Limit Theorems for Poisson U-Statistics

Singapore, May 21, 2015

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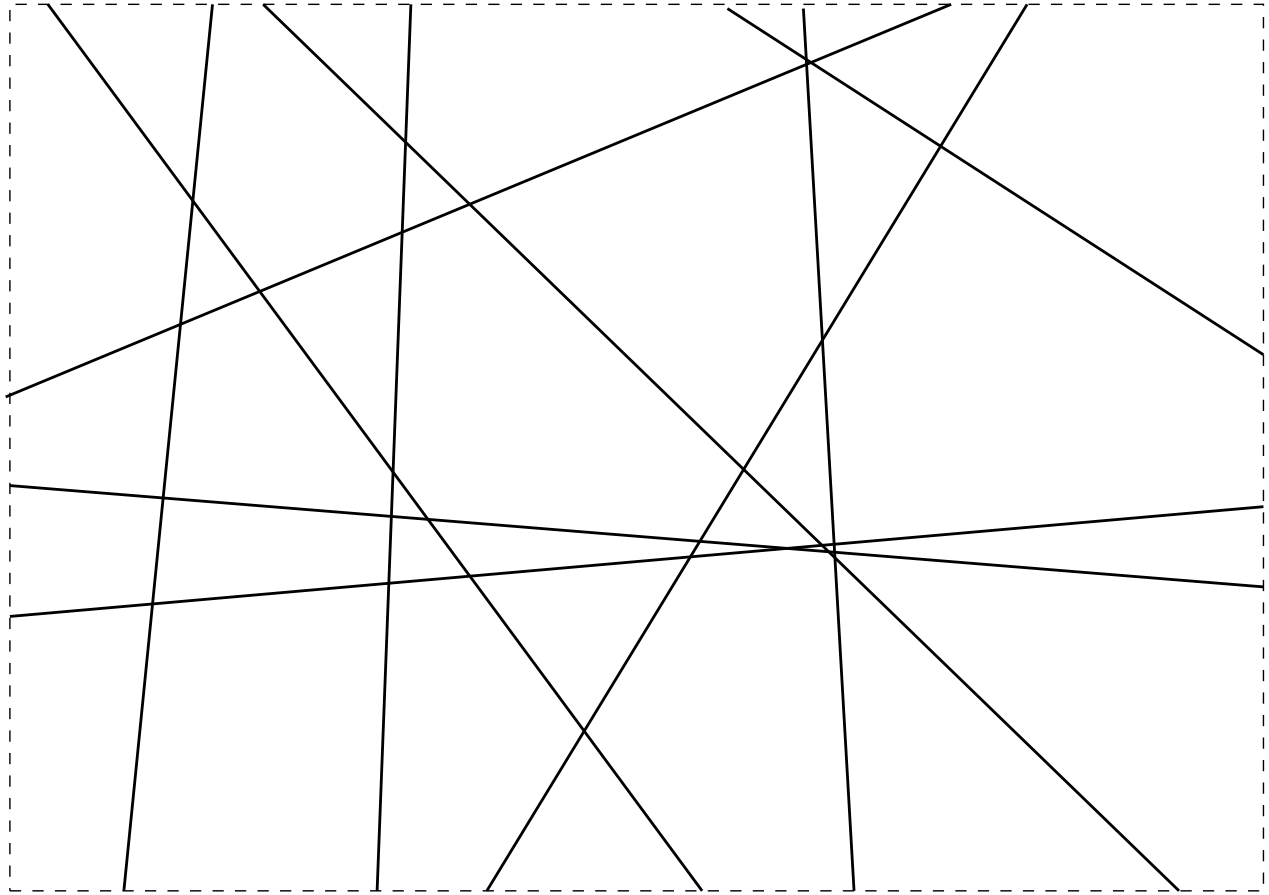
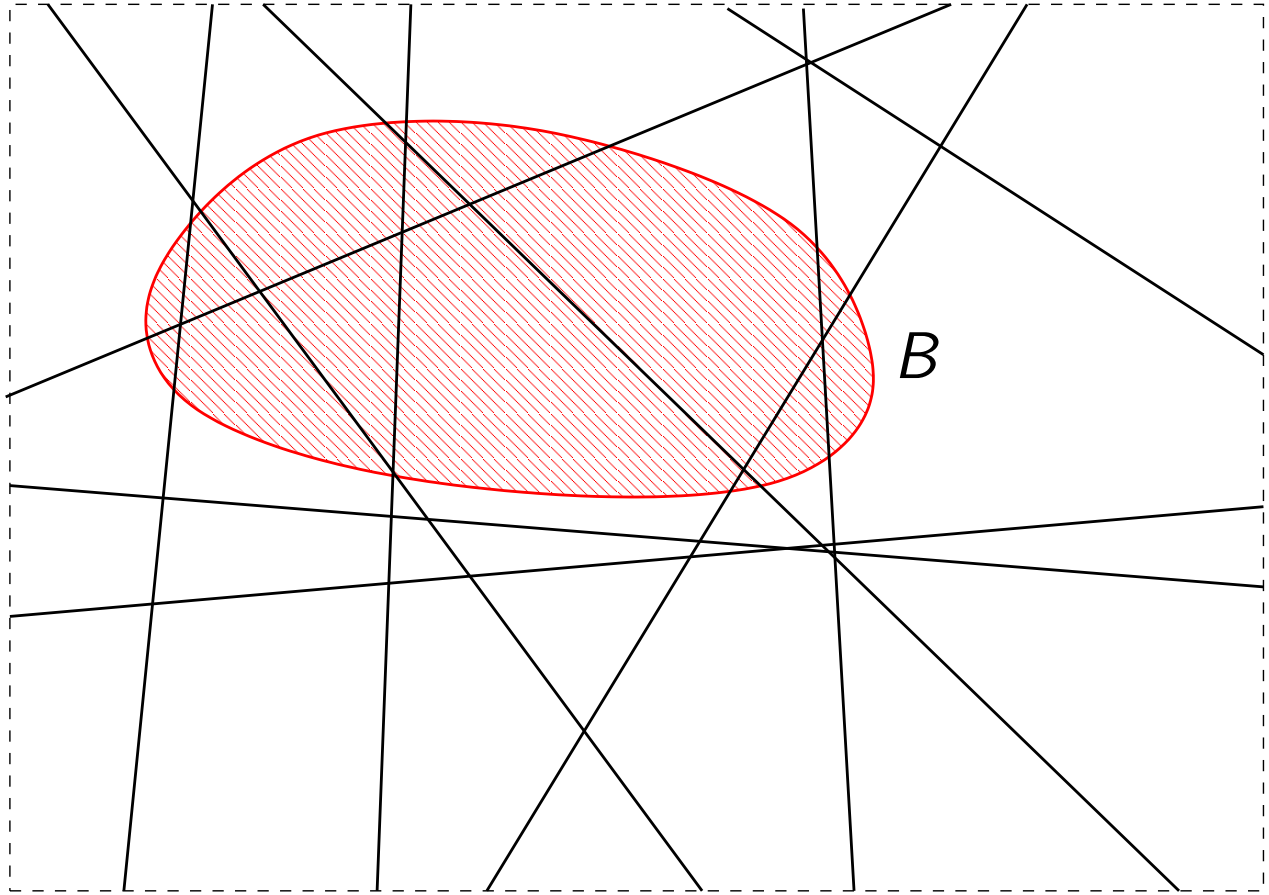
UNIVERSITÄT OSNABRÜCK

