Conditional distribution approximation with birth-death processes

Han Liang Gan

Washington University in St. Louis

May 21, 2015

Joint work with Aihua Xia

Conditional?

Motivational pictures.

Regular Stein setup

For the Poisson distribution the Stein identity is

$$\mathbb{E}\left[\lambda g(W+1) - Wg(W)\right].$$

Regular Stein setup

For the Poisson distribution the Stein identity is

$$\mathbb{E}\left[\lambda g(W+1) - Wg(W)\right].$$

For any function f, define g_f as the solution to

$$\lambda g_f(i+1) - ig_f(i) = f(i) - Ef(Z), \tag{1}$$

where Z follows Poisson distribution.

Regular Stein setup

For the Poisson distribution the Stein identity is

$$\mathbb{E}\left[\lambda g(W+1) - Wg(W)\right].$$

For any function f, define g_f as the solution to

$$\lambda g_f(i+1) - ig_f(i) = f(i) - Ef(Z), \tag{1}$$

where Z follows Poisson distribution.

If we take $f \in \mathcal{F}_{TV}$, that is $f = \mathbf{1}_A(x)$ for some $A \subset \mathbb{Z}_+$, then it follows from (1)

$$\mathbb{P}(W \in A) - \mathbb{P}(Z \in A) = \mathbb{E}\left[\lambda g_A(W+1) - Wg_A(W)\right] \tag{2}$$

The generator approach

If we set $g(w) = h(w) - h(w-1)$ the Stein identity becomes:

$$\mathbb{E}\left[\lambda\left(h(W+1) - h(W)\right) - W(h(W) - h(W-1))\right]$$

The generator approach

If we set $g(w) = h(w) - h(w-1)$ the Stein identity becomes:

$$\mathbb{E}\left[\lambda\left(h(W+1) - h(W)\right) - W(h(W) - h(W-1))\right]$$

Set

$$\mathcal{A}h(i) = \lambda\left[h(i+1) - h(i)\right] - i\left[h(i) - h(i-1)\right].$$

The generator approach

If we set $g(w) = h(w) - h(w-1)$ the Stein identity becomes:

$$\mathbb{E}\left[\lambda\left(h(W+1) - h(W)\right) - W(h(W) - h(W-1))\right]$$

Set

$$\mathcal{A}h(i) = \lambda\left[h(i+1) - h(i)\right] - i\left[h(i) - h(i-1)\right].$$

This is the generator for an immigration death process.

Moreover, the stationary distribution for this generator is $\mathrm{Po}(\lambda)$.

Stein Identity for the conditional Poisson distribution

Lemma

A random variable W has distribution $\mathrm{Po}^{(\mathrm{m})}(\lambda)$, if and only if for all functions g in a 'rich enough' family of functions $\mathcal{F}$,

$$\mathbb{E}\left[\lambda g(W+1) - Wg(W)\mathbf{1}_{W>m}\right] = 0. \tag{3}$$

The generator approach

If we set $g_m(w) = h_m(w) - h_m(w-1)$, equation (3) becomes:

$$\mathbb{E}\left[\lambda\left(h_m(W+1) - h_m(W)\right) - W(h_m(W) - h_m(W-1))\mathbf{1}_{W>m}\right]$$

The generator approach

If we set $g_m(w) = h_m(w) - h_m(w-1)$, equation (3) becomes:

$$\mathbb{E}\left[\lambda\left(h_m(W+1) - h_m(W)\right) - W(h_m(W) - h_m(W-1))\mathbf{1}_{W>m}\right]$$

Set

$$\mathcal{A}^{(m)} h_m(i) = \lambda\left[h_m(i+1) - h_m(i)\right] + i\left[h_m(i-1) - h_m(i)\right]\mathbf{1}_{i>m}.$$

The generator approach

If we set $g_m(w) = h_m(w) - h_m(w-1)$, equation (3) becomes:

$$\mathbb{E}\left[\lambda\left(h_m(W+1) - h_m(W)\right) - W(h_m(W) - h_m(W-1))\mathbf{1}_{W>m}\right]$$

Set

$$\mathcal{A}^{(m)}h_m(i) = \lambda\left[h_m(i+1) - h_m(i)\right] + i\left[h_m(i-1) - h_m(i)\right]\mathbf{1}_{i>m}.$$

This is the generator for a (censored) immigration death process.

Moreover, the stationary distribution for this generator is $\mathrm{Po}^{(\mathrm{m})}(\lambda)$.

Stein's magic factors

Theorem

For conditional (compound) Poisson approximation, both $\|g_{m,f}\|$ and $\|\Delta g_{m,f}\|$ are decreasing in m.

Conditional Stein's magic factors

Theorem

For conditional Poisson approximation, the solution to Stein's equation satisfies

$$\|g_{m,f}\| \leq 1 \wedge \sqrt{\frac{2}{\lambda e}},$$

$$\|\Delta g_{m,f}\| = \frac{\mathbb{P}(Z > m)}{\lambda \mathbb{P}(Z \geq m)},$$

where $Z \sim \mathrm{Po}(\lambda)$.

For example, in the case where $m = 1$, $\|\Delta g_{m,f}\| = \frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda(1-e^{-\lambda})}$.

