Stein's method for Gibbs process approximation in the total variation metric

Dominic Schuhmacher
Institute for Mathematical Stochastics
University of Göttingen

Joint work with Kaspar Stucki

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A short history of Stein's method for point process approximation

Poisson process approximation:

Barbour and Brown (1992); Brown, Weinberg and Xia (2000); Chen and Xia (2004); S and Xia (2008); S (2009).

Compound poisson process approximation:

Barbour and Månsson (2002).

"Polynomial birth-death" point process approximation:

Xia and Zhang (2012).

In all of these articles the locations of (multi-)points are independent.

Outline

- Gibbs processes
- A coupling of two spatial birth-death processes
- Generator approach
- Upper bounds in the total variation metric

Gibbs point processes

Point process preliminaries

- $(\mathcal{X}, d)$ compact metric space with Borel σ-algebra.
- α diffuse, finite measure on $\mathcal{X}$.
- $(\mathfrak{N}, \mathcal{N})$ space of finite counting measures on $\mathcal{X}$ with usual σ-algebra.
- PoP(α) distribution of Poisson process with expectation measure α. **Will be our reference measure for point process distributions, write $P_0 :=$ PoP(α).**

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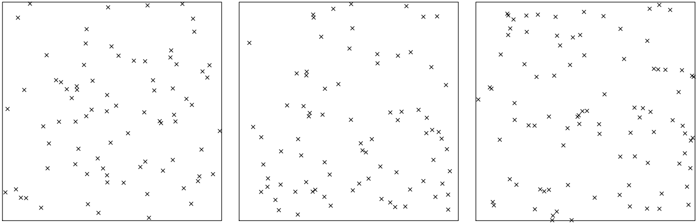
E.g. $\mathcal{X} \subset \mathbb{R}^d$, α Lebesgue measure, P_0 homogeneous Poisson process on $\mathcal{X}$.



Gibbs processes

A function $u\colon \mathfrak{N} \to \mathbb{R}_+$ is called **hereditary** if $u(\xi) = 0$ implies $u(\eta) = 0$ for all point configurations $\xi, \eta \in \mathfrak{N}$ with $\xi \subset \eta$.

A point process Ξ is called a **Gibbs process** if it has a hereditary density $u\colon \mathcal{F} \to \mathbb{R}_+$ with respect to the standard Poisson process distribution P_0.

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A point process Ξ is called a **Gibbs process** if it has a hereditary density $u\colon \mathcal{F} \to \mathbb{R}_+$ with respect to the standard Poisson process distribution P_0.

A Gibbs process is completely described by its **conditional intensity** $\lambda(\cdot \,|\, \cdot)$, where

$$\lambda(x \,|\, \xi) = \frac{u(\xi + \delta_x)}{u(\xi)} \quad \text{for all } \xi \in \mathfrak{N},\ x \in \mathcal{X} \text{ with } \xi(\{x\}) = 0.$$

Intuitively the pressure to have a point at x given we know that the point pattern everywhere else is ξ.

We write Gibbs(λ) for the distribution of this Gibbs process.

Pairwise interaction process (PIP)

A pairwise interaction process (PIP) has density of the form

$$u(\xi) = c \left(\prod_{x \in \xi} \beta(x) \right) \left(\prod_{\{x,y\} \subset \xi} \varphi_2(x, y) \right)$$

for suitable functions $\beta \colon \mathcal{X} \to \mathbb{R}_+$ and $\varphi_2 \colon \mathcal{X}^2 \to \mathbb{R}_+$ (symmetric).

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E.g. Homogeneous Strauss process:

