittaker measure. In this sense we have not found a determinantal structure for the OY polymer model itself.
- The proof is by generalizing Warren's process on Gelfand-Tsetlin cone. There are interesting generalizations of Dyson's Brownian motion and reflective Brownian motions.