ON THE PROBABILITY APPROXIMATION OF SPATIAL POINT PROCESSES

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Point processes

X locally compact metric space (serving as the state space of the points), $\mathcal{B}(X)$ Borel σ-field on X.

Γ_X set of locally finite point configurations of X:

$$\Gamma_X := \{\mathbf{x} \subseteq X : \sharp(\mathbf{x}_K) < \infty \ \forall \text{ relatively compact } K \in \mathcal{B}(X)\}$$

where $\mathbf{x}_K := \mathbf{x} \cap K$. Here for any subset $\mathbf{x} \subseteq X$, $\sharp(\mathbf{x})$ is the cardinality of $\mathbf{x}$, setting $\sharp(\mathbf{x}) := \infty$ if $\mathbf{x}$ is not finite. We endow Γ_X with the vague topology and denote by $\mathcal{B}(\Gamma_X)$ the Borel σ-field on Γ_X.

A point process is a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$.

Point processes with conditional intensity: definition

We assume that the probability measure μ on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ has conditional (or Papangelou) intensity π, i.e. $\pi : X \times \Gamma_X \to [0, +\infty]$ is a measurable function such that

$$\int_{\Gamma_X} \sum_{x \in \mathbf{x}} \varphi(x, \mathbf{x} \setminus \{x\})\, \mu(\mathrm{d}\mathbf{x}) = \int_{\Gamma_X} \int_X \varphi(x, \mathbf{x})\, \pi(x, \mathbf{x}) \sigma(\mathrm{d}x) \mu(\mathrm{d}\mathbf{x})$$

for functions $\varphi(x, \mathbf{x})$ which are non-negative, Papangelou (1974), Georgii (1976), Nguyen and Zessin (1979). Here σ is a diffuse and locally finite measure on $(X, \mathcal{B}(X))$.

μ is a Poisson process with intensity measure σ if and only if $\pi \equiv 1$.

POINT PROCESSES WITH CONDITIONAL INTENSITY: INTERPRETATION

The Papangelou intensity $\pi_x(\mathbf{x})$ has the following interpretation: $\pi_x(\mathbf{x})\,\sigma(\mathrm{d}x)$ is the infinitesimal probability of finding a point of the process in the region $\mathrm{d}x$ around x, given that the point process agrees with the configuration $\mathbf{x}$ outside $\mathrm{d}x$.

Main example: Gibbs point processes with pair potential

A pair potential is a function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(x) = \phi(-x)$. We define the relative energy of interaction between a particle at $x \in \mathbb{R}^d \setminus \mathbf{x}$ and the configuration $\mathbf{x} \in \Gamma_{\mathbb{R}^d}$ by

$$E(x, \mathbf{x}) := \begin{cases} \sum_{y \in \mathbf{x}} \phi(x - y) & \text{if } \sum_{y \in \mathbf{x}} |\phi(x - y)| < \infty \\ +\infty & \text{otherwise.} \end{cases}$$

μ on $\Gamma_{\mathbb{R}^d}$ is called Gibbs point process with activity $z > 0$ and pair potential ϕ if it has Papangelou intensity of the form

$$\pi_x(\mathbf{x}) := z \exp(-E(x, \mathbf{x})) \quad \text{with reference measure } \mathrm{d}x.$$

$\mathcal{G}_s(z, \phi) \neq \emptyset$ if ϕ is superstable, lower-regular and $1 - \mathrm{e}^{-\phi} \in L^1(\mathbb{R}^d, \mathrm{d}x)$, Ruelle (1970). μ is called inhibitory if $\phi \geq 0$ and finite range if $1 - \mathrm{e}^{-\phi}$ has compact support.

INNOVATION

We define the (raw) innovation as

$$\delta_{\mathbf{x}}(\varphi) := \sum_{x \in \mathbf{x}} \varphi(x) - \int_X \varphi(x)\pi_x(\mathbf{x})\,\sigma(\mathrm{d}x)$$

for any measurable function $\varphi : X \to \mathbb{R}$ for which $|\delta_{\mathbf{x}}(\varphi)| < \infty$ μ-a.s..
The innovation $\delta(\varphi)$ is well-defined for all $\varphi \in L^1(X, \mathrm{E}[\pi_x]\sigma(\mathrm{d}x))$. Here E denotes the mean with respect to μ.