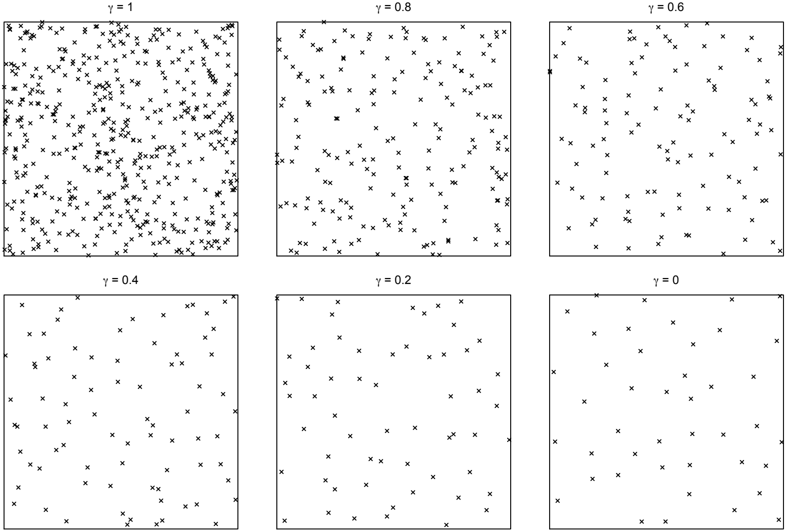
$$\varphi_2(x, y) = \begin{cases} \gamma & \text{if } d(x, y) \leq R, \\ 1 & \text{otherwise,} \end{cases}$$

where $\gamma \in [0, 1]$, $R > 0$; furthermore $\beta > 0$ constant.

In this case

$$u(\xi) = c\, \beta^{|\xi|}\, \gamma^{\#\{\{x,y\} \subset \xi \colon d(x,y) \leq R\}}$$

Simulated Strauss processes



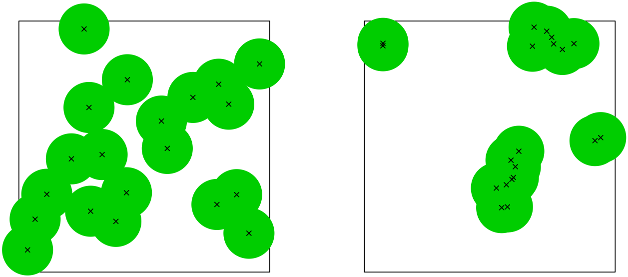
Other parameters are $R = 0.1$ and $\beta = 500$.

Interactions of arbitrary order: the homogeneous AIP

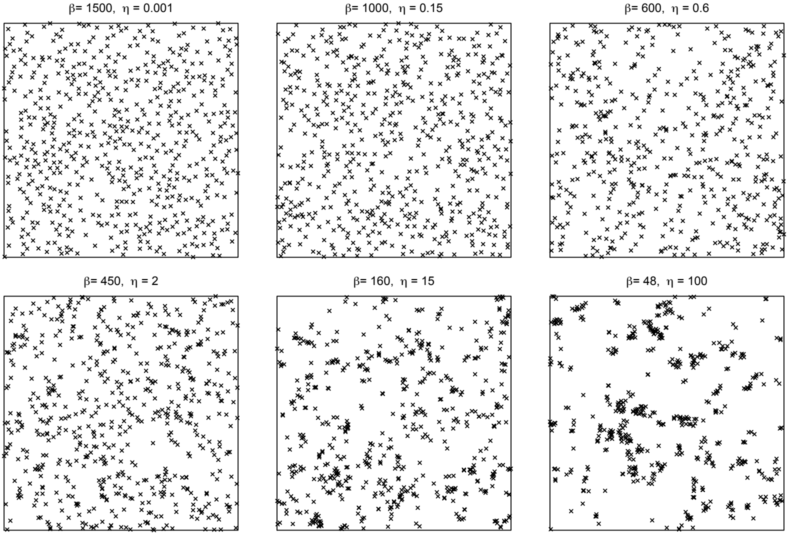
The homogeneous area-interaction process has density

$$u(\xi) := c\,\beta^{|\xi|}\,\gamma^{-\alpha(U_R(\xi))},$$

where $\gamma > 0$, $R > 0$, and $\beta > 0$.
$U_R(\xi) = \bigcup_{x\in\xi} B(x,R)$ denotes the green area.



Simulated area-interaction processes



$R = 0.02$ and β was adjusted so that expected number of points remains the same.

Conditional intensities

For the Strauss($\beta, \gamma; R$)-Process

$$\lambda(x \mid \xi) = \beta\, \gamma^{\#\{y \in \xi \setminus \{x\}:\; d(y,x) \leq R\}}$$

For the AIP($\beta, \gamma; R$)

$$\lambda(x \mid \xi) = \beta\, \gamma^{-\alpha\left(\mathbb{B}(x, R) \setminus U_R(\xi \setminus \{x\})\right)}$$

The conditional intensity is (usually) explicit, without the “unknown” factor c.

