Stein's Method and Convex Hulls

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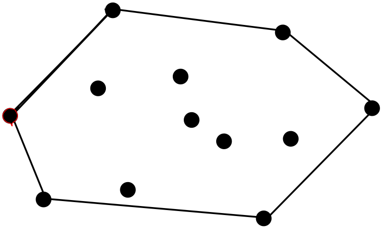
New Directions in Stein's Method, Singapore, May 2015

Introduction

convex hull of random points



extreme points = vertices

We assume points are i.i.d. uniform on a convex body K.

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$\mathrm{Vol}(K_n) =$ volume of K_n.

Expectation asymptotics

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Investigation has focussed on behavior of $f_l(K_n)$, $\ell \in \{0, ..., d-1\}$, as input size $n \to \infty$.

Behavior of $f_l(K_n)$ is sensitive to geometry of the boundary of K.

Expectation asymptotics ($d \geq 2, \ell \in \{0, 1, ..., d-1\}$)

K_n is convex hull of n i.i.d. uniform r.v. on K.

- Reitzner (2005) ∂K of class C^2, κ:= Gaussian curvature

$$\lim_{n \to \infty} n^{-\frac{d-1}{d+1}} \mathbb{E}\, f_\ell(K_n) = c_{d,\ell} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} dx;$$

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(flag is a maximal chain of faces, each a sub-face of the next in the chain)

- K_n is convex hull of n i.i.d. standard normal r.v. on $\mathbb{R}^d$:

$$\lim_{n \to \infty} (\log n)^{-\frac{d-1}{2}} \mathbb{E}\, f_\ell(K_n) = g_{d,\ell}.$$

Auffentranger + Schneider ('92), Baryshnikov + Vitale ('94).

Central limit theorems via Stein's method

$\ell \in \{0, ..., d-1\}, d \geq 2$, then

$$\sup_{t \in \mathbb{R}} \left| P\left[\frac{f_\ell(K_n) - \mathbb{E}\, f_\ell(K_n)}{\sqrt{\mathrm{Var}\, f_\ell(K_n)}} \leq t \right] - \Phi(t) \right| \leq c(K)\epsilon(n) = o(1).$$

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- Gaussian input: Bárány and Vu (2007)
- K convex: Bárány and Reitzner (2008)
- CLT for $\mathrm{Vol}(K_n)$.

Normal approximation under local dependence

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This yields nearly optimal rates of normal convergence for $\sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda)$.

In some cases local dependence gives optimal rates: Last, Pecatti, and Schulte (2014).

Questions

$\mathcal{P}_\lambda$ intensity λ PPP; let K_λ be convex hull of $P_\lambda \cap K$.

Can one write $f_\ell(K_\lambda)$ as a sum $\sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda)$ of stabilizing scores yielding a CLT via Chen-Shao methods and also yielding:

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We answer these questions positively when:

(i) the input is uniform on K, where either ∂K is of class C^2, or K is convex polytope, or

(ii) input consists of i.i.d. standard normal r.v. on $\mathbb{R}^d$, $d \geq 2$.

Normal approximation in the re-scaled picture

$\mathcal{X}$: locally finite set in upper half-space $\mathbb{R}^{d-1} \times \mathbb{R}^+$.

$$\xi_{\mathrm{Par}}(x, \mathcal{X}) := \begin{cases} 1 \text{ if } x \text{ is extreme in } \mathcal{X} \text{ wrt parabolas} \\ 0 \text{ otherwise.} \end{cases}$$

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· $\xi_{\text{Par}}(x, \mathcal{P}^{(\lambda)})$ is locally defined: its value is determined by input on a cylinder centered at x with radius having an exponentially decaying tail (non-trivial).

· Thus by the results of either (i) Penrose and Y. ('05) or (ii) Barbour and Xia ('06), one deduces rates of normal convergence for the total number of extreme points ... and the total number of ℓ faces in K_λ.

Main results (w. Pierre Calka): K = unit ball in $\mathbb{R}^d$

Thm (CLT) Let K be the unit ball, $\ell \in \{0, ..., d-1\}$, then

$$\sup_{t \in \mathbb{R}} \left| P\left[\frac{f_\ell(K_\lambda) - \mathbb{E}\, f_\ell(K_\lambda)}{\lambda^{(d-1)/2(d+1)}} \le t \right] - P[N(0, \sigma_\ell^2) \le t] \right| \le \epsilon(\lambda) = o(1),$$

where $\sigma_\ell^2 = \lim_{\lambda \to \infty} \lambda^{-\frac{d-1}{d+1}} \mathrm{Var} f_\ell(K_\lambda)$.

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Thm (scaling limit for boundary) For all $L \in (0, \infty)$, the interface $T^{(\lambda)}(\partial K_\lambda)$ converges in law as $\lambda \to \infty$ to parabolic festoon on $\mathcal{H}$ in $\mathcal{C}([-L, L])$ equipped with the sup norm.

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$\mathcal{H}$: rate one PPP in upper half-space.

$$\xi(x,\mathcal{H}) := \xi_{\mathrm{Par}}(x,\mathcal{H}) := \begin{cases} 1 \text{ if } x \text{ is extreme in } \mathcal{H} \quad \text{wrt} \quad \text{parabolas} \\ 0 \text{ otherwise.} \end{cases}$$

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$$c^{\xi}(w_1,w_2) :=$$
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and

$$V_{0,d} := \int_{-\infty}^{\infty} \mathbb{E}\,\xi((\mathbf{0},h),\mathcal{H})dh$$
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Thm (functional CLT, $d = 2$) Define integrated defect volume

$$W_\lambda(\rho) := \int_0^\rho (1 - \partial K_\lambda(\theta)) d\theta, \ \ \rho \in [0, 2\pi].$$

Then after centering and scaling, as $\lambda \to \infty$, $W_\lambda(\rho)$ converges in law to a Brownian motion in the space $\mathcal{C}([0, 2\pi])$.

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$$\lim_{\lambda \to \infty} (\log \lambda)^{-(d-1)} \mathrm{Var} f_0(K_\lambda) = c(d) \text{ (number of vertices of K) } V_{0,d},$$

where $c(d)$ is explicit constant depending on d.

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(iv) re-scaled picture yields variance asymptotics and scaling limits.

Scaling limits for Gaussian polytopes

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Define scaling transform $T^{(n)} : \mathbb{R}^d \to \mathbb{R}^{d-1} \times \mathbb{R}$

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THANK YOU