Solution to the Stein equation

It can be verifed that the solution to the Stein equation is

$$h_{m,f}(i) = -\int_0^\infty \left(\mathbb{E}f(Z_i^{(m)}(t)) - \mathbb{E}f(Z^{(m)})\right) dt,$$

where $Z^{(m)} \sim \mathrm{Po}^{(\mathrm{m})}(\lambda)$, and $Z_i^{(m)}(t)$ is an immigration-death process with generator $\mathcal{A}^{(m)}$ and $Z_i^{(m)}(0) = i$.

Complications with couplings

The standard coupling arguments do not work due to a lack of independence in the conditional case.

Complications with couplings

The standard coupling arguments do not work due to a lack of independence in the conditional case.

However, if we follow the procedure used in Brown & Xia (2001), and couple in time as opposed to coupling in location, we achieve the previously mentioned results.

Relating unconditional and conditional approximation

Lemma

If a nonnegative integer valued random variable W can be approximated by $\mathrm{Po}(\lambda)$ and it can be shown that

$$|\mathbb{E}\mathcal{A}g(W)| \leq \epsilon_1 \|g_{0,f}\| + \epsilon_2 \|g_{0,f}\|,$$

for all functions g_0 on $\mathbb{Z}_0$ for which $\|g_{0,f}\|$ and $\|\Delta g_{0,f}\|$ are finite, and ϵ_1, ϵ_2 are positive, then $W^{(m)}$ can be approximated by $\mathrm{Po}^{(\mathrm{m})}(\lambda)$ with

$$d_{TV}(\mathcal{L}(W^{(m)}), \mathrm{Po}^{(\mathrm{m})}(\lambda)) \leq \frac{1}{\mathbb{P}(\mathrm{W} \geq \mathrm{m})} \left\{\epsilon_1 \|\mathrm{g_{m,f}}\| + \epsilon_2 \|\Delta \mathrm{g_{m,f}}\|\right\}.$$

Take home message

For anything that has a good unconditional approximation, there exists a good conditional approximation.

Furthermore, this direct conditional approach should be better due to the monotonicity of the Stein factors.

Conditional Poisson point process approximation

Perhaps we are not just interested in how many points we have, but where in space they appear.

So we should try to work on conditional Poisson point process approximation.

Spatial immigration-death processes

Similarly to random variable approximation, we have our generator for a conditional point process.

$$\mathcal{A}^{(m)}h(\xi) = \int_{\Gamma} [h(\xi + \delta_\alpha) - h(\xi)] \boldsymbol{\lambda}(\delta_\alpha) + \int_{\Gamma} [h(\xi - \delta_\alpha) - h(\delta_\alpha)] \xi(d\alpha) \cdot \mathbf{1}_{|\xi|>m}. \tag{4}$$

Solution to the Stein equation

The solution to the Stein equation is

$$h_{m,f}(\xi) = \int_0^\infty \mathbb{E}\left[f(Z_\xi^{(m)}(t)) - \mathrm{Po}^{(\mathrm{m})}(\boldsymbol{\lambda})(\mathrm{f})\right] dt,$$

Metric for point process approximation

Definition

Let $\overline{d}_1 : \mathcal{H}^2 \to \mathbb{R}_+$, be

$$\overline{d}_1(\xi, \eta) = \frac{1}{n}\left(\min_{\pi \in \Pi_n} \sum_{i=1}^{m} d_0(x_i, y_{\pi(i)}) + (n-m)\right),$$

where Π_n is the permutation group of n elements, $|\xi| = n$, $|\eta| =$ m and without loss of generality $n \geq m$.

Definition

Let $\mathcal{F}_{\bar{d}} = \{f : \mathcal{H} \to [0,1]; |f(\xi) - f(\eta)| \leq \overline{d}_1(\xi, \eta) \text{ for all } \xi, \eta \in \mathcal{H}\}$. Set,

$$\overline{d}_2(P, Q) = \sup_{f \in \mathcal{F}} \left| \int_{\mathcal{H}} f dP - \int_{\mathcal{H}} f dQ \right|.$$

First difference of h

Let

$$K_1 := \min\left(1, \frac{0.95 + \log^+ \lambda}{\lambda}\right).$$

First difference of h

Let

$$K_1 := \min\left(1, \frac{0.95 + \log^+ \lambda}{\lambda}\right).$$

Theorem

$$\|\Delta h_{1,f}\| \le \frac{1}{\lambda} + 2K_1,$$

and if $\lambda > 3$,

$$\|\Delta h_{1,f}\| \le \frac{1}{\lambda(\lambda - 1)} + \frac{\lambda}{\lambda - 1} K_1.$$

Theorem

$$\|\Delta^2 h_{1,f}\| < \frac{4}{\lambda} + 6K_1 + K_2.$$

Furthermore, if $\lambda > 3$,

$$\|\Delta^2 h_{1,f}\| < C_1(\lambda) + C_2(\lambda)K_1 + K_2.$$

where

$$C_1(\lambda) := \frac{3\lambda + 10}{\lambda(\lambda - 1)(\lambda + 1)}, \quad C_2(\lambda) := \frac{4\lambda + 9}{(\lambda - 1)(\lambda + 1)},$$

and

$$K_2 := \min\left\{\frac{2\log\lambda}{\lambda}\mathbf{1}_{\lambda \geq 1.76}, \frac{1}{2}\mathbf{1}_{\lambda < 1.76}\right\}.$$