The innovation is used in spatial statistics e.g. to check the validity of a point process model fitted to data, Baddeley, Møller and Pakes (2008).

Finite difference operator

We define the finite difference operator by

$$D_x F(\mathbf{x}) := F(\mathbf{x} \cup \{x\}) - F(\mathbf{x}), \quad x \in X, \mathbf{x} \in \Gamma_X.$$

Here $F : \Gamma_X \to \mathbb{R}$ is a measurable function.

Wasserstein distance

Let $F \in L^1(\Gamma_X, \mu)$ and Z a standard normal random variable defined on $(\Omega, \mathcal{F}, P)$. We denote by E_P the mean with respect to P. The Wasserstein distance between F and Z is

$$d_W(F, Z) := \sup_{h \in \mathrm{Lip}(1)} |\mathrm{E}[h(F)] - \mathrm{E}_P[h(Z)]|,$$

where $\mathrm{Lip}(1)$ is the class of real-valued Lipschitz functions with Lipschitz constant less than or equal to 1.

Gaussian approximation: general bound

If μ has conditional intensity π and

$$\varphi \in L^{1,2}(X, \mathrm{E}[\pi_x]\,\sigma(\mathrm{d}x)),$$

then

$$
\begin{aligned}
& d_W(\delta(\varphi), Z) \\
& \leq \sqrt{2/\pi}\sqrt{1 - 2\int_X |\varphi(x)|^2 \mathrm{E}[\pi_x]\,\sigma(\mathrm{d}x) + \int_{X^2} |\varphi(x)\varphi(y)|^2 \mathrm{E}[\pi_x \pi_y]\,\sigma_2(\mathrm{d}x\mathrm{d}y)} \\
& \quad + \|\varphi\|^3_{L^3(X, \mathrm{E}[\pi_x]\,\sigma(\mathrm{d}x))} + \sqrt{2/\pi}\int_{X^2} |\varphi(x)\varphi(y)| \mathrm{E}[|D_x \pi_y| \pi_x]\,\sigma_2(\mathrm{d}x\mathrm{d}y) \\
& \quad + 2\int_{X^2} |\varphi(x)|^2 |\varphi(y)| \mathrm{E}[|D_x \pi_y| \pi_x]\,\sigma_2(\mathrm{d}x\mathrm{d}y) \\
& \quad + \int_{X^3} |\varphi(x)\varphi(y)\varphi(w)| \mathrm{E}[|D_x \pi_y||D_x \pi_w| \pi_x]\,\sigma_3(\mathrm{d}x\mathrm{d}y\mathrm{d}w).
\end{aligned}
$$

POISSON CASE

If μ is a Poisson process with mean measure σ and $\varphi \in L^{1,2}(X, \sigma)$, then

$$d_W(\delta(\varphi), Z) \leq |1 - \|\varphi\|^2_{L^2(X,\sigma)}| + \|\varphi\|^3_{L^3(X,\sigma)},$$

Peccati, Sole', Taqqu and Utzet (2010).

Gibbs case

If μ is a Gibbs point process with activity $z > 0$ and pair potential ϕ, and $\varphi \in L^{1,2}(\mathbb{R}^d, \mathrm{E}[\pi_x]\,\mathrm{d}x)$, then

