BEATING LOG-SOBOLEV: ONE STEIN'S KERNEL AT A TIME

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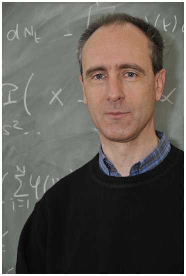
CREDITS

Based on two joint works: **(1)** Nourdin, Peccati and Swan (*JFA*, 2014), and **(2)** Ledoux, Nourdin and Peccati (*GAFA*, 2015).



A General Question

I shall use the notation $(\Omega, \mathscr{F}, \mathbf{P})$ for a generic probability space, with $\mathbf{E}$ indicating expectation with respect to $\mathbf{P}$. Fix two integers $k, d \geq 1$.

- ⋆ $G = (G_1, ..., G_k)$ is a k-dimensional vector of i.i.d. standard Gaussian random variables (the "underlying Gaussian field")
- ⋆ $F = (f_1(G), ..., f_d(G))$ is a d-dimensional vector of smooth (non-linear) transformations of G (the "unknown distribution") with identity covariance matrix. We assume that F has a density.
- ⋆ $N = (N_1, ..., N_d)$ is a d-dimensional vector of i.i.d. standard Gaussian random variables (the "target distribution").

Question: Can we (meaningfully!) bound the quantity

$$\mathbf{TV}(F, N) = \sup_{A \in \mathscr{B}(\mathbb{R}^d)} |\mathbf{P}(F \in A) - \mathbf{P}(N \in A)| \; ?$$

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NOTATION

The law of N is denoted by γ_d, that is

$$\begin{aligned}\gamma_d(\mathrm{d}x_1,...,\mathrm{d}x_d) &= \frac{1}{(2\pi)^{d/2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{d}x_i^2\right\}\mathrm{d}x_1\cdots\mathrm{d}x_d \\ &:= \phi(x)\mathrm{d}x.\end{aligned}$$

We shall assume that the law of F has a smooth density h with respect to γ_d, that is:

$$\mathbf{P}[F \in A] = \int_A h(x)\,\gamma_d(\mathrm{d}x), \quad A \subseteq \mathbb{R}^d.$$

This implies, in particular, that

$$\mathbf{TV}(F,N) = \frac{1}{2}\int_{\mathbb{R}^d} |h(x)-1|\,\gamma_d(\mathrm{d}x).$$

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The Ornstein-Uhlenbeck semigroup

It will be useful to consider the standard interpolation:

$$F_t := e^{-t}F + \sqrt{1-e^{-2t}}N, \quad t \geq 0,$$

so that $F_0 = F$ and $F_\infty = N$.

It is easy to prove that the density of F_t is given by $P_t h$, where, for a given test function g,

$$P_t g(x) = \mathbf{E}[g(e^{-t}x + \sqrt{1-e^{-2t}}N)], \quad t \geq 0,$$

defines the **Ornstein-Uhlenbeck semigroup** (Mehler's form).

We denote by L the **generator** of $\{P_t\}$. It is well known that L (as an operator on $L^2(\gamma_k)$) has eigenvalues $0, -1, -2, ...,$; the corresponding eigenspaces $\{C_n : n \geq 0\}$ are the so-called **Wiener chaoses** associated with γ_k.

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The Stein Kernel

An application of integration by parts shows that, for every smooth mapping $g : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbf{E}[Fg(F)] = \mathbf{E}[\tau_h \cdot \nabla g(F)] \quad \text{(as vectors)}$$

where τ_h is the $d \times d$ matrix given by

$$\tau_h^{i,j} = \tau_h^{i,j}(F) = \mathbf{E}\left[\langle \nabla f_j(G), -\nabla L^{-1} f_i(G) \rangle_{\mathbb{R}^k} \,|\, F \right].$$

(L^{-1} is the (pseudo)inverse of the OU semigroup on $L^2(\gamma_k)$).

We call τ_h the **Stein kernel** associated with F. Note that, if $h = 1$ (and therefore $F \overset{LAW}{=} N$), then necessarily $\tau_h =$ identity.

