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# A second order Poincaré inequality for functionals of general Poisson processes

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# 1. Fock space representation of Poisson functionals

## Setting

$\eta$ is a Poisson process on some measurable space $(\mathbb{X}, \mathcal{X})$ with intensity measure $\lambda$. This is a random element in the space **N** of all integer-valued $\sigma$-finite measures on $\mathbb{X}$, equipped with the usual $\sigma$-field with the following two properties:

- The random variables $\eta(B_1), \dots, \eta(B_m)$ are stochastically independent whenever $B_1, \dots, B_m$ are measurable and pairwise disjoint.
- $$\mathbb{P}(\eta(B) = k) = \frac{\lambda(B)^k}{k!} \exp[-\lambda(B)], \quad k \in \mathbb{N}_0, B \in \mathcal{X},$$

  where $\infty^k e^{-\infty} := 0$ for all $k \in \mathbb{N}_0$.

## Definition (Difference operator)

For a measurable function $f : \mathbf{N} \to \mathbb{R}$ and $x \in \mathbb{X}$ we define a function $D_x f : \mathbf{N} \to \mathbb{R}$ by

$$D_x f(\mu) := f(\mu + \delta_x) - f(\mu).$$

For $x_1, \dots, x_n \in \mathbb{X}$ we define $D^n_{x_1,\dots,x_n} f : \mathbf{N} \to \mathbb{R}$ inductively by

$$D^n_{x_1,\dots,x_n} f := D^1_{x_1} D^{n-1}_{x_2,\dots,x_n} f,$$

where $D^1 := D$ and $D^0 f = f$.

## Definition

Let $L^2_\eta$ denote the space of all random variables $F \in L^2(\mathbb{P})$ such that $F = f(\eta)$ $\mathbb{P}$-almost surely, for some measurable function (representative) $f : \mathbf{N} \to \mathbb{R}$. In this case we define

$$D^n_{x_1,\dots,x_n} F := (D^n_{x_1,\dots,x_n} f)(\eta), \quad x_1, \dots, x_n \in \mathbb{X}.$$

## Theorem (Ito, Y. '88, L. and Penrose '11)

*For any $F, G \in L^2_\eta$,*

$$\mathbb{C}\mathrm{ov}(F, G) = \sum_{n=1}^{\infty} \frac{1}{n!} \int (\mathbb{E} D^n_{x_1,\dots,x_n} F)(\mathbb{E} D^n_{x_1,\dots,x_n} G) \lambda^n(d(x_1, \dots, x_n)).$$

# 2. The Poincaré inequality

**Theorem (Chen '85; Wu '00; L. and Penrose '11)**

*For any $F \in L^2_\eta$,*

$$\mathbb{V}\mathrm{ar}[F] \le \mathbb{E} \int (D_x F)^2 \, \lambda(dx).$$

*Equality holds iff $F$ is a linear function of $\eta$.*

**Theorem**

*For any $F \in L^2_\eta$,*

$$\mathbb{V}\mathrm{ar}[F] \ge \int (\mathbb{E} D_x F)^2 \, \lambda(dx).$$

# 3. A perturbation formula

**Theorem (Molchanov and Zuyev '00, L. '14)**

*Let $\lambda$ be a $\sigma$-finite measure and $\nu$ be a finite signed measure such that $\lambda + \nu$ is a measure. Let $\eta_\lambda$ and $\eta_{\lambda+\nu}$ be Poisson processes with intensity measure $\lambda$ and $\lambda + \nu$, respectively. Suppose that $f : \mathbf{N} \to \mathbb{R}$ is measurable and satisfies a suitable integrability assumption. Then*

$$\mathbb{E} f(\eta_{\lambda+\nu}) = \mathbb{E} f(\eta_\lambda) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathbb{E} D^n_{x_1,\dots,x_n} f(\eta_\lambda)\, \nu^n(d(x_1,\dots,x_n)).$$

# 4. A covariance identity

**Definition**

For $F \in L^2_\eta$ with representative $f$ we define,

$$P_s F := \int \mathbb{E}[f(\eta^{(s)} + \chi) \mid \eta]\Pi_{(1-s)}(d\chi), \quad s \in [0,1],$$

where $\eta^{(s)}$ is a $s$-thinning of $\eta$ and $\Pi_{(1-s)}$ is the distribution of a Poisson process with intensity measure $(1-s)\lambda$.