$$
\begin{aligned}
&d_W(\delta(\varphi), Z) \\
&\leq \sqrt{2/\pi}\sqrt{1 - 2\int_{\mathbb{R}^d} |\varphi(x)|^2 \mathrm{E}[\pi_x]\,\mathrm{d}x + \int_{\mathbb{R}^{2d}} |\varphi(x)\varphi(y)|^2 \mathrm{E}[\pi_x\pi_y]\,\mathrm{d}x\mathrm{d}y} \\
&\quad + \|\varphi\|^3_{L^3(\mathbb{R}^d, \mathrm{E}[\pi_x]\,\mathrm{d}x)} + \sqrt{2/\pi}\int_{\mathbb{R}^{2d}} |\varphi(x)\varphi(y)||1 - \mathrm{e}^{-\phi(y-x)}|\mathrm{E}[\pi_x\pi_y]\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\int_{\mathbb{R}^{2d}} |\varphi(x)|^2|\varphi(y)||1 - \mathrm{e}^{-\phi(y-x)}|\mathrm{E}[\pi_x\pi_y]\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^{3d}} |\varphi(x)\varphi(y)\varphi(w)||1 - \mathrm{e}^{-\phi(y-x)}||1 - \mathrm{e}^{-\phi(w-x)}|\mathrm{E}[\pi_x\pi_y\pi_w]\,\mathrm{d}x\mathrm{d}y\mathrm{d}w.
\end{aligned}
$$

An explicit bound for s.i.f.r. Gibbs point processes

Suppose μ stationary Gibbs point process with $z > 0$ and $\phi \geq 0$ with compact support and $\varphi \in L^{1,2}(\mathbb{R}^d, \mathrm{d}x)$. Then, for any $p', q', p'', q'' > 1$ such that $p'^{-1} + q'^{-1} = p''^{-1} + q''^{-1} = 1$,

$$d_W(\delta(\varphi), Z) \leq \sqrt{2/\pi}\sqrt{1 - 2c_1\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)} + zc_2\|\varphi\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}} + c_2 A,$$

where

$$\begin{aligned} A :=& \|\varphi\|^3_{L^3(\mathbb{R}^d,\mathrm{d}x)} + z\sqrt{2/\pi}\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)}\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ &+ 2z\|\varphi^2\|_{L^{p'}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\ &+ z^2\|\varphi\|_{L^{p'p''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{p'q''}(\mathbb{R}^d,\mathrm{d}x)}\|\varphi\|_{L^{q'}(\mathbb{R}^d,\mathrm{d}x)}\|1 - \mathrm{e}^{-\phi}\|^2_{L^1(\mathbb{R}^d,\mathrm{d}x)} \end{aligned}$$

and

$$c_1 := \frac{z}{1 + z\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}}, \quad c_2 := \frac{z}{2 - \exp(-z\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d,\mathrm{d}x)})}.$$

EXAMPLE

Define

$$\phi(x) := -\log(1 - r^d \mathrm{e}' \rho_r(x)), \quad r > 0, \mathrm{e}' < \mathrm{e}, x \in \mathbb{R}^d$$

where $\rho_r(x) := r^{-d}\rho(x/r)$ is the classical mollifier, i.e.

$$\rho(x) := \mathbf{1}\{\|x\| \leq 1\}\mathrm{e}^{(\|x\|^2-1)^{-1}}.$$

Then

$$\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} = \mathrm{e}' r^d.$$

EXAMPLE

Let μ be the Strauss process, i.e. take

$$\phi(x) := (-\log \nu)\mathbf{1}\{\|x\| \leq r\}, \quad \nu \in (0,1), r > 0, x \in \mathbb{R}^d.$$

Then

$$\|1 - \mathrm{e}^{-\phi}\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} = (1-\nu)\alpha_d r^d,$$

where α_d denotes the volume of the unit ball of $\mathbb{R}^d$.

A QUANTITATIVE CLT FOR S.I.F.R. GIBBS POINT PROCESSES

Assume μ_n, $n \geq 1$, stationary Gibbs point process with $z_n > 0$ and $\phi_n \geq 0$ with compact support, $z_n \to z > 0$ and $\|1 - \mathrm{e}^{-\phi_n}\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} \to 0$. In addition, assume that $\{\varphi_n\}_{n \geq 1}$ is a sequence of integrable and square integrable functions such that, for some $p', q', p'', q'' > 1$ with $p'^{-1} + q'^{-1} = p''^{-1} + q''^{-1} = 1$,

$$\|\varphi_n\|^2_{L^2(\mathbb{R}^d, \mathrm{d}x)} \to z^{-1}, \quad \|\varphi_n\|_{L^3(\mathbb{R}^d, \mathrm{d}x)} \to 0$$