Poisson hyperplane process in $\mathbb{R}^d$



$$\eta = \{H_1, H_2, \dots\}$$

Poisson hyperplane process in $\mathbb{R}^d$



$$\eta(B) = \{H \in \eta : H \cap B \neq \emptyset\}$$

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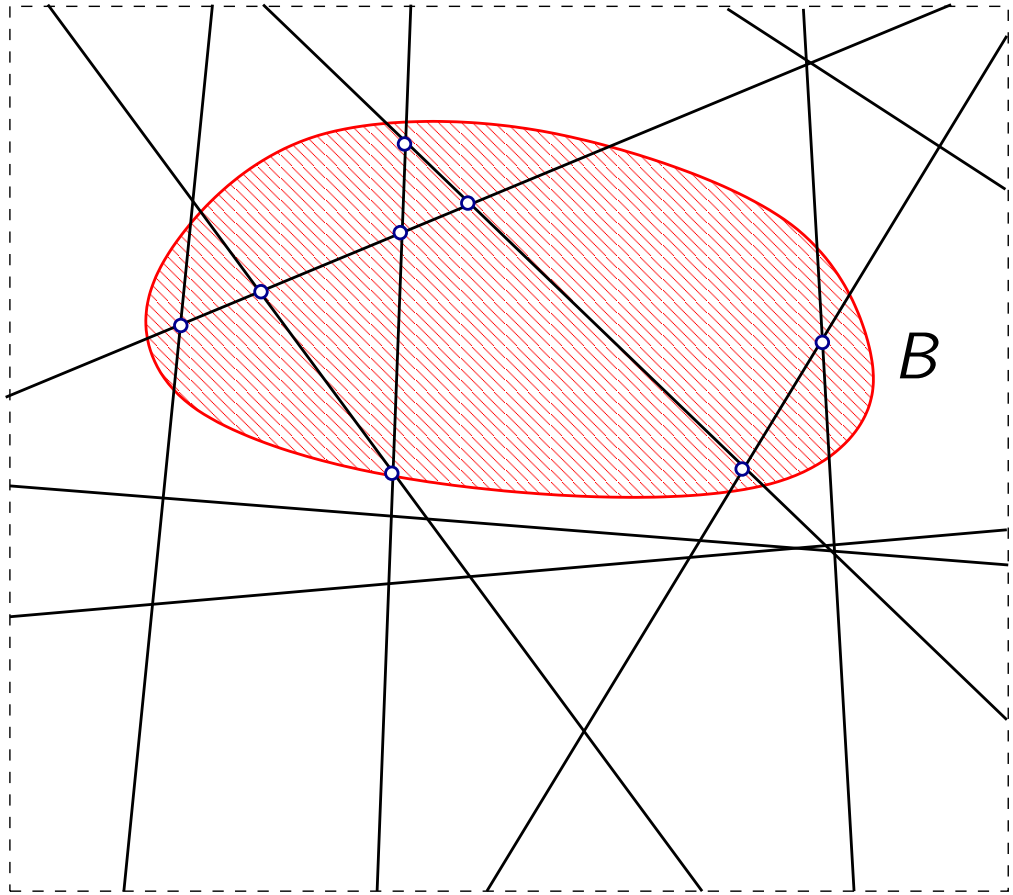
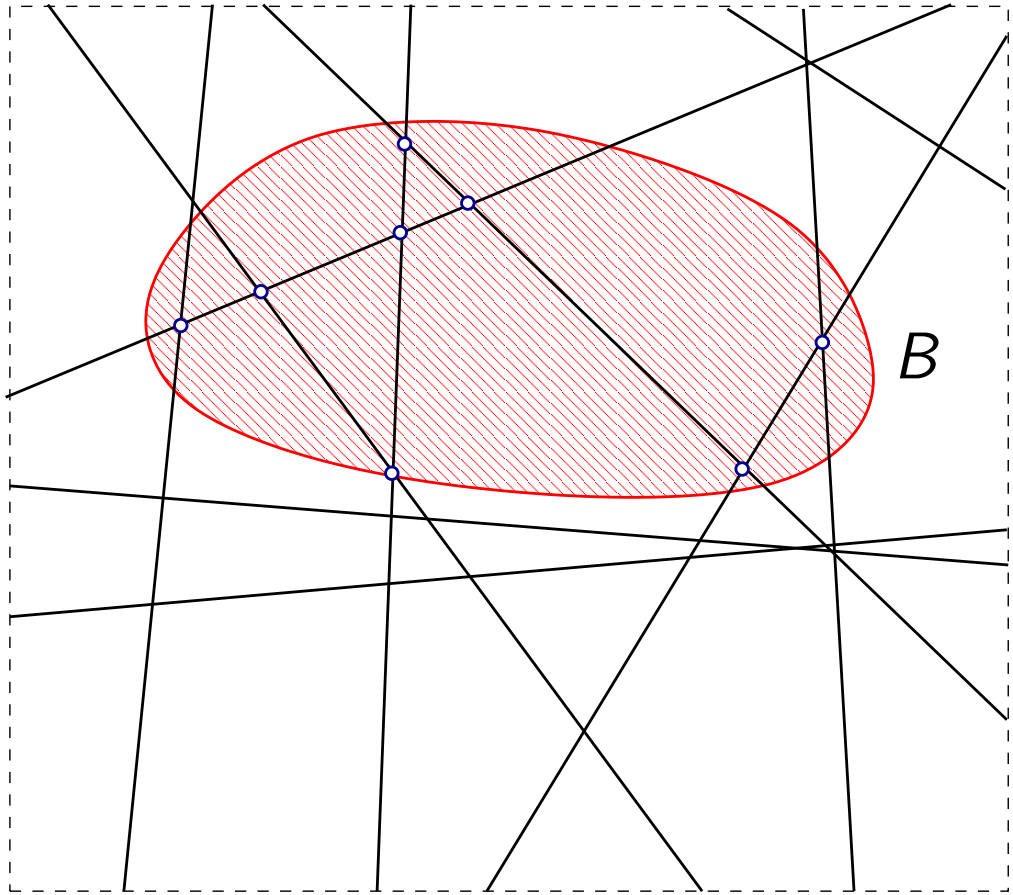


$$Y = \sharp\{\bigcap_{i=1}^{d} H_i : H_i \in \eta\}$$

Poisson hyperplane process in $\mathbb{R}^d$



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$\mathbb{E}Y$ ... integral geometry

$\mathbb{V}Y = ?$

$Y \sim ?$

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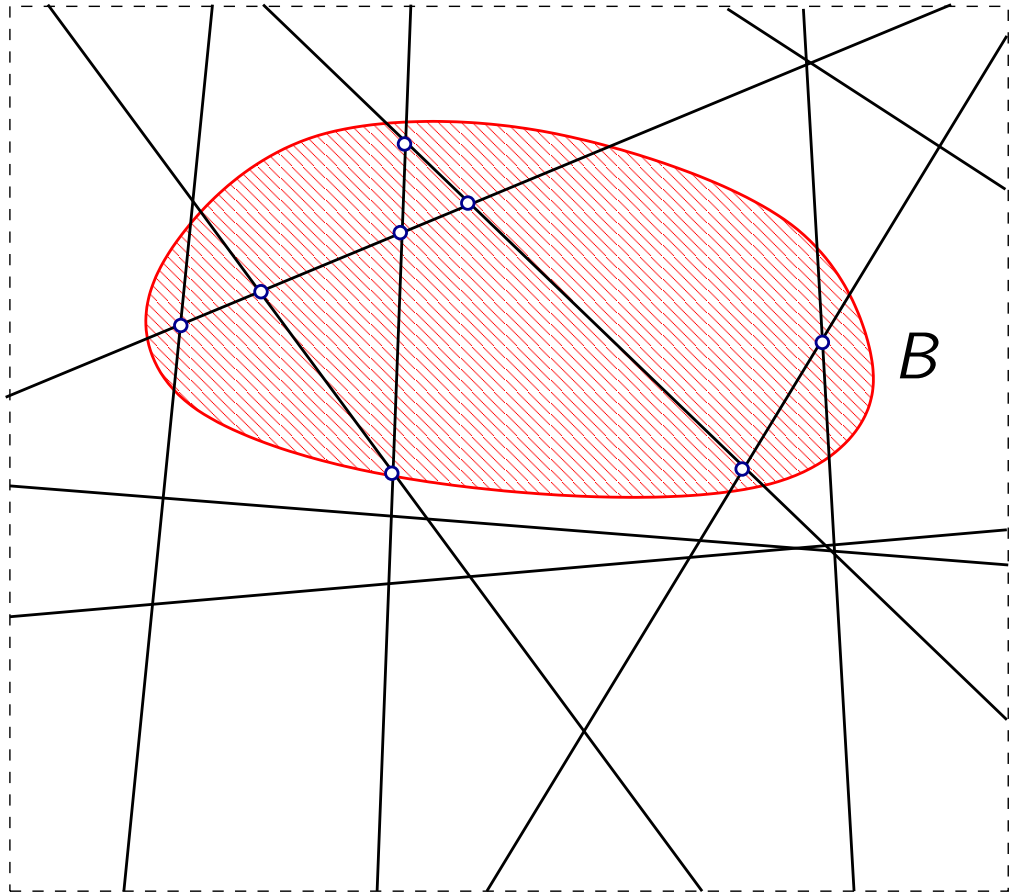