**Theorem**

*For any $F, G \in L^2_\eta$ such that $DF, DG \in L^2(\mathbb{P} \otimes \lambda)$,*

$$\mathbb{C}\mathrm{ov}(F, G) = \mathbb{E} \iint_0^1 (D_x F)(P_t D_x G)\, dt\, \lambda(dx).$$

# 5. The Stein-Malliavin method

## Definition

The Wasserstein distance between the laws of two random variables $X$, $Y$ is defined as

$$d_1(X,Y) = \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}h(X) - \mathbb{E}h(Y)|.$$

## Theorem (Peccati, Solé, Taqqu and Utzet '10)

*Suppose that $F \in L^2_\eta$ satisfies $\mathbb{E}F = 0$ and $DF \in L^2(\mathbb{P} \otimes \lambda)$. Let $N$ be standard normal. Then*

$$\begin{aligned} d_1(F,N) \le \mathbb{E}\Big|1 - \iint_0^1 (P_t D_x F)(D_x F)\,dt\,\lambda(dx)\Big| \\ + \mathbb{E}\iint_0^1 |P_t D_x F|(D_x F)^2\,dt\,\lambda(dx). \end{aligned}$$

## Idea of the proof:

- Use Stein's method:

$$d_1(F,N) \le \sup_{g \in \mathbf{AC}_{1,2}} |\mathbb{E}[g'(F) - Fg(F)]|.$$

- Use the covariance identity

$$\mathbb{E}Fg(F) = \mathbb{E}\iint_0^1 (P_t D_x F)(D_x g(F))\, dt\, \lambda(dx).$$

- Use

$$D_x g(F) = g(F + D_x F) - g(F)$$

and the properties of $g$.

## Remark

Schulte '12 and Eichelsbacher and Thäle '13 derived a similar (but more complicated bound) for the Kolmogorov distance

$$d_K(F,N) = \sup_{x\in\mathbb{R}} |\mathbb{P}(F \le x) - \mathbb{P}(N \le x)|$$

between the laws of $F$ and $N$.

# 6. The second order Poincaré inequality

**Theorem (L., Peccati and Schulte '14)**

*Let $F \in L^2_\eta$ be such $DF \in L^2(\mathbb{P} \otimes \lambda)$, $\mathbb{E}F = 0$ and $\mathbb{V}\mathrm{ar}\, F = 1$, and let $N$ be standard normal. Then,*

$$d_1(F, N) \le \gamma_1 + \gamma_2 + \gamma_3,$$

*where*

$$\gamma_1^2 := 4 \int \left[\mathbb{E}(D_{x_1}F)^2(D_{x_2}F)^2\right]^{1/2} \left[\mathbb{E}(D^2_{x_1,x_3}F)^2(D^2_{x_2,x_3}F)^2\right]^{1/2} d\lambda^3,$$

$$\gamma_2^2 := \int \mathbb{E}(D^2_{x_1,x_3}F)^2(D^2_{x_2,x_3}F)^2 \, d\lambda^3,$$

$$\gamma_3 := \int \mathbb{E}|D_x F|^3 \, \lambda(dx).$$

## Theorem (L., Peccati and Schulte '14)

*Let $F \in L^2_\eta$ be such that $DF \in L^2(\mathbb{P} \otimes \lambda)$, $\mathbb{E}F = 0$ and $\mathbb{V}\mathrm{ar}\, F = 1$, and let $N$ be standard normal. Then,*

$$d_K(F, N) \le \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6,$$

*where*

$$\gamma_4 := \frac{1}{2}\left[\mathbb{E}F^4\right]^{1/4} \int \left[\mathbb{E}(D_x F)^4\right]^{3/4} \lambda(dx),$$

$$\gamma_5^2 := \int \mathbb{E}(D_x F)^4 \lambda(dx),$$

$$\gamma_6^2 := \int 6\left[\mathbb{E}(D_{x_1} F)^4\right]^{1/2}\left[\mathbb{E}(D^2_{x_1,x_2} F)^4\right]^{1/2} \lambda^2(d(x_1, x_2))$$
$$+ \int 3\mathbb{E}(D^2_{x_1,x_2} F)^4 \lambda^2(d(x_1, x_2)).$$

## Theorem (L., Peccati and Schulte '14)

*For* $F, G \in \operatorname{dom} D$ *with* $\mathbb{E}F = \mathbb{E}G = 0$, *we have*

$$
\begin{aligned}
&\mathbb{E}\left( \mathbb{C}\mathrm{ov}(F,G) - \iint_0^1 (D_x F)(P_t D_x G)\, dt\, \lambda(dx) \right)^2 \\
&\le 3 \int \left[ \mathbb{E}(D^2_{x_1,x_3} F)^2 (D^2_{x_2,x_3} F)^2 \right]^{1/2} \left[ \mathbb{E}(D_{x_1} G)^2 (D_{x_2} G)^2 \right]^{1/2} d\lambda^3 \\
&\quad + \int \left[ \mathbb{E}(D_{x_1} F)^2 (D_{x_2} F)^2 \right]^{1/2} \left[ \mathbb{E}(D^2_{x_1,x_3} G)^2 (D^2_{x_2,x_3} G)^2 \right]^{1/2} d\lambda^3 \\
&\quad + \int \left[ \mathbb{E}(D^2_{x_1,x_3} F)^2 (D^2_{x_2,x_3} F)^2 \right]^{1/2} \left[ \mathbb{E}(D^2_{x_1,x_3} G)^2 (D^2_{x_2,x_3} G)^2 \right]^{1/2} d\lambda^3.
\end{aligned}
$$

# 7. Additive functionals of the Boolean model

## Setting

$\xi$ is an independent marking of a stationary Poisson process $\eta$ with intensity $\gamma > 0$. The marks come from the space of all non-empty convex bodies in $\mathbb{R}^d$ and have distribution $\mathbb{Q}$. The Boolean model is the random closed set given by

$$Z := \bigcup_{(x,K)\in\xi} (K + x).$$

We consider a measurable, additive and locally bounded functional $\varphi$ on the convex ring.

