On bounds in multivariate Poisson approximation

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Talk

Title

On bounds in multivariate Poisson approximation

Note: This talk is based on two our papers published in years 2013 and 2014.

1. T. L. Hung and V. T. Thao, (2013), Bounds for the Approximation of Poisson-binomial distribution by Poisson distribution, Journal of Inequalities and Applications, 2013:30.
2. T. L. Hung and L. T. Giang, (2104), On bounds in Poisson approximation for integer-valued independent random variables, Journal of Inequalities and Applications, 2014:291.

Outline

1. A few words about Poisson approximation.
2. Motivation (Multivariate Poisson approximation).
3. Mathematical tools (A linear oprator, A probability distance).
4. Results (Le Cam type bounds for sums and random sums of d-dimensional independent Bernoulli distributed random vectors)
5. Remarks
6. References

Bernoulli's formula

Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli random variables with success probability p, such that

$$P(X_j = 1) = p = 1 - P(X_j = 0), p \in (0,1)$$

Set $S_n = X_1 + \ldots + X_n -$ the number of successes. Then

Formula

$$P(S_n = k) = \frac{n!}{k!(n-k)!} \times p^k \times (1-p)^{n-k}; n \geq 1, k = 1, 2, \ldots, n. \tag{1}$$

Approximation formula

Suppose that the number n is large enough, and the success probability p is small, such that $n \times p = \lambda$. Then,

Formula

$$P(S_n = k) \approx P(Z_\lambda = k); k = 0, 1, 2, \ldots, . \qquad (2)$$

where $\lambda = E(S_n) = np = E(Z_\lambda)$ and $P(Z_\lambda = k) = \frac{e^{-\lambda}\lambda^k}{k!}$. We denote $Z_\lambda \sim Poi(\lambda)$ the Poisson random variable with mean λ.

Poisson limit theorem

Suppose that $n \to \infty, p \to 0$ such that $n \times p \to \lambda$. Then,

Formula

$$S_n \xrightarrow{d} Z_\lambda, \quad as \quad n \to \infty \tag{3}$$

where the notation $\xrightarrow{d}$ denotes the convergence in distribution.

Note:The Poisson limit theorem suggests that the distribution of a sum of independent Bernoulli random variables with small success probabilities (p) can be approximated by the Poisson distribution with the same mean (np), if the success probabilities (p) are small and the number (n) of random variables is large.

Le Cam's bound, (1960)

Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli distributed random variables such that

$$P(X_j = 1) = p_j = 1 - P(X_j = 0) = 1 - p_j; p_j \in (0, 1).$$

Let $S_n = X_1 + \ldots + X_n$ and $\lambda_n = E(S_n) = \sum_{j=1}^{n} p_j$. Then,

Le Cam's result

$$\sum_{k=0}^{\infty} | P(S_n = k) - P(Z_{\lambda_n} = k | \leq 2 \sum_{j=1}^{n} p_j^2. \tag{4}$$

Chen's bound, (1975)

Chen's result

$$\sum_{k=0}^{\infty} \mid P(S_n = k) - P(Z_{\lambda_n} = k \mid \leq \frac{2\left(1 - e^{-\lambda_n}\right)}{\lambda_n} \sum_{j=1}^{n} p_j^2 \qquad (5)$$

Mathematical tools

- Operator theoretic method (Le Cam, (1960))
- Stein-Chen method:
  1. the local approach (Chen (1975), Arratia, Goldstein and Gordon (1989, 1990)
  2. the size-bias coupling approach (Barbour((1982), Barbour, Holst and Janson (1992), Chatterjee, Diaconis and Meckes (2005))

Total variation distance

A measure of the accuracy of the approximation is the total variation distance. For two distributions P and Q over $Z_+ = \{0, 1, 2, ...\}$, the total variation distance between them is defined by

$$\boxed{d_{TV}(P,Q) = \sup_{A \subset Z_+} \mid P(A) - Q(A) \mid}$$

which is also equal to

$$\frac{1}{2} \sup_{|h|=1} \mid \int hdP - \int hdQ \mid = \frac{1}{2} \sum_{i \in Z_+} \mid P\{i\} - Q\{i\} \mid .$$

Well-known results

- For the binomial distribution $Bin_n(p)$, Prohorov ((1953)) proved that

$$d_{TV}(Bin_n(p), Poi(np)) \leq p \times \left[\frac{1}{\sqrt{2\pi e}} + O(\inf(1, \frac{1}{\sqrt{n}}) + p\right] \tag{6}$$