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for $t \to \infty$?

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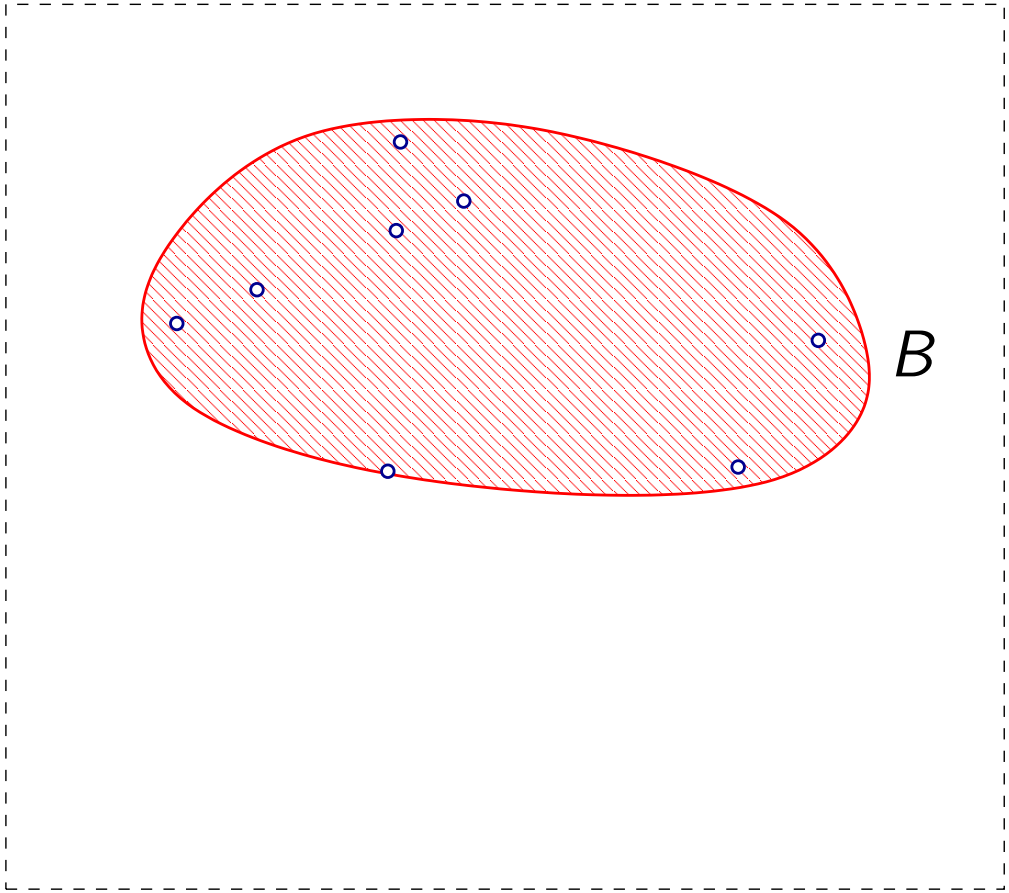

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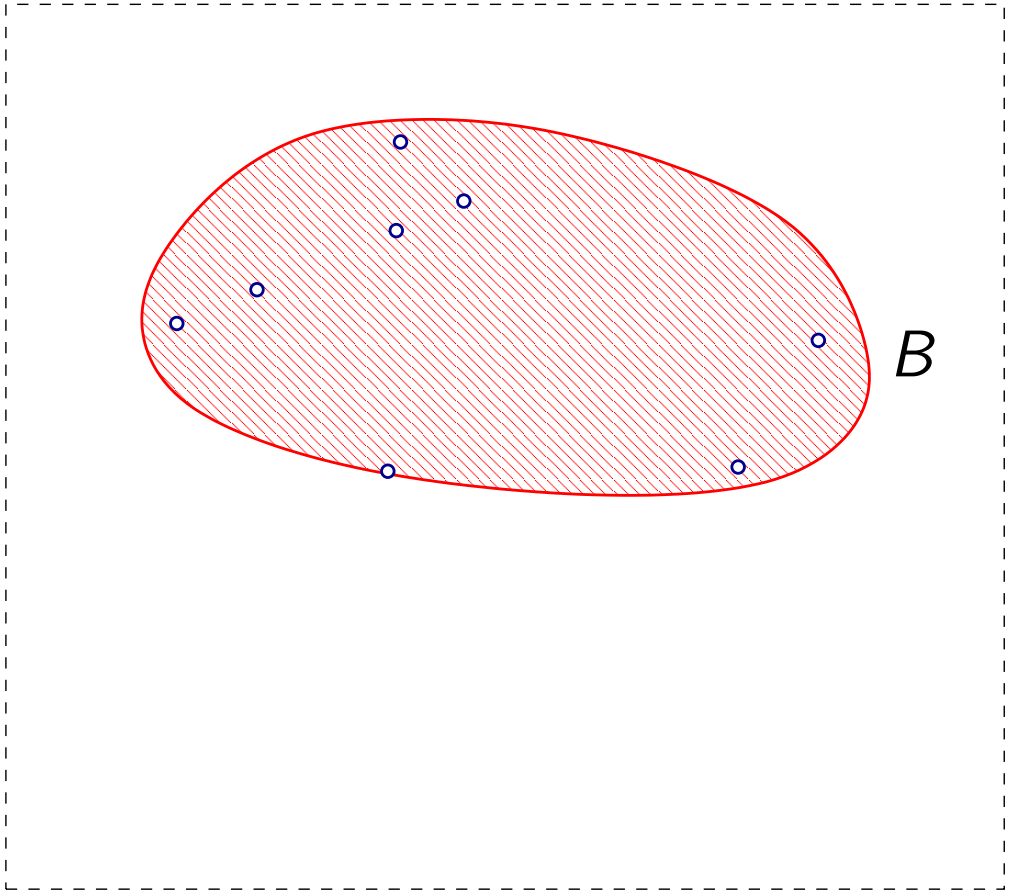