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The conditional intensity is (usually) explicit, without the "unknown" factor c.

Furthermore the **Georgii–Nguyen–Zessin equation** holds

$$\mathbb{E}\left(\int_{\mathcal{X}} h(x, \Xi - \delta_x) \, \Xi(dx)\right) = \int_{\mathcal{X}} \mathbb{E}\big(h(x, \Xi)\lambda(x \mid \Xi)\big) \, \alpha(dx)$$

for every measurable $h \colon \mathcal{X} \times \mathfrak{N} \to \mathbb{R}_+$.

Stability condition

Let H always be a Gibbs process with conditional intensity λ that satisfies the following “stability condition”:

$$\textbf{(S)} \qquad \sup_{\xi \in \mathfrak{N}} \int_{\mathcal{X}} \lambda(x \mid \xi)\, \alpha(dx) < \infty.$$

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Is satisfied if H is *locally stable*, i.e. if

$$\lambda(x \,|\, \xi) \leq \psi^*(x),$$

where $\psi^* \colon \mathcal{X} \to \mathbb{R}_+$ is an integrable function.

Satisfied e.g. if H is an AIP or an inhibitory PIP (i.e. $\varphi_2 \leq 1$).

A coupling of two spatial birth-death processes

Spatial birth-death processes

Suppose that we have birth rates and death rates

$$b(\cdot \,|\, \cdot) \colon \mathcal{X} \times \mathfrak{N} \to \mathbb{R}_+ \quad \text{with } \bar{b}(\xi) := \int_{\mathcal{X}} b(x \,|\, \xi) \; \alpha(dx) < \infty;$$

$$d(\cdot \,|\, \cdot) \colon \mathcal{X} \times \mathfrak{N} \to \mathbb{R}_+ \quad \text{with } \bar{d}(\xi) := \sum_{x \in \xi} d(x \,|\, \xi) < \infty.$$

Let $\bar{a}(\xi) = \bar{b}(\xi) + \bar{d}(\xi)$.

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Let $\bar{a}(\xi) = \bar{b}(\xi) + \bar{d}(\xi)$.

A $\text{SBD}^{(\xi_0)}(b, d)$-process is a pure-jump Markov process on $\mathfrak{N}$ that starts in $\xi_0 \in \mathfrak{N}$ and holds each state ξ for an $\text{Exp}(\bar{a}(\xi))$-distributed time, after which

- with probability $\bar{b}(\xi)/\bar{a}(\xi)$ a point is added, positioned according to the density $b(\cdot \,|\, \xi)/\bar{b}(\xi)$, or
- with probability $d(x \,|\, \xi)/\bar{a}(\xi)$ the point at x is deleted.

SBD(λ,1)-process

In what follows always $b = \lambda$, $d \equiv 1$ ("unit per-capita death rate").

Let $Z = \big(Z(t)\big)_{t \geq 0} \sim \text{SBD}(\lambda, 1)$. Under Condition (S)

- Z is non-explosive;

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- Z is non-explosive;
- Z has *Gibbs*(λ) as its unique stationary distribution.
- Z has the infinitesimal generator

$$\mathcal{A}h(\xi) = \int_{\mathcal{X}} \big[h(\xi + \delta_x) - h(\xi)\big]\, \lambda(x \,|\, \xi)\, \alpha(dx) + \int_{\mathcal{X}} \big[h(\xi - \delta_x) - h(\xi)\big]\, \xi(dx)$$

for certain functions $h \colon \mathfrak{N} \to \mathbb{R}$.

A coupling of SBD$(\lambda, 1)$-processes

Goal: For $\xi, \eta \in \mathfrak{N}$ find a coupling (Z_1, Z_2) with $Z_1 \sim \text{SBD}^{(\xi)}(\lambda, 1)$, $Z_2 \sim \text{SBD}^{(\eta)}(\lambda, 1)$ so that Z_1 and Z_2 coincide "as soon as possible".