$$\|\varphi_n^2\|_{L^{p'}(\mathbb{R}^d, \mathrm{d}x)} \|\varphi_n\|_{L^{q'}(\mathbb{R}^d, \mathrm{d}x)} \|1 - \mathrm{e}^{-\phi_n}\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} \to 0$$

$$\|\varphi_n\|_{L^{p'p''}(\mathbb{R}^d, \mathrm{d}x)} \|\varphi_n\|_{L^{p'q''}(\mathbb{R}^d, \mathrm{d}x)} \|\varphi_n\|_{L^{q'}(\mathbb{R}^d, \mathrm{d}x)} \|1 - \mathrm{e}^{-\phi_n}\|^2_{L^1(\mathbb{R}^d, \mathrm{d}x)} \to 0.$$

Then

$$d_W(\delta^{(n)}(\varphi_n), Z) \leq \sqrt{2/\pi} \sqrt{1 - 2c_1^{(n)} \|\varphi_n\|^2_{L^2(\mathbb{R}^d, \mathrm{d}x)} + z_n c_2^{(n)} \|\varphi_n\|^4_{L^2(\mathbb{R}^d, \mathrm{d}x)} + c_2^{(n)} A_n} \to 0.$$

Example

$\{z_n\}_{n\geq 1}$ sequence of positive numbers converging to $z>0$, ϕ_n, $n\geq 1$, mollifiers or Strauss with $r=n^{-1}$, and

$$\varphi_n(x) := \mathbf{1}_{K_n}(x)(z\ell(K_n))^{-1/2}, \quad n\geq 1, x\in\mathbb{R}^d$$

ℓ Lebesgue measure, $K_n\subset\mathbb{R}^d$ with $\ell(K_n)\to\infty$. We have the quantitative CLT

$$d_W(\delta^{(n)}(\varphi_n), Z) \leq \sqrt{2/\pi}\sqrt{1-2z^{-1}c_1^{(n)}+z^{-2}z_n c_2^{(n)}}+c_2^{(n)}A_n \to 0$$

where

$$c_1^{(n)} = z_n(1+\alpha z_n n^{-d})^{-1}, \qquad c_2^{(n)} = z_n(2-\mathrm{e}^{-\alpha z_n n^{-d}})^{-1}$$

and

$$\begin{aligned} A_n =& z^{-3/2}\,\ell(K_n)^{-\frac{1}{2}} + \alpha(\sqrt{2/\pi}z^{-1} + 2z^{-3/2}\,\ell(K_n)^{-\frac{1}{2}})z_n n^{-d} \\ &+ \alpha^2 z^{-3/2} z_n^2\,\ell(K_n)^{-\frac{1}{2}}\,n^{-2d}. \end{aligned}$$

Example (continued)

If e.g. $z_n = z$ and $\ell(K_n) \sim n^d$, elementary computations show that the bound is asymptotically equivalent to

$$\left(\frac{1}{\sqrt{z}} + \sqrt{\frac{2\alpha z}{\pi}} \right) n^{-d/2}$$

Proof of the explicit bound for s.i.f.r. Gibbs point processes

By the general bound for Gibbs point processes we have

$$
\begin{aligned}
&d_W(\delta(\varphi), Z) \\
&\leq \sqrt{2/\pi}\sqrt{1 - 2\mathrm{E}[\pi_0]\|\varphi\|^2_{L^2(\mathbb{R}^d,\mathrm{d}x)} + z\mathrm{E}[\pi_0]\|\varphi\|^4_{L^2(\mathbb{R}^d,\mathrm{d}x)}} \\
&\quad + \mathrm{E}[\pi_0]\|\varphi\|^3_{L^3(\mathbb{R}^d,\mathrm{d}x)} + z\sqrt{2/\pi}\mathrm{E}[\pi_0]\||\varphi|(|\varphi| * |1 - \mathrm{e}^{-\phi}|)\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\
&\quad + 2z\mathrm{E}[\pi_0]\||\varphi|^2(|\varphi| * |1 - \mathrm{e}^{-\phi}|)\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} \\
&\quad + z^2\mathrm{E}[\pi_0]\||\varphi|(|\varphi| * |1 - \mathrm{e}^{-\phi}|)^2\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}.
\end{aligned}
$$

