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Notes for talk on "Some applications of Stein's method to number theory", Workshop on New Directions in Stein's Method, IMS Singapore, 26th May 2015

Plan of the talk: ⓪ A little bit of number theory notation / background

① Erdős-Kac Central Limit Theorem

② Random multiplicative functions

Other Topics: ③ Some ideas about prime number races / $\zeta(\frac{1}{2}+it)$

0) Number Theory background

Let $\omega(n) := \#\{\text{distinct prime factors of } n\}$ (e.g. $\omega(12) = \omega(2^2 \cdot 3) = 2$)

Let $P(n) :=$ largest prime factor of n (≥ 2).

Let $\mu(n) := \begin{cases} 0 & \text{if } n \text{ is divisible by a non-trivial square} \\ 1 & \text{if } n \text{ is sqfree and } \omega(n) \text{ is } \underline{\text{even}} \\ -1 & \text{if } n \text{ is sqfree and } \omega(n) \text{ is } \underline{\text{odd}} \end{cases}$, the Möbius function

(e.g. $\mu(2) = \mu(3) = \mu(\text{prime}) = -1$, $\mu(4) = 0$, $\mu(6) = 1$)

Theorem: (Prime Number Theorem, Hadamard, de la Vallée Poussin, 1896).

$$\#\{p \leq x : p \text{ prime}\} = (1+o(1)) \frac{x}{\log x} \quad \text{as } x \to \infty.$$

~~Vinogradov's Notation: we write $f(x) \ll g(x)$ to mean $|f(x)| \leq C g(x)$ (i.e. the same as "big~~ ~~not "much much less than")~~

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1) Erdös – Kac Central Limit Theorem

We can write $w(n) = \sum_{\text{primes } p \leq n} \delta_p(n)$, where $\delta_p(n) = \begin{cases} 1 & \text{if } p \mid n \\ 0 & \text{o/w} \end{cases}$

If we randomly choose an integer $n \leq x \in \mathbb{N}$, then $\mathbb{P}(\delta_p(n) = 1) = \frac{1}{x} \#\{n \leq x : p \mid n\}$
$= \frac{1}{x} \#\{m \leq \frac{x}{p}\} = \frac{1}{x}(\frac{x}{p} + O$
$= \frac{1}{p} + O(\frac{1}{x})$,

and $\mathbb{P}(\delta_p = 1 \text{ and } \delta_q = 1) = \frac{1}{x} \#\{m \leq \frac{x}{pq}\} = \frac{1}{pq} + O(\frac{1}{x})$ $(p \neq q)$.

So $w(n)$ is a sum of indicators that behave "roughly independently" — so maybe there is a CLT here?

Theorem: (Erdös – Kac, 1939–40)

If $n \leq x$ is chosen discrete uniformly at random, then $\frac{w(n) - \log\log x}{\sqrt{\log\log x}} \xrightarrow{d} N(0,1)$ as $x \to \infty$.

Particular consequence: $w(n) \approx \log\log x$ with very high prob

This is now very classical, but in my opinion it is still an open problem to give a "nice" proof of the sharp rate of convergence:

- the correct rate is a Berry – Esseen type rate $\frac{1}{\sqrt{\log\log x}}$ (Rényi – Turán, 1958 proved this, but proof is difficult complex and

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- there is a very simple qualitative proof using moments, by Billingsley, 1969 [Erdös-Kac also used moments, but it was much messier].
- I (2009) used the Chen-Stein method to give a total variation bound $O(\frac{1}{\log\log x})$ between a truncated version of $\omega(n)$ (don't count very large prime factors) and Poisson — this leads to a rate $O(\frac{\log\log\log x}{\sqrt{\log\log x}})$ in Erdös-Kac.
- Nice open problem: prove such a total variation approximation (maybe with a weaker rate) without truncating $\omega(n)$? This would show that $\omega(n)$ is odd and even asymptotically $\frac{1}{2}$ the time, which is equivalent to PNT.