## Assumption

We assume that

$$\int \bar{V}(K)^3 \, \mathbb{Q}(dK) < \infty$$

where $\bar{V} := V_0 + \cdots + V_d$ is the sum of intrinsic volumes (Wills functional).

## Theorem (Hug, L. and Schulte '15+)

*Let $W \subset \mathbb{R}^d$ be a convex body and assume that $\sigma_W^2 := \mathbb{V}\mathrm{ar}[F_W] > 0$, where $F_W := \varphi(Z \cap W)$. Let $\hat{F}_W := \sigma_W^{-1}(F_W - \mathbb{E}[F_W])$. Then*

$$d_1(\hat{F}_W, N) \le c_1 \sigma_W^{-2} \bar{V}(W)^{1/2} + c_2 \sigma_W^{-3} \bar{V}(W),$$

*where $c_1, c_2$ do not depend on $W$.*

## Example

In the case $\varphi := V_d$ we have

$$d_1(\hat{F}_W, N) \le \left[3(\gamma\gamma_2)^{3/2} c_W + \gamma\gamma_3^{1/2}(c_W)^{3/2}\right] V_d(W)^{-1/2},$$

where $\gamma_i := \int V_d(K)^i \, \mathbb{Q}(dK)$ and

$$c_W := V_d(W)(1-p)^{-2}\left[\int V_d(W \cap (W+x))\left(e^{\gamma\beta_d(x)} - 1\right) dx\right]^{-1}.$$

Here $p = 1 - \exp[-\gamma\gamma_1]$ is the volume fraction of $Z$ and $\beta_d(x) := \int \lambda_d(K \cap (K+x)) \, \mathbb{Q}(dK)$.

# 8. Clusters of the Gilbert graph

## Setting

$\xi$ is an independent marking of a stationary Poisson process $\eta$ with intensity $\gamma > 0$. The marks are non-negative random variables (radii). The Gilbert graph has vertex set $\eta$ and an edge between two different points $x, y \in \eta$ if the balls centred at $x$ and $y$ overlap. A $k$-cluster is a connected component with $k$ points.

## Theorem

*Let $k \geq 1$ and $F_W$ be the number of $k$-clusters within a compact observation window $W$. Let $\hat{F}_W := \mathbb{V}\mathrm{ar}[F_W]^{-1}(F_W - \mathbb{E}[F_W])$. Then, under a suitable (polynomial) moment assumption on the radius distribution,*

$$\max\{d_K(\hat{F}_W, N), d_1(\hat{F}_W, N)\} \leq c\lambda_d(W)^{-1/2},$$

*where $c > 0$ does not depend on $W$.*

## Remark

Penrose '03 proved the (multivariate) CLT for the Gilbert graph with deterministic radii.

# 9. Cluster counting in the random connection model (Work in progress)

**Setting**

Let $\eta$ be a stationary Poisson process with intensity $\gamma > 0$ and let $\varphi : \mathbb{R}^d \to [0, 1]$ be a measurable and symmetric connection function. Given $\eta$, connect any two points $x, y \in \eta$ with probability $\varphi(x - y)$ independently of all other pairs. This gives the random connection model $(\eta, \chi)$, where $\chi$ is the point process of edges.

## Theorem (van de Brug and Meester '04)

*Suppose that* $\varphi(x) = \tilde{\varphi}(|x|)$ *for a decreasing function* $\tilde{\varphi}$ *satisfying* $\int r^{d-1}\tilde{\varphi}(r)dr < \infty$. *Let* $F_W$ *be the number of isolated points within a compact observation window* $W$. *Let* $\hat{F}_W := \mathbb{V}\mathrm{ar}[F_W]^{-1/2}(F_W - \mathbb{E}[F_W])$. *Then* $\hat{F}_W \xrightarrow{d} N$ *as* $W \uparrow \mathbb{R}^d$. *If* $\varphi$ *has bounded support, then the same holds for the number of* $k$*-clusters.*

## Goal

The number $F_W$ of $k$-clusters within a compact observation window $W$ satisfies

$$\max\{d_1(\hat{F}_W, N), d_K(\hat{F}_W, N)\} \le c\lambda_d(W)^{-1/2}.$$

This should hold under minimal assumptions on $\varphi$ and also for other suitable localizing (stabilizing) functionals $F_W$.

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