Proof of the explicit bound for s.i.f.r. Gibbs point processes (continued)

We conclude the proof using Hölder's inequality,

$$\|f * g\|_{L^p(\mathbb{R}^d,\mathrm{d}x)} \leq \|f\|_{L^1(\mathbb{R}^d,\mathrm{d}x)}\|g\|_{L^p(\mathbb{R}^d,\mathrm{d}x)}$$

and that for s.i.f.r. Gibbs point processes one has the crucial estimates

$$c_1 \leq \mathrm{E}[\pi_0] \leq c_2,$$

Stucki and Schuhmacher (2014).

Integration by parts formulas for point processes

Peccati et al. (2010) used the IBP formula of the Malliavin calculus on the Poisson space due to Nualart and Vives (1990).

In Privault and T. (2013), for the purpose of probability approximation of Poisson functionals, we proposed an IBP formula based on the Clark-Ocone covariance representation.

I cite also the IBP formula on the Poisson space due to Albeverio, Kondratiev and Röckner (1998) (construction of diffusion) and the one in Privault and T. (2011), which holds for general finite point processes (density estimation). These latter two formulas involve "gradients" and "divergences" (very) different from those in Nualart and Vives (1990) and are difficult to use for probability approximation.

Proof of the general bound: the integration by parts formula

For all $F : \Gamma_X \to \mathbb{R}$ and $\varphi : X \to \mathbb{R}$ such that

$$\int_X |\varphi(x)| \mathrm{E}[\pi_x] \, \sigma(\mathrm{d}x) < \infty$$

and

$$\int_X |\varphi(x)| \mathrm{E}[|D_x F \pi_x|] \, \sigma(\mathrm{d}x) < \infty, \quad \int_X |\varphi(x)| \mathrm{E}[|F| \pi_x] \, \sigma(\mathrm{d}x) < \infty,$$

we have

$$\mathrm{E}\left[\int_X \varphi(x) D_x F \pi_x \, \sigma(\mathrm{d}x)\right] = \mathrm{E}[F \delta(\varphi)].$$

Proof of the general bound: the integration by parts formula (continued)

The following "sample path" identity holds

$$\delta_{\mathbf{x}}(\varphi DF) = F(\mathbf{x})\delta_{\mathbf{x}}(\varphi) - \delta_{\mathbf{x}}(\varphi F) - \int_X \varphi(x) D_x F(\mathbf{x}) \pi_x(\mathbf{x})\, \sigma(\mathrm{d}x),$$

where

$$\delta_{\mathbf{x}}(\varphi DF) := \sum_{x \in \mathbf{x}} \varphi(x) D_x F(\mathbf{x} \setminus \{x\}) - \int_X \varphi(x) D_x F(\mathbf{x}) \pi_x(\mathbf{x})\, \sigma(\mathrm{d}x)$$

and the term $\delta_{\mathbf{x}}(\varphi F)$ is defined similarly. The claim follows taking the mean.

Proof of the general bound: the Stein method

It is known that by the Stein equation, for any $F \in L^1(\Gamma_X, \mu)$, one has

$$d_W(F, Z) \le \sup_{f \in \mathcal{F}_W} |\mathrm{E}[f'(F) - Ff(F)]|,$$

where $\mathcal{F}_W$ denotes the class of twice differentiable functions f such that

$$\|f\|_\infty \le 2, \quad \|f'\|_\infty \le \sqrt{2/\pi}, \quad \|f''\|_\infty \le 2.$$

Proof of the general bound: Malliavin meets Stein

Take $F = \delta(\varphi)$, $f \in \mathcal{F}_W$. Following Peccati et al. (2010), we deduce