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Definition
The **Stein discrepancy** *S between F and N is defined as follows:*

$$S = S(F\|N) := \sqrt{\sum_{i,j=1}^{d} \mathbf{E}[(\tau_h^{i,j} - \delta_j^i)^2]}$$

A direct computation shows that

$$S(F_t\|N) \leq e^{-2t} S(F\|N), \quad t \geq 0.$$

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Some estimates

Can one use S to rigorously measure the distance between F and N?

Stein's method (Stein '72, '86) allows one to obtain the following bounds (see Nourdin and Peccati, '09, '12; Nourdin and Rosinski, '12).

- $\star$ When $d = 1$,

$$\mathbf{TV}(F, N) \leq 2S(F \| N).$$

- $\star$ When $d = 1$ and F belongs to the qth Wiener chaos of γ_k ($q \geq 2$), then

$$\mathbf{TV}(F, N) \leq 2S(F \| N) \leq 2\sqrt{\frac{q-1}{3q}}\sqrt{E[F^4] - 3}$$

(note that $3 = \mathbf{E}[N^4]$). See Nualart and Peccati (2005).

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- ⋆ In general, for any $d \geq 1$,

$$\mathbf{Wass}_1(F,N) := \sup_{g \in \mathbf{Lip}(1)} |\mathbf{E}(g(F)) - \mathbf{E}(g(N))| \leq S(F\|N).$$

- ⋆ Finally, for any $d \geq 1$ and when each F_i is chaotic,

$$\mathbf{Wass}_1(F,N) \leq S(F\|N) \leq \sqrt{\mathbf{E}[\|F\|^4_{\mathbb{R}^d}] - \mathbf{E}[\|N\|^4_{\mathbb{R}^d}]}.$$

- ⋆ Very large scope of applications, e.g.: **(1)** *variations of Gaussian-subordinated processes*, **(2)** *local times of fractional processes*, **(3)** *zeros of random polynomials*, **(4)** *statistical analysis of spherical fields*, **(5)** *excursion sets of random fields on homogeneous spaces*, **(6)** *random matrices*, **(7)** *universality principles*.

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From Wasserstein to Total Variation

- ⋆ For a general dimension d, going from $\mathbf{Wass}_1$ to **TV** is a remarkably difficult task.
- ⋆ For instance, Nourdin, Nualart and Polly (2013), have proved that, if $F_n \overset{LAW}{\to} N$, and F_n lives in the sum of the first q chaoses

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for every

$$\alpha < \frac{1}{1 + (d+1)(3 + 4d(q+1))}.$$

- ⋆ We will address this task by using the notions of **entropy** (H), **Fisher information** (I), and **2-Wasserstein distance** (W).

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$$H(F\|N) := \int_{\mathbb{R}^d} h(x)\log h(x)\,\gamma_d(\mathrm{d}x)$$
$$= \mathbf{E}[-\log\phi(N)] - \mathbf{E}[-\log p(F)] = \mathbf{Ent}(N) - \mathbf{Ent}(F).$$

(ϕ = density of N; $p = h\phi$ = density of F)

Recall **Pinsker's inequality**:

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It is well-known that:

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Important relations

- ⋆ A famous formula, due to **de Bruijn**, states that

$$H(F\|N) = \int_0^\infty I(F_t\|N)\,\mathrm{d}t$$

(the proof is based on the use of the heat equation).

- ⋆ Using the exponential decay of $t \mapsto I(F_t\|N)$, we deduce immediately the **log-Sobolev inequality** (Gross, 1972)

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2-Wasserstein distance

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$$W(F,N) = \inf \sqrt{\mathbf{E}\|X - Y\|_{\mathbb{R}^d}^2}$$

where the infimum runs over all pairs (X, Y) *such that* $X \overset{LAW}{=} F$ *and* $Y \overset{LAW}{=} N$.