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Construction: For $\zeta_1, \zeta_2 \in \mathfrak{N}$ let

$$\lambda_{\max}(x \,|\, \zeta_1, \zeta_2) = \max(\lambda(x \,|\, \zeta_1), \lambda(x \,|\, \zeta_2)) \quad \bar{\lambda}_{\max}(\zeta_1, \zeta_2) = \int_{\mathcal{X}} \lambda_{\max}(x \,|\, \zeta_1, \zeta_2)\, \alpha(dx)$$

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and furthermore $\bar{a}(\zeta_1, \zeta_2) = \bar{\lambda}_{\max}(\zeta_1, \zeta_2) + |\zeta_1 \cup \zeta_2|$.

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and furthermore $\bar{a}(\zeta_1, \zeta_2) = \bar{\lambda}_{\max}(\zeta_1, \zeta_2) + |\zeta_1 \cup \zeta_2|$.

Let (Z_1, Z_2) be the pure-jump Markov process on $\mathfrak{N} \times \mathfrak{N}$ that starts in (ξ, η) and holds each state (ζ_1, ζ_2) for an $\mathrm{Exp}(\bar{a}(\zeta_1, \zeta_2))$-distributed time, after which

- with probability $\bar{\lambda}_{\max}(\zeta_1, \zeta_2)/\bar{a}(\zeta_1, \zeta_2)$ a birth occurs at $X \sim \lambda_{\max}(\cdot \mid \zeta_1, \zeta_2)/\bar{\lambda}_{\max}(\zeta_1, \zeta_2)$. Given X, a point at X is added to the process Z_i with probability $\lambda(X \mid \zeta_i)/\lambda_{\max}(X \mid \zeta_1, \zeta_2)$.
- with probability $1/\bar{a}(\zeta_1, \zeta_2)$ a death occurs at $X \in \zeta_1 \cup \zeta_2$. Given X, a point at X is deleted from the process Z_i if it has such a point.

A coupling of SBD$(\lambda, 1)$-processes

How can we control how quickly the two processes meet?

A coupling of SBD$(\lambda, 1)$-processes

How can we control how quickly the two processes meet?

Consider the jump chain $(Z_1(T_j), Z_2(T_j))_{j\in\mathbb{N}}$. At each step j there are three possibilities:

(1) Z_1 and Z_2 both have the same birth or death event.

(2) Only one process has a birth ("bad birth"). We have

$$\mathbb{P}\big(\text{“bad birth”} \,\big|\, Z_1(T_{j-1}), Z_2(T_{j-1})\big) = \frac{\bar{\lambda}_{\max}(Z_1(T_{j-1}), Z_2(T_{j-1})) - \bar{\lambda}_{\min}(Z_1(T_{j-1}), Z_2(T_{j-1}))}{\bar{\lambda}_{\max}(Z_1(T_{j-1}), Z_2(T_{j-1})) + |Z_1(T_{j-1}) \cup Z_2(T_{j-1})|}.$$

(3) One of the non-common points of $Z_1(T_{j-1})$ and $Z_2(T_{j-1})$ dies ("good death"). Then

$$\mathbb{P}\big(\text{“good death”} \,\big|\, Z_1(T_{j-1}), Z_2(T_{j-1})\big) = \frac{\|Z_1(T_{j-1}) - Z_2(T_{j-1})\|}{\bar{\lambda}_{\max}(Z_1(T_{j-1}), Z_2(T_{j-1})) + |Z_1(T_{j-1}) \cup Z_2(T_{j-1})|}.$$

Probability for becoming closer

Suppose that in the total variation norm $\|\zeta_1 - \zeta_2\| = n$, i.e. ζ_1 and ζ_2 differ in n points.
Let A_n be the event that good death occurs before bad birth, i.e. the next time something interesting happens the two BDPs come closer together.

Lemma

The probability of the event A_n is bounded from below as

$$\mathbb{P}(A_n) \geq \left(1 + \frac{1}{n} \sup_{||\xi' - \eta'||=n} \int_{\mathcal{X}} |\lambda(x \mid \xi') - \lambda(x \mid \eta')| \, \alpha(dx)\right)^{-1},$$

which is > 0 by the stability condition (S).