$$
\begin{aligned}
&|\mathrm{E}[f'(\delta(\varphi)) - \delta(\varphi) f(\delta(\varphi))]| \\
&= \left| \mathrm{E}\left[f'(\delta(\varphi)) - \int_X \varphi_x D_x f(\delta(\varphi)) \pi_x \, \sigma(\mathrm{d}x) \right] \right| \\
&= \left| \mathrm{E}\left[f'(\delta(\varphi)) - \int_X \varphi(x) \left(f'(\delta(\varphi)) D_x \delta(\varphi) + R(D_x \delta(\varphi)) \right) \pi_x \, \sigma(\mathrm{d}x) \right] \right| \\
&\leq \sqrt{2/\pi}\, \mathrm{E}\left[\left| 1 - \int_X \varphi(x) D_x \delta(\varphi) \pi_x \, \sigma(\mathrm{d}x) \right| \right] \\
&\qquad + \mathrm{E}\left[\int_X |\varphi(x)| |D_x \delta(\varphi)|^2 \pi_x \, \sigma(\mathrm{d}x) \right].
\end{aligned}
$$

Conclusion of the proof of the general bound

Taking the supremum over f, we have

$$d_W(\delta(\varphi), Z) \leq \sqrt{2/\pi}\,\mathrm{E}\left[\left|1 - \int_X \varphi(x) D_x\delta(\varphi)\pi_x\,\sigma(\mathrm{d}x)\right|\right] + \mathrm{E}\left[\int_X |\varphi(x)||D_x\delta(\varphi)|^2\pi_x\,\sigma(\mathrm{d}x)\right].$$

Total variation distance

Take $F : \Gamma_X \to \mathbb{N}$ and let $\mathrm{Po}(\lambda)$ be a Poisson random variable with mean $\lambda > 0$ defined on the probability space $(\Omega, \mathcal{F}, P)$. The total variation distance between F and $\mathrm{Po}(\lambda)$ is defined by

$$d_{TV}(F, \mathrm{Po}(\lambda)) := \sup_{A \subseteq \mathbb{N}} |\mu(F \in A) - P(\mathrm{Po}(\lambda) \in A)|.$$

Poisson approximation: general bound

Assume $\varphi : X \to \mathbb{N}$ such that

$$\int_X \varphi(x)\mathrm{E}[\pi_x]\,\sigma(\mathrm{d}x) < \infty$$

and set $N_{\mathbf{x}}(\varphi) := \sum_{x\in\mathbf{x}} \varphi(x)$. Then

$$d_{TV}(N(\varphi),\mathrm{Po}(\lambda)) \leq \frac{1-\mathrm{e}^{-\lambda}}{\lambda}\int_X \varphi(x)(\varphi(x)-1)\mathrm{E}[\pi_x]\,\sigma(\mathrm{d}x)$$

$$+\frac{1-\mathrm{e}^{-\lambda}}{\lambda^2}\int_X (\varphi(x))^2(\varphi(x)-1)\mathrm{E}[\pi_x]\,\sigma(\mathrm{d}x)$$

$$+\min\left(1,\sqrt{\frac{2}{\lambda\mathrm{e}}}\right)\sqrt{\int_{X^2}\varphi(x)\varphi(y)(\mathrm{E}[\pi_x\pi_y]-\mathrm{E}[\pi_x]\mathrm{E}[\pi_y])\,\sigma(\mathrm{d}x)\sigma(\mathrm{d}y)}$$

where

$$\lambda := \mathrm{E}[N(\varphi)] = \int_X \varphi(x)\mathrm{E}[\pi_x]\,\sigma(\mathrm{d}x).$$

Poisson case

If μ is a Poisson process with mean measure σ and $\varphi \in L^1(X, \sigma)$ and $\mathbb{N}$-valued, then

$$d_{TV}(N(\varphi), \mathrm{Po}(\lambda)) \leq \frac{1 - \mathrm{e}^{-\lambda}}{\lambda} \int_X \varphi(x)(\varphi(x) - 1)\, \sigma(\mathrm{d}x)$$
$$+ \frac{1 - \mathrm{e}^{-\lambda}}{\lambda^2} \int_X (\varphi(x))^2 (\varphi(x) - 1)\, \sigma(\mathrm{d}x),$$

where

$$\lambda := \int_X \varphi(x)\, \sigma(\mathrm{d}x),$$

Peccati (2011).