- Using the method of convolution operators, Le Cam ((1960)) obtained the error bounds

$$d_{TV}(\mathcal{L}(S_n), Poi(\lambda_n)) \leq 2\sum_{j=1}^{n} p_j^2 \tag{7}$$

- and, if $\max\limits_{1\leq j\leq n} p_j \leq \frac{1}{4}$,

$$d_{TV}(\mathcal{L}(S_n), Poi(\lambda_n)) \leq \frac{8}{\lambda_n}\sum_{j=1}^{n} p_j^2 \tag{8}$$

Motivation

A. Renyi, Probability Theory, ((1970))

Poisson approximation Theorem: Let $X_{n1}, X_{n2}, \ldots$ be a sequence of independent random variables which assume only the values 0 and 1, with

$$P(X_{nj}=1)=p_{nj}=1-P(X_{nj}=0).$$

Put $\lambda_n=\sum_{j=1}^{n} p_{nj}$ and suppose that

$$\lim_{n\to\infty}\lambda_n=\lambda \tag{9}$$

and

$$\lim_{n\to\infty}\max_{1\le j\le n} p_{nj}=0. \tag{10}$$

Then, $X_{n1}+\ldots+X_{nn}\xrightarrow{d} Z_\lambda$ as $n\to\infty$.

Linear Operator

- Let K denote the set of all real-valued bounded functions $f(x)$, defined on the nonnegative integers $Z_+ = \{0, 1, 2, \ldots\}$. Put $\| f \| = \sup | f(x) |$.
- Let there be associated with every probability distribution $\mathcal{P} = \{p_0, p_1, \ldots\}$ an operator defined by

$$\boxed{A_{\mathcal{P}} f = \sum_{r=0}^{\infty} f(x+r) p_r,} \tag{11}$$

for every $f \in K$.

Properties

- Operator $A_{\mathcal{P}}$ maps the set K into itself.
- $A_{\mathcal{P}}$ is a linear contraction operator
- If $\mathcal{P}$ and $\mathcal{Q}$ are any two distributions defined on the nonnegative integers, then $A_{\mathcal{P}}A_{\mathcal{Q}} = A_{\mathcal{R}}$, where $\mathcal{R}$ is the convolution of the distributions $\mathcal{P}$ and $\mathcal{Q}$.
- The sufficient condition for proof of Poisson approximation Theorem is

$$\lim_{n\to\infty} \| A_{\mathcal{P}_{S_n}} - A_{\mathcal{P}_{Z_\lambda}} \| = 0 \tag{12}$$

for $f \in K, S_n = \sum_{j=1}^{n} X_{nj}, \lambda_n = E(S_n), \lambda = \lim_{n\to\infty} \lambda_n, Z_\lambda \sim Poi(\lambda)$.

Le Cam's type bounds in Poisson approximation via operator introduced by A. Renyi

Hung T. L. and Thao V. T., Journal of Inequalities and Applications, ((2013))

Theorem 1. Let $X_{n1}, X_{n2}, \dots$ be a row-wise triangular array of independent, Bernoulli random variables with success probabilities $P(X_{nj} = 1) = 1 - P(X_{nj} = 0) = p_{nj}, p_{nj} \in (0,1), j = 1, 2, ..., n; n = 1, 2, ...$ Let us write $S_n = \sum_{j=1}^{n} X_{nj}, \lambda_n = E(S_n)$. We denote by Z_{λ_n} the Poisson random variable with the parameter λ_n. Then, for all real-valued bounded functions $f \in K$, we have

$$\boxed{\| A_{\mathcal{P}_{S_n}} f - A_{\mathcal{P}_{Z_{\lambda_n}}} \| \leq 2 \| f \| \sum_{j=1}^{n} p_{nj}^2.} \qquad (13)$$

Le Cam's type bounds in Poisson approximation for Random sums

Theorem 2. Let $X_{n1}, X_{n2}, \dots$ be a row-wise triangular array of independent, Bernoulli random variables. Let $\{N_n, n \geq 1\}$ be a sequence of non-negative integer-valued random variables independent of all X_{nj}. Let us write $S_{N_n} = \sum_{j=1}^{N_n} X_{nj}, \lambda_{N_n} = E(S_{N_n})$. Then, for all real-valued bounded functions $f \in K$, we have

$$\boxed{\| A_{\mathcal{P}_{S_{N_n}}} f - A_{\mathcal{P}_{Z_{\lambda_{N_n}}}} \| \leq 2 \| f \| \times E\left(\sum_{j=1}^{N_n} p_{N_n j}^2\right).} \qquad (14)$$

Le Cam's type bounds in Poisson approximation for sum of independent integer-valued random variables