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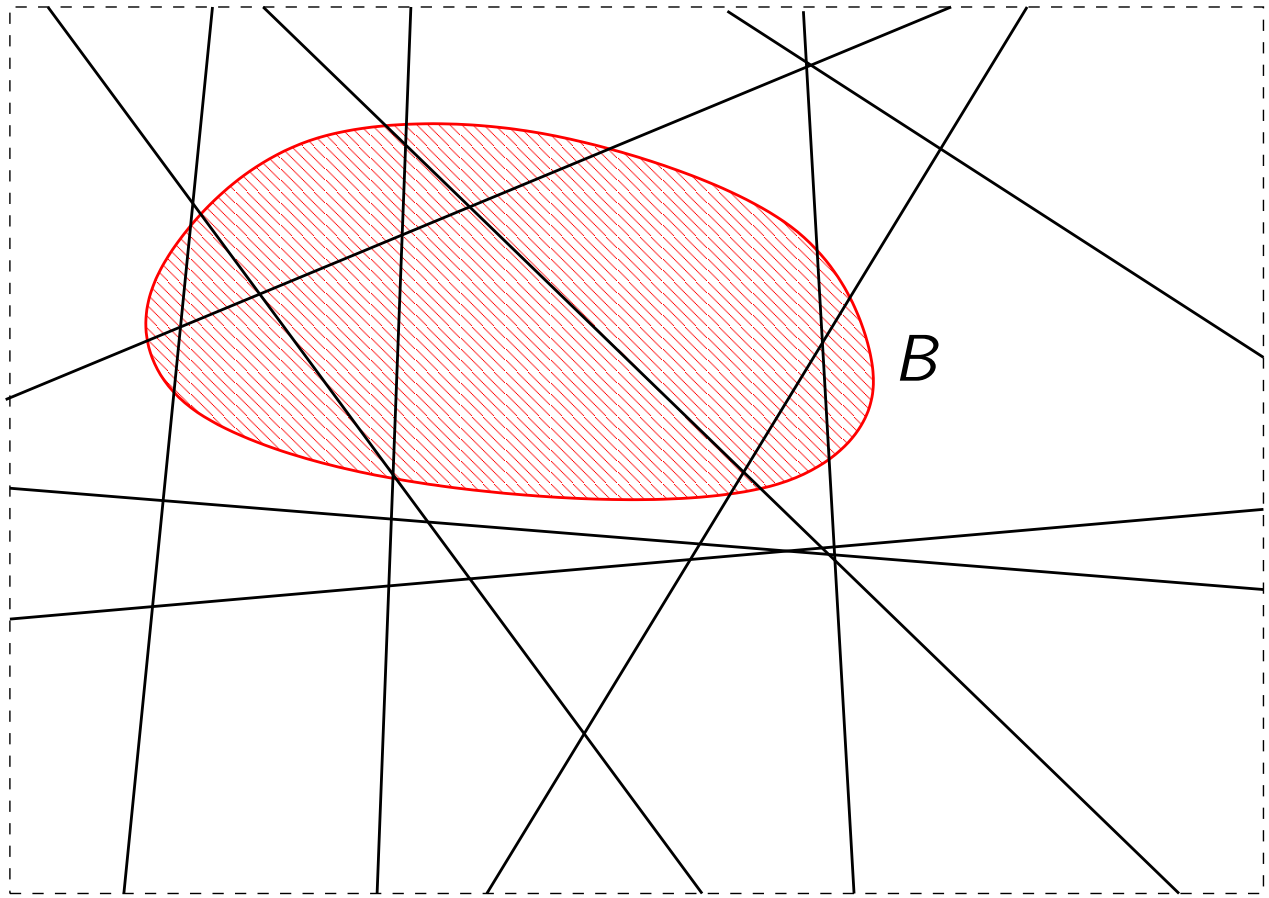
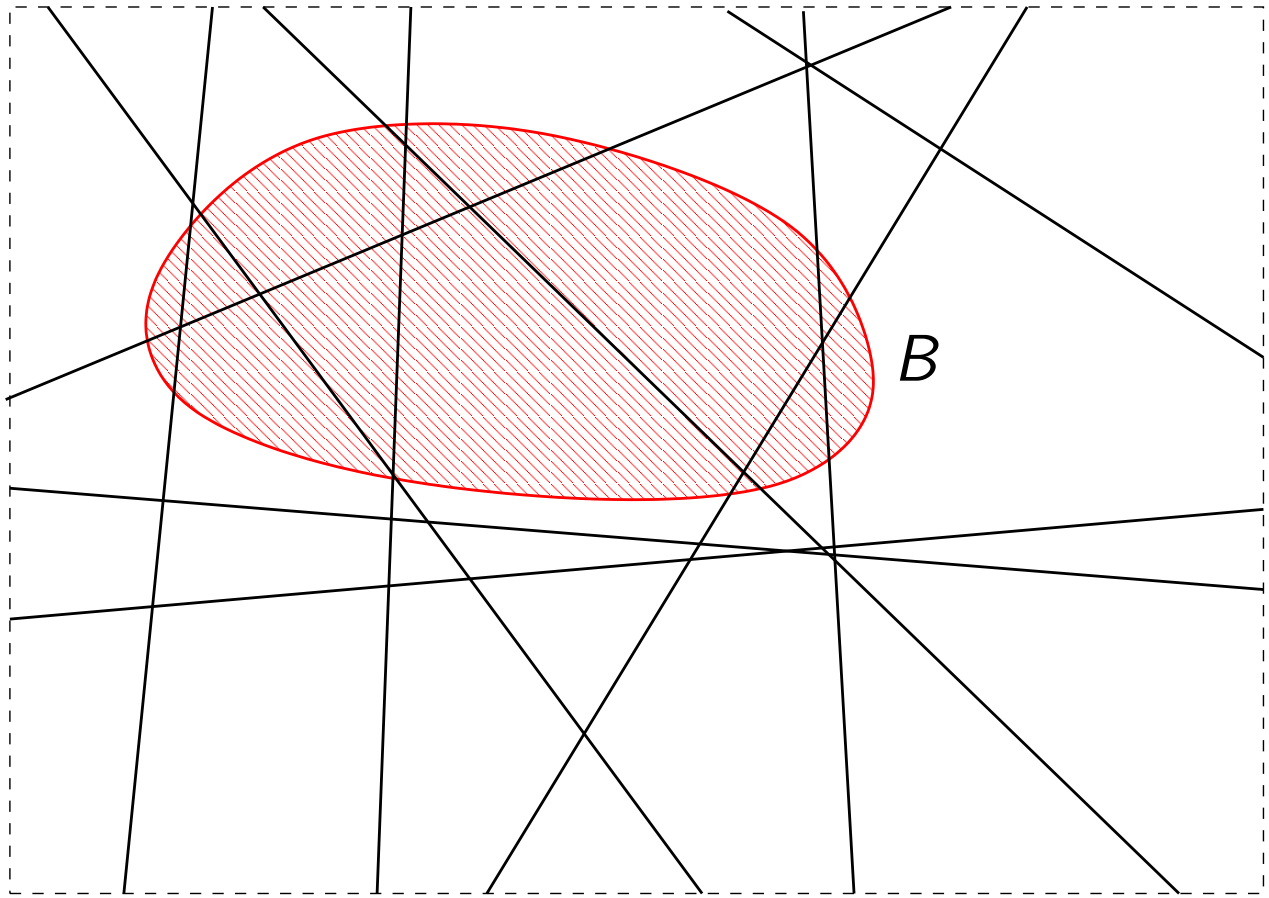
Heinrich, Schmidt, Schmidt