Coupling time

Theorem

For all configurations ξ, η *the coupling time* $\tau_{\xi,\eta} := \inf\{t \geq 0\colon Z_1^{(\xi)}(t) = Z_2^{(\eta)}(t)\}$ *has finite expectation. In particular if* ξ *and* η *differ in only one point, we have*

$$\mathbb{E}\tau_{\xi,\eta} \leq \frac{e^c - 1}{c} + \int_0^c \frac{e^s - 1}{s}\, ds,$$

where $c = \sup_{\xi',\eta' \in \mathfrak{N}} \int_{\mathcal{X}} |\lambda(x \mid \xi') - \lambda(x \mid \eta')|\, \alpha(dx)$*, which is finite by the stability condition (S).*

Idea of the proof

- Denote by $X(t) = (||Z_1^{(\xi)}(t) - Z_2^{(\eta)}(t)||)_{t \geq 0}$ the process counting the non-common points of $Z_1^{(\xi)}$ and $Z_2^{(\eta)}$. Let $p_n = (1 + c/n)^{-1} \leq \mathbb{P}(A_n)$.

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- Construct a (non-spatial) birth-death process $(Y(t))_{t\geq 0}$ with birth rate $(1 - p_n)n$ and death rate $p_n n$ such that

$$\tau_{\xi,\eta} = \tau^{(X,0)} \leq_{st} \tau^{(Y,0)},$$

where $\tau^{(X,0)}, \tau^{(Y,0)}$ are the hitting times in 0.

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  where $\tau^{(X,0)}, \tau^{(Y,0)}$ are the hitting times in 0.
- Compute $\mathbb{E}\tau^{(Y,0)}$ by standard techniques for pure-jump Markov processes, i.e. as smallest solution of a certain recurrence relation.

Construction of $(Y(t))_{t\geq 0}$

1. Set $Y'(0) = X(0)$.
2. If $Y'(t) = X(t) = n$ for some t, let the next jump of $Y'(t)$ occur at the same time T_j as the next jump of $X(t)$. The jump for $Y'(t)$ goes to $n-1$ with probability $p_n = (1 + c/n)^{-1} \leq \mathbb{P}(A_n)$ and to $n+1$ with probability $1 - p_n$, coupled with $X(t)$ in such a way that $Y'(T_j) \geq X(T_j)$.
3. If $Y'(t) > X(t)$, let $Y'(t)$ behave like an independent birth-death process with birth rate $(1 - p_n)n$ and death rate $p_n n$ until $Y'(t)$ and $X(t)$ meet again.
4. $(Y'(t))$ has the right up-/down-probabilities, and its holding times in n can be stochastically dominated by $Exp(n)$-random variables. $\rightsquigarrow (Y(t))$.

A refined bound

Theorem

If ξ and η differ in only one point, we have for any $n^* \in \mathbb{N} \cup \{\infty\}$

$$\mathbb{E}\tau_{\xi,\eta} \le (n^*-1)! \left(\frac{\varepsilon}{c}\right)^{n^*-1} \left(\frac{1}{c}\sum_{i=n^*}^{\infty} \frac{c^i}{i!} + \int_0^c \frac{1}{s}\sum_{i=n^*}^{\infty}\frac{s^i}{i!}\,ds\right) + \frac{1+\varepsilon}{\varepsilon}\sum_{i=1}^{n^*-1}\frac{\varepsilon^i}{i} =: c_1(\lambda),$$

where

$$\varepsilon = \sup_{||\xi-\eta||=1} \int_{\mathcal{X}} |\lambda(x \mid \xi) - \lambda(x \mid \eta)|\,\alpha(dx) \quad \text{and}$$

$$c = c(n^*) = \sup_{||\xi-\eta||\ge n^*} \int_{\mathcal{X}} |\lambda(x \mid \xi) - \lambda(x \mid \eta)|\,\alpha(dx).$$

In particular, if $\varepsilon < 1$, we may choose $n^* = \infty$, so that

$$\mathbb{E}\tau_{\xi,\eta} \le \frac{1+\varepsilon}{\varepsilon}\log\left(\frac{1}{1-\varepsilon}\right) \le \frac{1+\varepsilon}{1-\varepsilon}.$$