Hung T. L. and Giang L. T., Journal of Inequalities and Applications, ((2014))

Theorem 3. Let $X_{n1}, X_{n2}, \ldots$ be a row-wise triangular array of independent, integer-valued random variables with success probabilities

$P(X_{nj} = 1) = p_{nj}, P(X_{nj} = 0) = 1 - p_{nj} - q_{nj}, p_{nj}, q_{nj} \in (0,1), p_{nj} + q_{nj} \in (0,1), j = 1, 2, ..., n; n = 1, 2, ...$ Then, for all real-valued bounded functions $f \in K$, we have

$$\boxed{\| A_{\mathcal{P}_{S_n}} f - A_{\mathcal{P}_{Z_{\lambda_n}}} \| \leq 2 \| f \| \sum_{j=1}^{n} (p_{nj}^2 + q_{nj}).} \qquad (15)$$

Le Cam's type bounds in Poisson approximation for random sum of independent integer-valued random variables

Theorem 4. Let $X_{n1}, X_{n2}, \ldots$ be a row-wise triangular array of independent, integer-valued random variables with success probabilities
$P(X_{nj} = 1) = p_{nj}, P(X_{nj} = 0) = 1 - p_{nj} - q_{nj}, p_{nj}, q_{nj} \in (0, 1), p_{nj} + q_{nj} \in (0, 1), j = 1, 2, ..., n; n = 1, 2, ...$ Let $\{N_n, n \geq 1\}$ be a sequence of non-negative integer-valued random variables independent of all X_{nj}. Then, for all real-valued bounded functions $f \in K$, we have

$$\boxed{\| A_{\mathcal{P}_{S_{N_n}}} f - A_{\mathcal{P}_{Z_{\lambda_{N_n}}}} \| \leq 2 \| f \| E\left(\sum_{j=1}^{N_n} (p_{N_nj}^2 + q_{N_nj})\right).} \quad (16)$$

Multivariate Poisson approximation

- Consider independent Bernoulli random d-vectors, $X_1, X_2, \ldots$ with

$$P(X_j = e^{(i)}) = p_{j,i}, P(X_j = 0) = 1 - p_j, 1 \leq i \leq d, 1 \leq j \leq n,$$

where $e^{(i)}$ denotes the ith coordinate vector in $\mathbb{R}^d$ and $p_j = \sum_{1 \leq i \leq d} p_{j,i}$.

- Let $S_n = \sum_{j=1}^{n} X_j, \lambda_n = \sum_{j=1}^{n} p_j$.

Multivariate Poisson approximation

McDonald (1980): for the independent Bernoulli d- random vectors

$$d_{TV}(\mathcal{P}_{S_n}, \mathcal{P}_{Z_{\lambda_n}}) := \sup_{A \in \mathcal{Z}_+^d} | P(S_n \in A) - P(Z_{\lambda_n} \in A) | \\ \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{d} p_{j,i} \right)^2 \tag{17}$$

for $S_n = X_1 + X_2 + \cdots + X_n$, and
$\lambda_n = E(S_n) = (\lambda_1, \dots, \lambda_d), Z_{\lambda_n} \sim Poi(\lambda_n)$,

Multivariate Poisson approximation

Using the Stein-Chen method, Barbour (1988) showed that

$$d_{TV}(\mathcal{P}_{S_n}, \mathcal{P}_Z) \\ \leq \sum_{j=1}^{n} \min\left\{ \left(\frac{c_{\lambda_n}}{\lambda_n} \sum_{i=1}^{n} \frac{p_{j,i}^2}{\mu_i} \right), \left(\sum_{i=1}^{d} p_{j,i} \right)^2 \right\} \tag{18}$$

where $c_{\lambda_n} = \frac{1}{2} + \log^+(2\lambda_n), \mu_i = \lambda^{-1} \sum\limits_{j=1}^{n} p_{j,i}, Z = (Z_1, \ldots, Z_d)$, and $Z_1, \ldots, Z_d$ are independent Poisson random variables with means $\lambda\mu_1, \ldots, \lambda\mu_d$. This bound is sharper than the bound in McDonald (1980).