Poisson plane process in $\mathbb{R}^d$



$\eta = \{E_1, E_2, \dots\}$ ... Poisson k-plane process

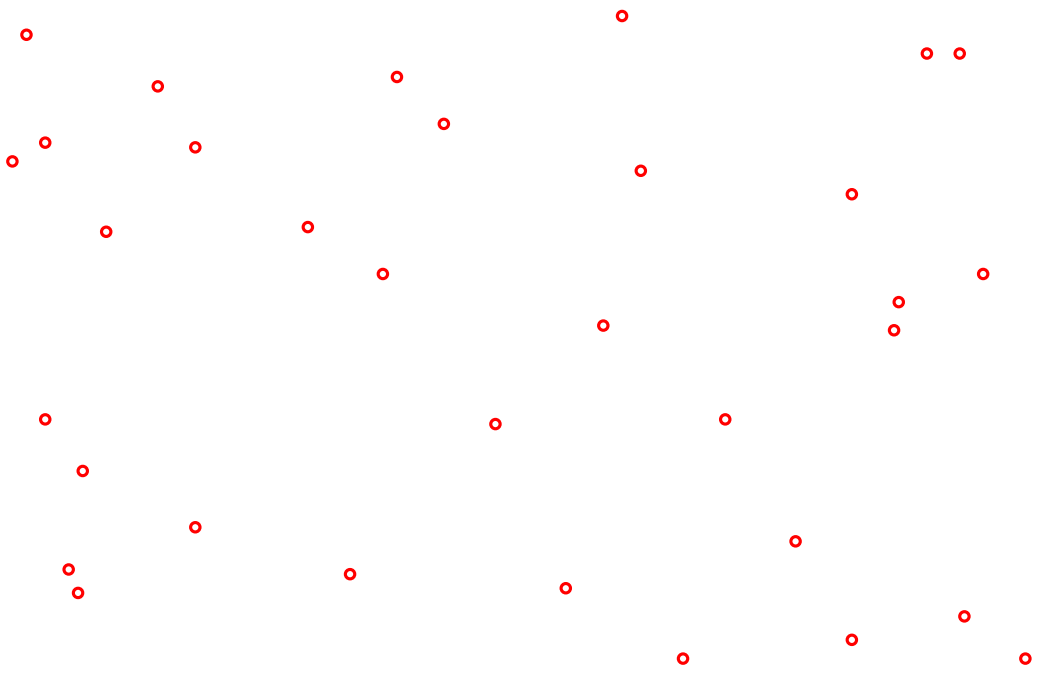
Poisson plane process in $\mathbb{R}^d$



$\eta = \{E_1, E_2, \dots\}$ ... Poisson k-plane process

$\Longrightarrow \tilde{\eta} = \{\bigcap_{i=1}^{m} E_i,\ E_i \in \eta\}, \quad X(B) = \sum_{L \in \tilde{\eta}} V_j(L \cap B)$

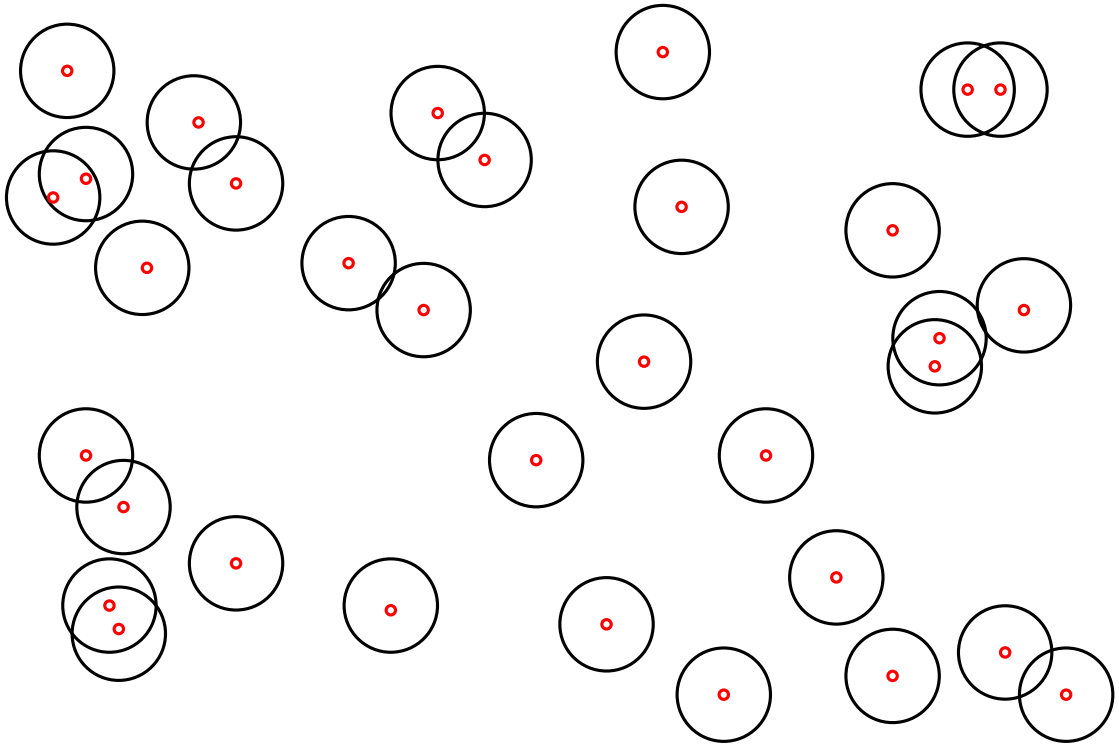
Poisson Point Process



η_t

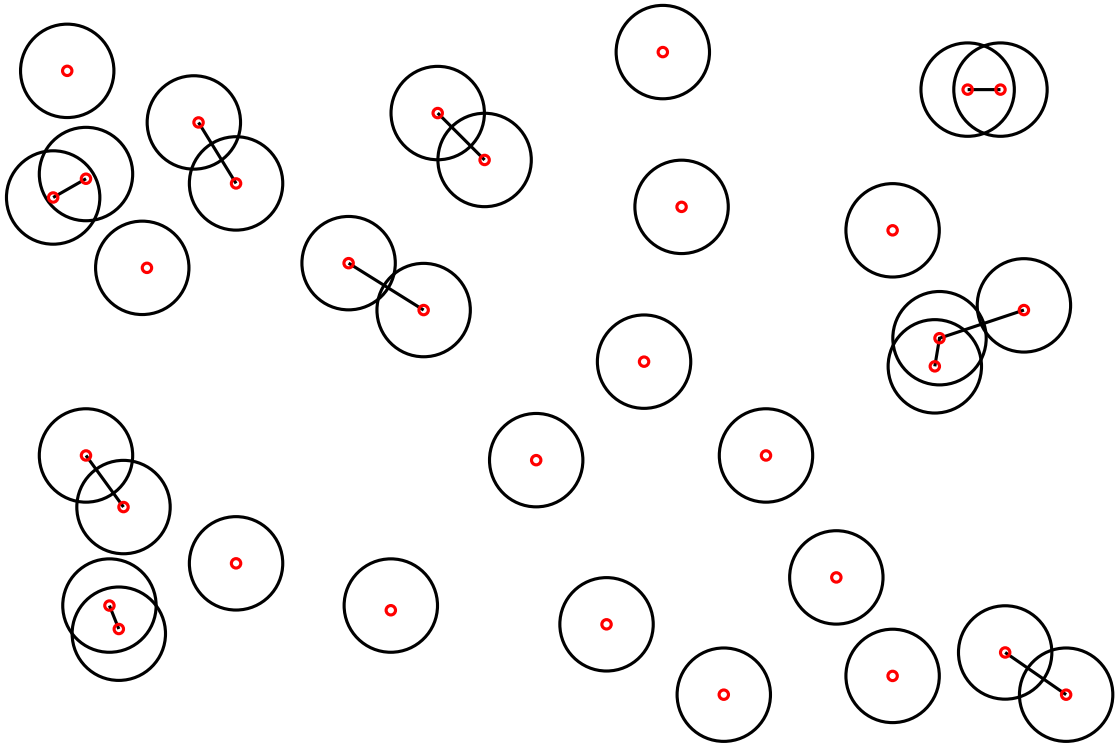
Poisson point process of intensity t (according to Lebesgue measure)