An explicit bound for s.i.f.r. Gibbs point processes

μ stationary Gibbs point process with $z > 0$ and $\phi \geq 0$ with compact support. Suppose $\varphi \in L^1(\mathbb{R}^d, \mathrm{d}x)$ and $\mathbb{N}$-valued. Then, for any $c \in [c_1, c_2]$,

$$d_{TV}(N(\varphi), \mathrm{Po}(c\|\varphi\|_1)) \leq B(1 - \mathrm{e}^{-\|\varphi\|_1 c_2}) + C\left(\frac{1 - \mathrm{e}^{-\|\varphi\|_1 c_2}}{c_1}\right)$$
$$+ \|\varphi\|_1 \min\left(1, \sqrt{\frac{2}{\|\varphi\|_1 c_1 \mathrm{e}}}\right)\sqrt{zc_2 - (c_1)^2} + \|\varphi\|_1(c_2 - c_1),$$

where $B := \|\varphi\|_1^{-1}\|\varphi(\varphi - 1)\|_1$, $C := \|\varphi\|_1^{-2}\|\varphi^2(\varphi - 1)\|_1$ and c_1, c_2 are as above.

A quantitative Poisson limit theorem for s.i.f.r. Gibbs point processes

Assume μ_n, $n \geq 1$, stationary Gibbs with $z_n > 0$ and $\phi_n \geq 0$ with compact support, $z_n \to z > 0$, $\|1 - \mathrm{e}^{-\phi_n}\|_{L^1(\mathbb{R}^d, \mathrm{d}x)} \to 0$. Assume $\{\varphi_n\}_{n \geq 1}$ $\mathbb{N}$-valued and integrable functions such that

$$\max\{\|\varphi_n\|_1, \|\varphi_n\|_2^2, \|\varphi_n\|_3^3\} \to \gamma \in (0, \infty).$$

Then

$$d_{TV}(N^{(n)}(\varphi_n), \mathrm{Po}(z\gamma)) \leq B_n(1 - \mathrm{e}^{-\|\varphi_n\|_1 c_2^{(n)}}) + C_n \left(\frac{1 - \mathrm{e}^{-\|\varphi_n\|_1 c_2^{(n)}}}{c_1^{(n)}} \right.$$
$$\left. + \|\varphi_n\|_1 \min\left(1, \sqrt{\frac{2}{c_1^{(n)} \|\varphi_n\|_1 \mathrm{e}}}\right) \sqrt{z_n c_2^{(n)} - (c_1^{(n)})^2} + \|\varphi_n\|_1 (c_2^{(n)} - c_1^{(n)}) \right)$$
$$+ \max\left\{ |z\gamma - c_1^{(n)}\|\varphi_n\|_1|, |z\gamma - c_2^{(n)}\|\varphi_n\|_1| \right\} \to 0.$$

Example

$\{z_n\}_{n\geq 1}$ sequence of positive numbers converging to $z>0$, ϕ_n, $n\geq 1$, mollifiers or Strauss with $r=n^{-1}$, and

$$\varphi_n(x) := \mathbf{1}_{K_n}(x), \quad n\geq 1,\, x\in\mathbb{R}^d$$

$K_n\subset\mathbb{R}^d$ with $\ell(K_n)\to\gamma>0$, ℓ Lebesgue measure. We have the quantitative Poisson Limit Theorem

$$d_{TV}(N^{(n)}(\mathbf{1}_{K_n}),\mathrm{Po}(z\gamma))\leq \ell(K_n)\min\left(1,\sqrt{\frac{2}{c_1^{(n)}\ell(K_n)\mathrm{e}}}\right)\sqrt{z_n c_2^{(n)}-(c_1^{(n)})^2}$$
$$+\ell(K_n)(c_2^{(n)}-c_1^{(n)})+\max\left\{|z\gamma-c_1^{(n)}\ell(K_n)|,|z\gamma-c_2^{(n)}\ell(K_n)|\right\}\to 0.$$

Example (continued)

If e.g. $z_n = z$ and $\ell(K_n) \sim \gamma + n^{-d}$, elementary computations show that the bound is asymptotically equivalent to

$$\gamma\sqrt{\alpha z^3}\min\left(1, \sqrt{\frac{2}{\gamma \mathrm{e} z}}\right) n^{-d/2}.$$

MANY, MANY THANKS!