Multivariate Poisson approximation

Using the multivariate adaption of Kerstan's generating function method, Roos (1999) proved that

$$d_{TV}(\mathcal{P}_{S_n}, \mathcal{P}_Z) \leq 8.8 \sum_{j=1}^{n} \min\left\{p_j 2, \left(\frac{1}{\lambda_n}\sum_{i=1}^{d}\frac{p_{j,i}^2}{\mu_i}\right)\right\} \qquad (19)$$

which improved over (18)in removing c_{λ_n} from the error bound although the absolute constant is increased to 8.8

Operator for d-dimensional random vector

Definition

Let X be a d-dinensional random vector with probability distribution $\mathcal{P}_X$. A linear operator $A_{\mathcal{P}_X} : K \to K$ such that

$$\left(A_{\mathcal{P}_X} f\right)(x) := \sum_{m \in \mathcal{Z}_+^d} f(x+m) P(X=m) \tag{20}$$

where $f \in K$, the class of all real-valued bounded functions f on the set of all non-negative integers $\mathcal{Z}_+^d$ and $x = (x_1, \ldots, x_d), m = (m_1, \ldots, m_d) \in \mathcal{Z}_+^d$. The norm of function f is $\| f \| = \sup_{x \in \mathcal{Z}_+^d} | f(x) |$

Properties

1. Operator $A_{\mathcal{P}_X}$ is a contraction operator
$$\| A_{\mathcal{P}_X} f \| \leq \| f \|$$
2. For the convolution of two distributions:
$$A_{\mathcal{P}_X * \mathcal{P}_Y} f = A_{\mathcal{P}_X} \times (A_{\mathcal{P}_Y} f) = A_{\mathcal{P}_Y} \times (A_{\mathcal{P}_X} f)$$
3. Let $\| A_{\mathcal{P}_{X_n}} f - A_{\mathcal{P}_X} f \| = o(1)$ as $n \to \infty$. Then $X_n \xrightarrow{d} X$ as $n \to \infty$.

Inequalities

1. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be two sequences of independent d-random vectors in each group (and they are independent). Then, for $f \in K$,

$$\| A_{\sum\limits_{j=1}^{n} \mathcal{P}_{X_j}} f - A_{\sum\limits_{j=1}^{n} \mathcal{P}_{Y_j}} f \| \leq \sum_{j=1}^{n} \| A_{\mathcal{P}_{X_j}} f - A_{\mathcal{P}_{Y_j}} f \|$$

2. Let $N_n, n \geq 1$ be a sequence of non-negative integer random variables, independent from all X_j and Y_j. Then

$$\| A_{\sum\limits_{j=1}^{N_n} \mathcal{P}_{X_j}} f - A_{\sum\limits_{j=1}^{N_n} \mathcal{P}_{Y_j}} f \| \leq \sum_{k=1}^{\infty} P(N_n = k) \sum_{j=1}^{k} \| A_{\mathcal{P}_{Y_j}} f - A_{\mathcal{P}_{Y_j}} f \|$$

Inequalities for iid. d-random vectors

1. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sequences of iid. d-random vectors in each group (and they are independent). Then, for $f \in K$,

$$\| A_{\sum\limits_{j=1}^{n} \mathcal{P}_{X_j}} f - A_{\sum\limits_{j=1}^{n} \mathcal{P}_{Y_j}} f \| \leq n \| A_{\mathcal{P}_{X_1}} f - A_{\mathcal{P}_{Y_1}} f \|$$

2. Let $\{N_n, n \geq 1\}$ be a sequence of non-negative integer random variables, independent from all iid. X_j and Y_j. Then

$$\| A_{\sum\limits_{j=1}^{N_n} \mathcal{P}_{X_j}} f - A_{\sum\limits_{j=1}^{N_n} \mathcal{P}_{Y_j}} f \| \leq E(N_n) \| A_{\mathcal{P}_{Y_1}} f - A_{\mathcal{P}_{Y_1}} f \|$$

Notations

Denote by $\mathcal{P}_{Z^d_{\lambda_n}}$ by the distribution of Poisson d-random vector $Z^d_{\lambda_n}$, where $\lambda_n = (\lambda_1, \ldots, \lambda_d)$ and

$$P(Z_{\lambda_n} = m) = \prod_{j=1}^{d} \left(\frac{e^{-\lambda_j} \lambda_j^{m_j}}{m_j!} \right),$$

where $m = (m_1, \ldots, \lambda_d) \in \mathcal{Z}^d_+$.

Notations

Denote $\mathcal{P}_{S_n}$ by the distribution of sum $S_n^d = X_1^d + \ldots + X_n^d$, where $X_1^d, \ldots, X_n^d$ are independent Bernoulii d-random vectors with success probabilities

$$P(X_k^d = e_j) = p_{j,k} \in [0,1], k = 1, 2, \ldots, n; j = 1, 2, \ldots d.$$

and

$$P(X_k^d = (0, \ldots, 0)) = 1 - \sum_{j=1}^{d} p_{j,k} \in [0,1]$$

Here $e_j \in \mathcal{R}^d$ denotes the vector with entry 1 at position j and entry 0 otherwise.