Boolean Model



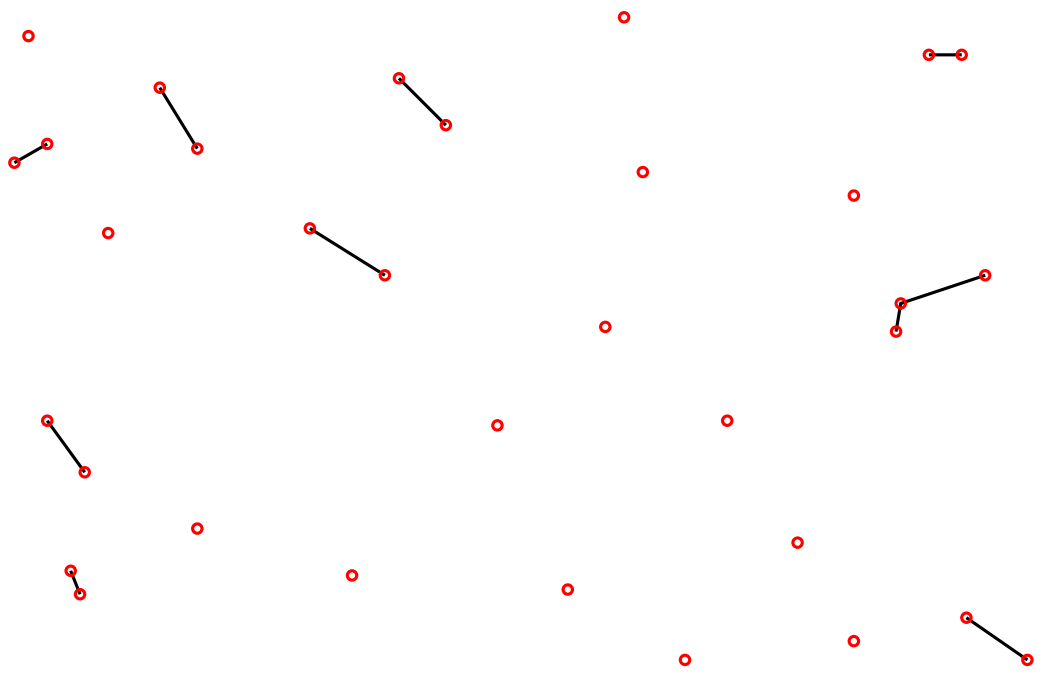
$$\eta_t + B_d(0, \tfrac{1}{2}\delta)$$

Boolean Model



$$\eta_t + B_d(0, \tfrac{1}{2}\delta)$$

Gilbert (or Random Geometric) Graph



Gilbert (or Random Geometric) Graph $G(\eta_t, \delta)$

Random geometric graph

geometric graph

$$\mathcal{G}(\eta_t, \delta_t) = (\mathcal{V}, \mathcal{E}) = \left(\eta, \{(x, y) \in \eta_{\neq}^2 : \ \|x - y\| \leq \delta_t\}\right)$$

number of edges: $N_t = \frac{1}{2} \sum_{(x_1, x_2) \in \eta_t{}_{\neq}^2} \mathbb{1}(\|x_1 - x_2\| \leq \delta_t)$

total edge length: $L_t = \frac{1}{2} \sum_{(x_1, x_2) \in \eta_t{}_{\neq}^2} \mathbb{1}(\|x_1 - x_2\| \leq \delta_t) \, \|x_1 - x_2\|$

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$$\frac{N_t - \mathbb{E}N_t}{\sqrt{\mathbb{V}N_t}} \sim \mathcal{N}(0, 1)$$

for $t \to \infty$

Penrose

Random geometric graph

geometric graph

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$$\frac{L_t - \mathbb{E} L_t}{\sqrt{\mathbb{V} L_t}} \sim \mathcal{N}(0, 1)$$

for $t \to \infty$

Penrose

Poisson U-statistic

Definition: U-statistic

$\eta = \{Z_1, \ldots, Z_n\}$:

$$F(\eta) = \sum_{\eta^k_{\neq}} f(x_1, \ldots, x_k)$$

Poisson U-statistic

Definition: Poisson U-statistic

η Poisson point process, $f \in L_1(\Omega)$:

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Slyvniak-Mecke formula

Poisson U-statistic

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$$\mathbb{E}F(\eta) = \int_{\Omega^k} f(x_1, \ldots, x_k) d\mu^k$$

Slyvniak-Mecke formula

Poisson U-statistic

Malliavin calculus for functions $F(\eta)$ of Poisson point processes:

Wiener-Itô chaos expansion

If $F \in L^2(\mathbb{P})$ then

$$F(\eta) = \sum_{i=0}^{\infty} I_i(f_i)$$

Wiener-Ito Chaos Expansion

The kernel f_i:

- $D_y F(\eta) = F(\eta \cup \{y\}) - F(\eta)$ ... Difference operator D_y

  $f_1 = f_1(y_1) = \mathbb{E} D_{y_1} F(\eta)$

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- $D^i_{y_1,\ldots,y_i} F(\eta) = D_{y_i} D^{i-1}_{y_1,\ldots,y_{i-1}} F(\eta)$

  $f_i = f_i(y_1, \ldots, y_i) = \frac{1}{i!} \mathbb{E} D^i_{y_1,\ldots,y_i} F(\eta)$

Wiener-Ito Chaos Expansion

The Wiener-Ito integral of $f_i(y_1, \ldots, y_i)$:

$$I_i(f_i) = \int\limits_{\Omega^i \setminus \mathrm{diag}} f_i \, d(\eta - \mu)^i$$

Wiener-Ito Chaos Expansion

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Isometry:

$$\mathbb{E}\left[I_n(f_n)\,(\eta)\right]^2 = n! \|f_n\|^2$$

Poisson U-statistic

$$F(\eta) = \sum_{\eta^k_{\neq}} f(x_1, \ldots, x_k)$$

$$f_i(y_1, \ldots, y_i) = \binom{k}{i} \int\limits_{\Omega^{k-i}} f(y_1, \ldots, y_i, x_1, \ldots, x_{k-i})\, d\mu(x_1, \ldots, x_{k-i})$$

$$I_i(f_i) = \int\limits_{\Omega^i \setminus \mathrm{diag}} f_i \, d(\eta - \mu)^i$$

Poisson U-statistic

Wiener-Itô chaos expansion

$F \in L^2(\mathbb{P})$ has a finite Wiener-Itô chaos expansion

$$F(\eta) = \sum_{i=0}^{k} I_i(f_i)$$

with kernels $f_i \in L^1 \cap L^2(\Omega)$, iff F is a sum of U-statistics.

Reitzner, Schulte

Poisson U-statistic

Example:

$$F = \sum_{\eta^2_{\neq}} f(x_1, x_2)$$

with $f \in L^1 \cap L^2(\Omega)$

Poisson U-statistic

Example:

$$\mathbb{E}F = \mathbb{E}\sum_{\eta^2_{\neq}} f(x_1, x_2)$$

$$\text{with } f \in L^1 \cap L^2(\Omega)$$

Poisson U-statistic

Example:

$$F = \sum_{\eta^2_{\neq}} f(x_1, x_2) \in L^1(\mathbb{P})$$

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Poisson U-statistic

Example:

$$F = \sum_{\eta^2_{\neq}} f(x_1, x_2) \in L^1(\mathbb{P})$$

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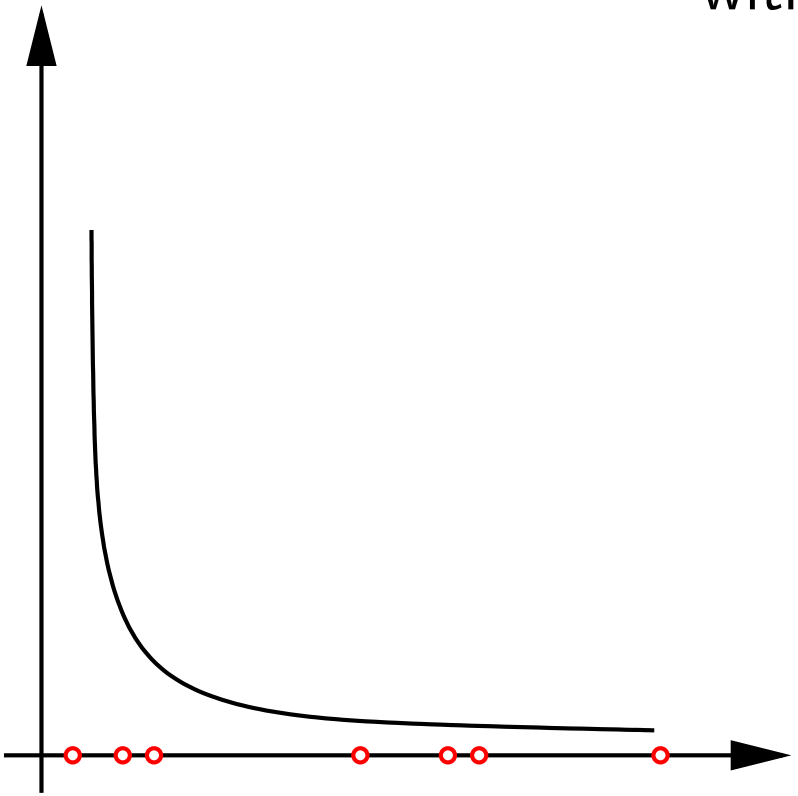


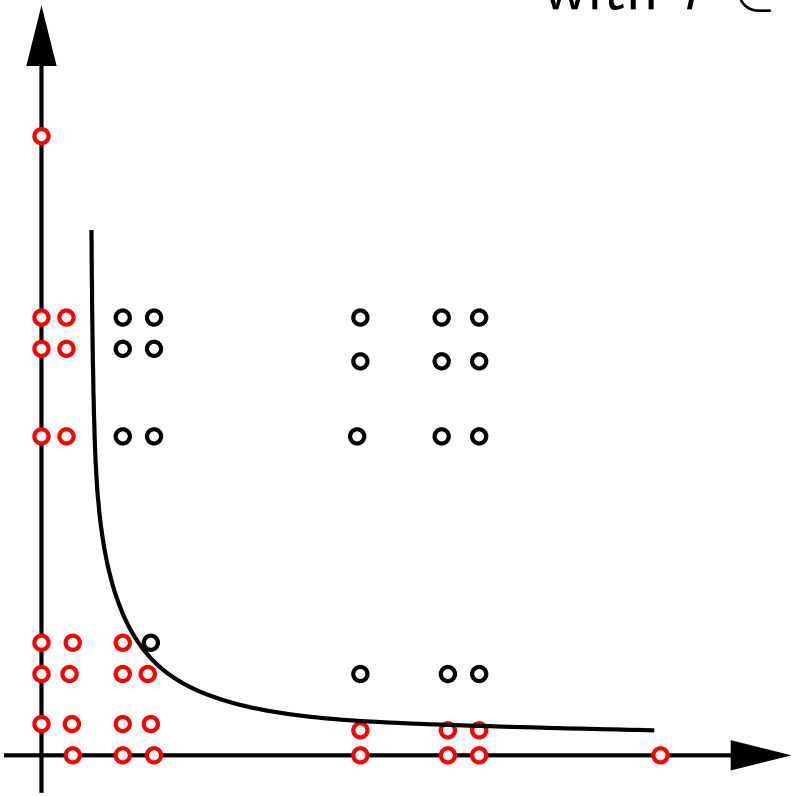
$$f = \mathbb{1}(x_1^2 \leq x_2^{-1})\mathbb{1}(x_2^2 \leq x_1^{-1})$$

Poisson U-statistic

Example:

$$F = \sum_{\eta^2_{\neq}} f(x_1, x_2) \in L^1(\mathbb{P})$$

with $f \in L^1 \cap L^2 \cap L^\infty(\Omega)$



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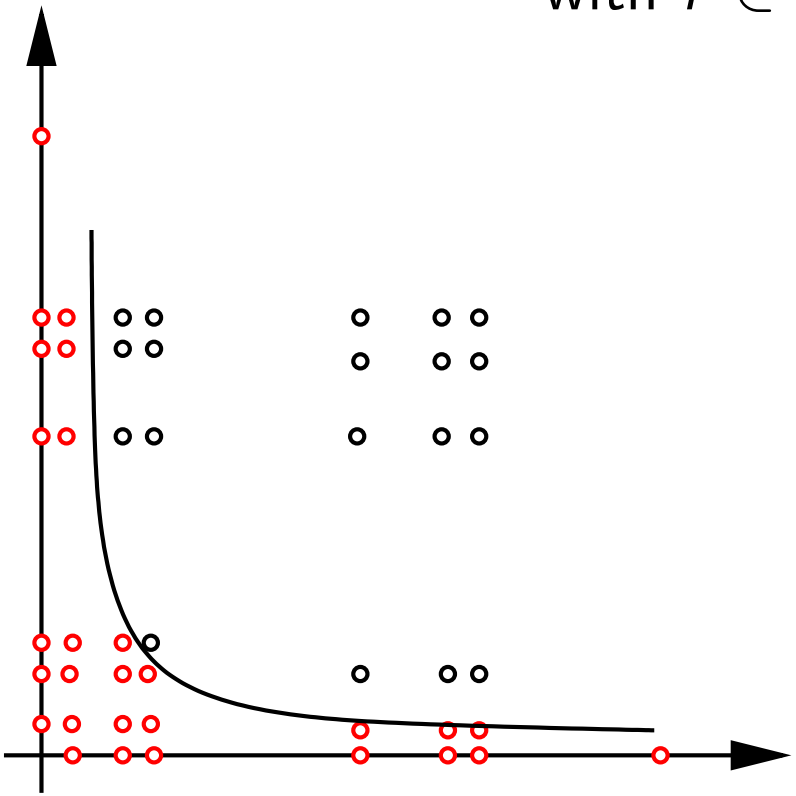
Poisson U-statistic

Example:

$$F = \sum_{\eta^2_{\neq}} f(x_1, x_2) \in L^1(\mathbb{P})$$

with $f \in L^1 \cap L^2 \cap L^\infty(\Omega)$ and with $F \notin L^2(\mathbb{P})$.

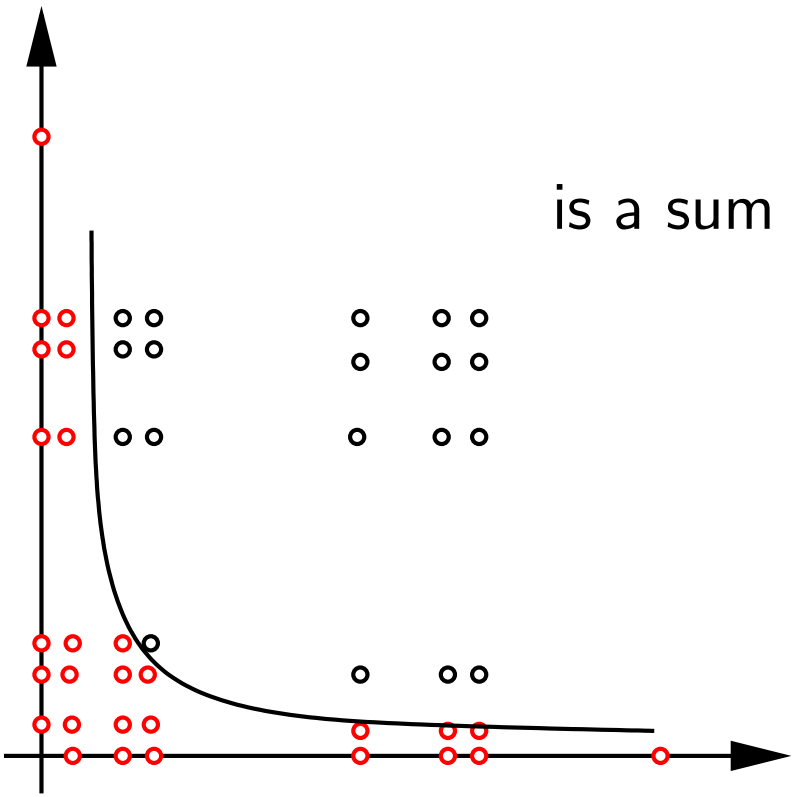


$$f = \mathbb{1}(x_1^2 \leq x_2^{-1})\mathbb{1}(x_2^2 \leq x_1^{-1})$$

Poisson U-statistic

Example (continued):