Some Examples

- **Poisson:** $\lambda(x \mid \xi) = \lambda(x)$, hence

$$\varepsilon = 0.$$

- **Inhibitory PIP:** $\lambda(x \mid \xi) = \beta(x) \prod_{y \in \xi \setminus \{x\}} \varphi_2(x, y)$, hence

$$\varepsilon \leq \sup_{y \in \mathcal{X}} \int_{\mathcal{X}} \beta(x)(1 - \varphi_2(x, y))\, \alpha(dx).$$

- **Homogeneous Strauss:** $\lambda(x \mid \xi) = \beta \prod_{y \in \xi \setminus \{x\}} \gamma^{1\{d_0(x,y) \leq R\}}$, hence

$$\varepsilon \leq \beta(1 - \gamma) \sup_{y \in \mathcal{X}} \alpha(B_R(y)).$$

The generator approach

Overview

Our goal:

Find upper bound for the total variation distance

$$d_{TV}\big(\mathrm{Gibbs}(\nu), \mathrm{Gibbs}(\lambda)\big) = \sup_{f \in \mathcal{F}} \big|\mathbb{E}f(\Xi) - \mathbb{E}f(\mathrm{H})\big|,$$

where λ satisfies the stability condition (S), $\mathcal{F} = \mathcal{F}_{TV} = \{1_C\,;\ C \in \mathcal{N}\}$ and $\Xi \sim \mathrm{Gibbs}(\nu), \mathrm{H} \sim \mathrm{Gibbs}(\lambda)$.

Generator approach (Barbour, 1988)

For every $f \in \mathcal{F}$ find $h = h_f : \mathfrak{N} \to \mathbb{R}$ such that

$$f(\xi) - \mathbb{E}f(\mathsf{H}) = \mathscr{A}h(\xi) \quad \text{for all } \xi \in \mathfrak{N}, \qquad \text{(Stein equation)}$$

where $\mathscr{A}$ is the generator of a Markov process with stationary distribution Gibbs(λ) (generator approach).

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Natural choice: the SBD$^{(\xi)}(\lambda, 1)$-process $Z^{(\xi)} := \big(Z_t^{(\xi)}\big)_{t \geq 0}$ from earlier.

Solution of the Stein equation

It can be shown for bounded f that the function $h = h_f : \mathfrak{N} \to \mathbb{R}$,

$$h(\xi) := -\int_0^\infty \left[\mathbb{E}f(Z_t^{(\xi)}) - \mathbb{E}f(\mathrm{H})\right] dt,$$

is well-defined and solves the Stein equation $f(\xi) - \mathbb{E}f(\mathrm{H}) = \mathscr{A}h(\xi)$.

Bounding the first differences

By our result on expected coupling times (coupling $(Z_t^{(\xi+\delta_x)})$ and $(Z_t^{(\xi)})$ accordingly) we have for every $\xi \in \mathfrak{N}$ and every $x \in \mathcal{X}$

$$\begin{aligned}
&|h_f(\xi+\delta_x) - h_f(\xi)| \\
&\quad = \left| \int_0^\infty \left[\mathbb{E}\big(f(Z_t^{(\xi+\delta_x)})\big) - \mathbb{E}f(\mathrm{H}) \right] dt - \int_0^\infty \left[\mathbb{E}\big(f(Z_t^{(\xi)})\big) - \mathbb{E}f(\mathrm{H}) \right] dt \right| \\
&\quad = \left| \mathbb{E} \int_0^\infty \left[f(Z_t^{(\xi+\delta_x)}) - f(Z_t^{(\xi)}) \right] 1\{\tau_{\xi+\delta_x,\xi} > t\}\, dt \right| \\
&\quad \leq \sup_{\xi',\eta' \in \mathfrak{N}} |f(\xi') - f(\eta')| \int_0^\infty \mathbb{P}(\tau_{\xi+\delta_x,\xi} > t)\, dt \\
&\quad \leq \mathbb{E}\tau_{\xi+\delta_x,\xi} \\
&\quad \leq c_1(\lambda).
\end{aligned}$$

(Same argument as in Barbour and Brown, 1992)