Le Cam type bound in multivariate Poisson approximation

Theorem 5. For the independent Bernoulli distributed d-dimensional random vectors. For $f \in K$,

$$\boxed{\| A_{\mathcal{P}_{S_n}} f - A_{\mathcal{P}_{Z_{\lambda_n}}} \| \leq 2 \sum_{j=1}^{n} \left(\sum_{i=1}^{d} p_{j,i} \right)^2} \qquad (21)$$

Le Cam type bound in multivariate Poisson approximation for random sum

Theorem 6. For the independent Bernoulli distributed d-dimensional random vectors. For $f \in K$, For $\{N_n, n \geq 1\}$ the sequence of non-negative integer-value random variables independent from all $X_1, X_2, \ldots, Z_{\lambda_1}, Z_{\lambda_2}, \ldots$

$$\boxed{\| A_{\mathcal{P}_{S_{N_n}}} f - A_{\mathcal{P}_{Z_{\lambda_{N_n}}}} \| \leq 2E\left(\sum_{j=1}^{N_n}\left(\sum_{i=1}^{d} p_{j,i}\right)^2\right)} \qquad (22)$$

Concluding Remarks

1. The operator method is elementary and elegant, but efficient in use.
2. This operator method can be applicable to a wide class of discrete d-dimensional random vectors as geometric, hypergeometric, negative binomial, etc.
3. This operator theoretic method can be applicaable to compound Poisson approximation and Poisson process approximation problems.
4. Based on this operator we can define a probability distance having a role in stydy of Poisson approximation problems.

Probability distance based on operator $A_{\mathcal{P}_X}$

A probability distance for two probability distributions $\mathcal{P}_X$ and $\mathcal{P}_Y$ is defined by

Definition

$$d(\mathcal{P}_X, \mathcal{P}_Y, f) := \| A_{\mathcal{P}_X} f - A_{\mathcal{P}_Y} \| = \sup_{x \in \mathbb{Z}_+^d} | Ef(X + x) - Ef(Y + x) | \tag{23}$$

for $f \in K$.

Properties

1. Distance $d(\mathcal{P}_X, \mathcal{P}_Y, f)$ is a probability distance, i.e.
   - $d(\mathcal{P}_X, \mathcal{P}_Y, f) \geq 0$.
   - $d(\mathcal{P}_X, \mathcal{P}_Y, f) = 0$, if $P(X = Y) = 1$.
   - $d(\mathcal{P}_X, \mathcal{P}_Y, f) = d(\mathcal{P}_Y, \mathcal{P}_X, f)$
   - $d(\mathcal{P}_X, \mathcal{P}_Z, f) \leq d(\mathcal{P}_X, \mathcal{P}_Y, f) + d(\mathcal{P}_Y, \mathcal{P}_Z, f)$.
2. Convergence $d(\mathcal{P}_{X_n}, \mathcal{P}_X, f) \to 0$ as $n \to \infty$ implies convergence in distribution for $X_n \xrightarrow{d} X$, as $n \to \infty$.

Properties

1. Suppose that $X_1, X_2, \dots X_n; Y_1, Y_2, \dots Y_n$ are d-dimensional independent random variables (in each group). Then, for every $f \in \mathbb{K}$,

$$d\left(\mathcal{P}_{\sum_{j=1}^{n} X_j}, \mathcal{P}_{\sum_{j=1}^{n} Y_j}; f\right) \le \sum_{j=1}^{n} d(\mathcal{P}_{X_j}, \mathcal{P}_{Y_j}; f). \qquad (24)$$

2. Moreover, let $\{N_n, n \ge 1\}$ be a sequence of positive integer-valued random variables that independent of $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$. Then, for every $f \in \mathbb{K}$,

$$d\left(\mathcal{P}_{\sum_{j=1}^{N_n} X_j}, \mathcal{P}_{\sum_{j=1}^{N_n} Y_j}; f\right) \le \sum_{k=1}^{\infty} P(N_n = k) \sum_{j=1}^{k} d(\mathcal{P}_{X_j}, \mathcal{P}_{Y_j}; f). \qquad (25)$$

Comparision with total variation distance

Use the simple inequality (A. D. Barbour), for every $f \in K$,

$$\boxed{d(\mathcal{P}_X, \mathcal{P}_Y, f) \leq 2 \parallel f \parallel d_{TV}(\mathcal{P}_X, \mathcal{P}_Y)}$$

Thus, we can apply the probability distance $d(\mathcal{P}_X, \mathcal{P}_Y, f)$ to some problems related to Poisson approximation.

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Thanks

Thanks for your attention!