2) Random multiplicative functions

This is a random model introduced by Wintner, 1940, as a heuristic for the Möbius function $\mu(n)$ [and can also think of it as a heuristic for the characters of $(\mathbb{Z}/q\mathbb{Z})^*$ that are real-valued, with q very large]

Model: Let $(f(p))_{p \text{ prime}}$ be a sequence of independent Rademacher RVs $(\mathbb{P}(f(p)=1)=\mathbb{P}(f(p)=-1)=\frac{1}{2})$
Define $f(n)=0$ if n has a non-trivial square divisor, $f(n)=\prod_{p|n} f(p)$ o/w.

example of Rademacher chaos

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Limit Theorems

Question: Is it true that $\sum_{n \le x} f(n) / \sqrt{\mathbb{E}(\sum_{n \le x} f(n))^2} \xrightarrow{d} N(0,1)$ as $x \to \infty$?

Answer: Note that $\sum_{n \le x, \, \omega(n)=1} f(n) = \sum_{\text{primes } p \le x} f(p)$ does satisfy the CLT (sum of indep. Rademache

Hough, 2011, proved using moments that $\sum_{n \le x, \, \omega(n)=k} f(n)$ satisfies CLT, provided $k = o(\log\log\log x)$

I proved [2013] using martingales that CLT holds provided $k = o(\log\log x)$ (i.e. just fewer than average prime factors).

(We can write $\sum_{n \le x, \, \omega(n)=k} f(n) = \sum_{p \le x} \sum_{n \le x, \, \omega(n)=k, \, P(n)=p} f(n)$, a martingale decomposition w.r.t. the "prime revealing filtration").

This reduces matters to a fourth moment calculation.

I also proved that $\sum_{n \le x} f(n) / \sqrt{\mathbb{E}(\sum_{n \le x} f(n))^2} \not\xrightarrow{d} N(0,1)$ as $x \to \infty$. (by a conditioning argument).

Chatterjee & Soundararajan, 2011: proved that CLT holds for $\sum_{x-y \le n \le x} f(n)$, if $C x^{\frac{1}{5}} \log x < y = o(\frac{x}{\log x})$

The proof uses Chatterjee's "new method of normal approximation"

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Open problems: • What is the correct range of y in Chatterjee & Soundararajan's result?

• Does $\sum_{n \leq x} f(n)$ have a limit distribution, and if so what?

Size of $\sum_{n \leq x} f(n)$

We know that $\sum_{n \leq x} f(n) = O_{\varepsilon}(\sqrt{x}(\log\log x)^{5/2+\varepsilon})$ $\boxed{\text{a.s. for any } \varepsilon > 0}$

(Lau, Tenenbaum & Wu, 2013) ← Uses Bona hypercontr ineq.

$\sum_{n \leq x} f(n) \neq O\left(\frac{\sqrt{x}}{(\log\log x)^{5/2+\varepsilon}}\right)$ (H., 2013) ← Uses Ga processe

Open problem: What is the true size of the fluctuations of $\sum_{n \leq x} f(n)$?

In particular, is it true that $\mathbb{E}|\sum_{n \leq x} f(n)| \geq c\sqrt{x}$?

— I conjecture yes, but other people might disa

3) Other topics

Prime number race: Let $q \in \mathbb{N}$, and let $a_1, \ldots, a_r$ be some (possibly all) of the coprime residue classes mod q.

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Let $\pi(x;q,a_i) := \#\{\text{primes } p \leq x : p \equiv a_i \bmod q\}$.

Question: As $x \to \infty$, how often do we have

$$\pi(x;q,a_1) \geq \pi(x;q,a_2) \geq \ldots \geq \pi(x;q,a_r)?$$ "a_1 wins, a_2 comes second, ..."

Assuming GRH & LI Rubinstein & Sarnak showed that as $x \to \infty$, the distribution of $(\pi(x;q,a_1), \ldots, \pi(x;q,a_r))$ has a limit $\mathcal{L}_q(a_1,\ldots,a_r)$ (when suitably centred and rescaled). This limit is the same as the distribution of certain weighted sums of independent RVs (RVs depend on q, weights depend on a_i).

So to answer the Question, we can try to approximate $\mathcal{L}_q(a_1,\ldots,a_r)$ by a multivariate Gaussian distribution, and then use tools like Gaussian comparison inequalities.

Can be done using Stein's method, cf. H & Lamzouri.