SOME CLASSIC RELATIONS

- ⋆ The famous **Talagrand's transportation inequality** (Talagrand, 1996) states that

$$W(F,N) \leq \sqrt{2H(F\|N)}.$$

- ⋆ In Otto-Villani (2001), it is proved that

$$W(F,N) \leq \int_0^\infty \sqrt{I(F_t\|N)}\,\mathrm{d}t.$$

- ⋆ Finally, in Otto-Villani (2001) one can find the well-known **HWI inequality**

$$H(F\|N) \leq W(F,N)\sqrt{I(F\|N)} - \frac{1}{2}W^2(F,N).$$

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SOME CLASSIC RELATIONS

- ⋆ The famous **Talagrand's transportation inequality** (Talagrand, 1996) states that

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Proposition (Ledoux, Nourdin, Peccati (2015))

Assume F admits a Stein's kernel τ_h:

$$\mathbf{E}[Fg(F)] = \mathbf{E}[\tau_h \cdot \nabla g(F)] \quad \textit{(g smooth).}$$

Then,

$$I(F_t \| N) \leq \frac{e^{-4t}}{1 - e^{-2t}} S^2(F \| N), \quad t > 0.$$

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Theorem (HSI inequality)

One has that:

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Under the above notation,

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Remark: this estimate improves Talagrand's transportation inequality $W \leq \sqrt{2H}$. The reason is that

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$$\textbf{HSI}, \ \textbf{HWI} \Rightarrow \textbf{Log-Sobolev} \Rightarrow \textbf{Talagrand},$$

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SOME APPLICATIONS & EXTENSIONS

⋆ Reinforced **convergence to equilibrium**:

$$H(F_t \,\|\, N) \leq \frac{e^{-4t}}{1-e^{-2t}} S^2(F\|N)$$

(compare with the estimate: $H(F_t \,\|\, N) \leq e^{-2t} H(F \,\|\, N)$).

⋆ **Concentration** via L^p norms of Lipschitz functions:

$$\mathbf{E}[|u(F)|^p]^{1/p} \leq C\left[S_p + \sqrt{p} + \sqrt{pS_p}\right],$$

where

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EXTENSIONS

- ⋆ HSI can be suitably extended to the case where the target distribution μ is the invariant measure of a symmetric Markov semigroup $\{P_t\}$ with generator

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$$\int b \cdot \nabla f \, \mathrm{d}\nu + \int \langle \tau_\nu, \mathrm{Hess} f\rangle_{HS} \, \mathrm{d}\nu = 0$$

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$$S^2(\nu \,\|\, \mu) = \int \|a^{-1/2}\tau_\nu a^{-1/2} - \mathbf{Id}\|^2 \mathrm{d}\nu.$$

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The problem with Fisher information

- ⋆ Recall that we are interested in bounding $H(F\|N)$, when

$$F = (f_1(G), ..., f_d(G)),$$

and the f_i's are non-linear transforms of the Gaussian field G. The problem is that, in such a general framework, there is actually **no available technique for properly bounding the relative Fisher information** $I(F\|N)$ without further heavy constraints on F.

- ⋆ This means that one has to be much more careful in studying the term

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An explicit example

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where the Gaussian sequence X_k has autocorrelation $\varrho(k) \sim |k|^\alpha$, for some $\alpha < -2/3$ (this corresponds to the increments of a fractional Brownian motion of Hurst index $H < 2/3$). Let $N \sim \mathcal{N}(0, \mathbb{I}_2)$.

Then, for a suitable constant $\sigma > 0$,

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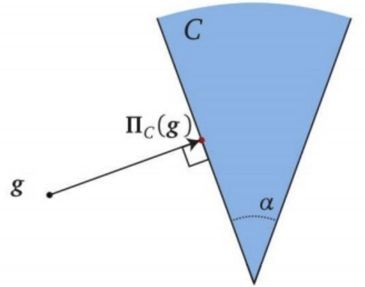

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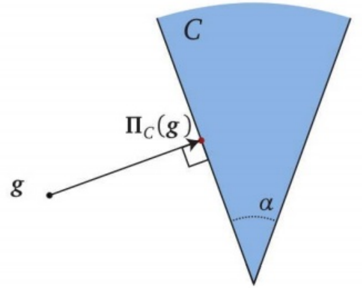

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THANK YOU FOR YOUR ATTENTION !