Bounding the Stein equation

If now Ξ is a Gibbs(ν) process (does not need to satisfy stability), we obtain by the Georgii–Nguyen–Zessin equation

$$
\begin{aligned}
&\left|\mathbb{E}f(\Xi) - \mathbb{E}f(\mathrm{H})\right| \\
&\quad = \left|\mathbb{E}\mathcal{A}h_f(\Xi)\right| \\
&\quad = \left|\mathbb{E}\int_{\mathcal{X}}\big[h_f(\Xi+\delta_x) - h_f(\Xi)\big]\lambda(x\,|\,\Xi)\;\alpha(dx) + \mathbb{E}\int_{\mathcal{X}}\big[h_f(\Xi-\delta_x) - h_f(\Xi)\big]\;\Xi(dx)\right| \\
&\quad = \left|\mathbb{E}\int_{\mathcal{X}}\big[h_f(\Xi+\delta_x) - h_f(\Xi)\big](\lambda(x\,|\,\Xi) - \nu(x\,|\,\Xi))\;\alpha(dx)\right| \\
&\quad \leq \sup_{\xi\in\mathfrak{N},\,x\in\mathcal{X}} |h_f(\xi+\delta_x) - h_f(\xi)| \int_{\mathcal{X}} \mathbb{E}|\nu(x\,|\,\Xi) - \lambda(x\,|\,\Xi)|\;\alpha(dx) \\
&\quad \leq c_1(\lambda)\int_{\mathcal{X}} \mathbb{E}|\nu(x\,|\,\Xi) - \lambda(x\,|\,\Xi)|\;\alpha(dx).
\end{aligned}
$$

Upper bounds in the total variation metric

Upper bound

Theorem (S and Stucki, 2014)

For any two Gibbs point processes

Ξ *with conditional intensity* $\nu(\cdot \,|\, \cdot)$,

H *with conditional intensity* $\lambda(\cdot \,|\, \cdot)$ *satisfying the stability condition (S),*

we have

$$d_{TV}\big(\mathscr{L}(\Xi), \mathscr{L}(H)\big) \leq c_1(\lambda) \int_{\mathcal{X}} \mathbb{E}\big|\nu(x \,|\, \Xi) - \lambda(x \,|\, \Xi)\big| \, \alpha(dx),$$

where the general formula for $c_1(\lambda)$ *was given earlier. E.g. if*

$$\varepsilon = \sup_{\xi \in \mathfrak{N},\, y \in \mathcal{X}} \int_{\mathcal{X}} \big|\lambda(x \,|\, \xi + \delta_y) - \lambda(x \,|\, \xi)\big| \, \alpha(dx) < 1,$$

we have

$$c_1(\lambda) = \frac{1+\varepsilon}{\varepsilon} \log\left(\frac{1}{1-\varepsilon}\right) \leq \frac{1+\varepsilon}{1-\varepsilon}.$$

Two consequences

- Suppose that $\mathcal{X} \subset \mathbb{R}^D$, and $\Xi \sim \mathrm{PIP}(\beta, \varphi_1)$ and $\mathrm{H} \sim \mathrm{PIP}(\beta, \varphi_2)$ are stationary and inhibitory, i.e. β is constant and $\varphi_i(x,y) = \varphi_i(x-y) \leq 1$ for all $x, y \in \mathcal{X}$. Then

$$d_{TV}(\mathscr{L}(\Xi), \mathscr{L}(\mathrm{H})) \leq c_1(\lambda)\,\beta\,\mathbb{E}|\Xi| \int_{\mathbb{R}^D} |\varphi_1(x) - \varphi_2(x)|\,dx.$$

- Suppose that $\mathcal{X} \subset \mathbb{R}^D$, and $\Xi \sim \mathrm{AIP}\big(\beta\gamma^{\alpha_D (R/2)^D}, \gamma; R/2\big)$ and $\mathrm{H} \sim \mathrm{Strauss}(\beta, 0; R)$, where α_D is the volume of the unit ball in $\mathbb{R}^D$. Then

$$d_{TV}(\mathscr{L}(\Xi), \mathscr{L}(\mathrm{H})) \leq c_1(\lambda)\,2D\alpha_D R^{D-1}\,\beta\,\mathbb{E}|\Xi|\big(\log\gamma^{-\alpha_D}\big)^{-1/D}.$$

Rate for the convergence result in Baddeley and Van Lieshout (1995).