$$F = I_2(f) \in L^2(\mathbb{P})$$



is a sum of U-statistics

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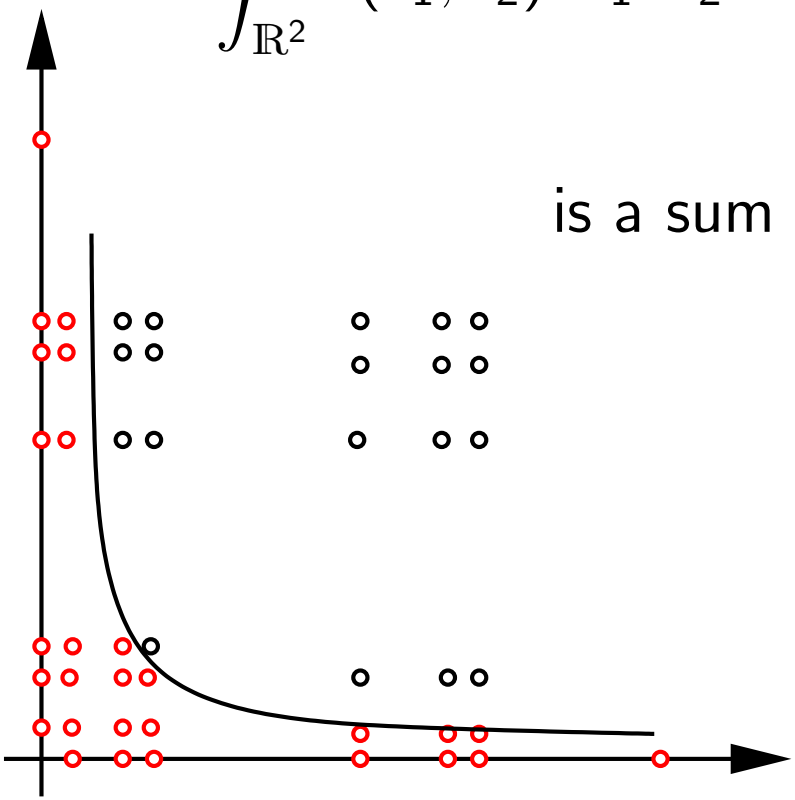
Poisson U-statistic

Example (continued):

$$
\begin{aligned}
F &= I_2(f) \in L^2(\mathbb{P}) \\
&= \int_{\mathbb{R}^2} f(x_1, x_2)\,dx_1\,dx_2 - 2\sum_{x_1 \in \eta} \int_{\mathbb{R}} f(x_1, x_2)\,dx_2 + \sum_{\eta^2_{\neq}} f(x_1, x_2)
\end{aligned}
$$

is a sum of U-statistics



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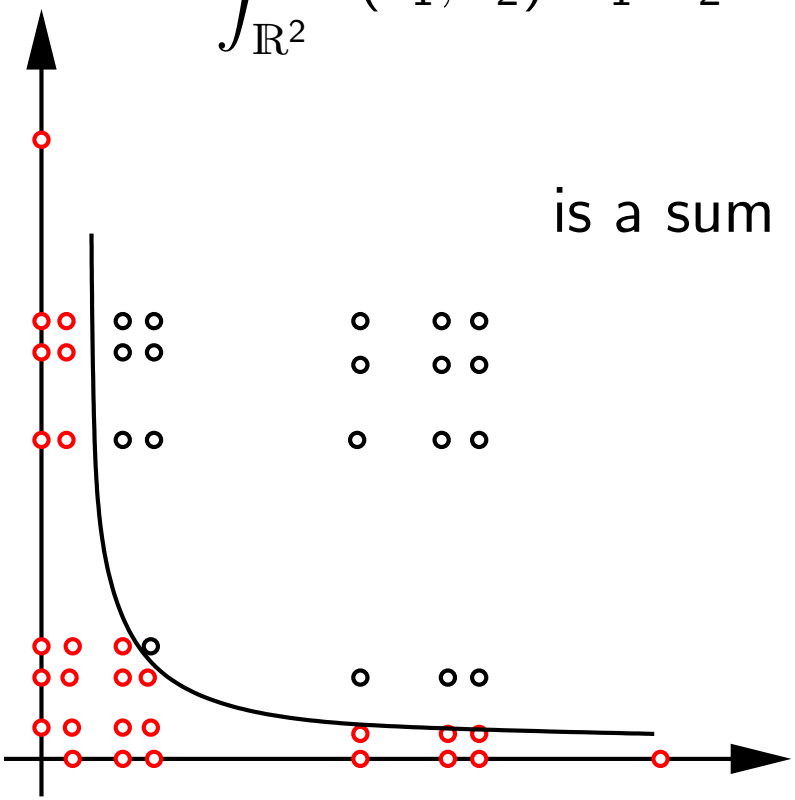

is a sum of U-statistics which are not in $L^2(\mathbb{P})$.

$$f = \mathbb{1}(x_1^2 \leq x_2^{-1})\mathbb{1}(x_2^2 \leq x_1^{-1})$$

Poisson U-statistic

Stein's methode and CLT's for Poisson functionals:

Central limit theorem

For $F \in L^2(\mathbb{P})$ we have

$$\begin{aligned} d_W(F,N) \;\leq\; & \frac{1}{\sqrt{\mathbb{V}F}}\sqrt{\mathbb{E}[(\mathbb{V}F - \langle D_xF, -D_xL^{-1}F\rangle_{L^2(\mu)})^2]} \\ & + \frac{1}{\mathbb{V}F}\int_\Omega \mathbb{E}[(D_xF)^2|D_xL^{-1}F|]\mu(dx). \end{aligned}$$

with $N \sim \mathcal{N}(0,1)$.

Peccati, Solé, Taqqu, and Utzet

Poisson U-statistic

Central limit theorem

F an absolutely convergent U-statistic of order k. Then

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq 2k^4 \frac{M_4(f)}{\mathbb{V}F}$$

with $N \sim \mathcal{N}(0,1)$.

Reitzner, Schulte

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$$F = \sum_0^k I_i(f_i)$$

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Reitzner, Schulte

$$\mathbb{V}F = \sum_{1}^{k} i!\|f_i\|^2$$

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Reitzner, Schulte

$$M_4(f) = \sum \int_\Omega \dots \int_\Omega |f(\dots)f(\dots)f(\dots)f(\dots)| \, d\mu^m(x_1, \dots, x_m)$$

Poisson U-statistic

Central limit theorem

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with $N \sim \mathcal{N}(0,1)$.

F is absolutely convergent if

$$\bar{F}(\eta) = \sum_{\eta^k_{\neq}} |f(x_1, \ldots, x_k)| \in L^2(\mathbb{P}).$$

Poisson U-statistic

Central limit theorem

$F \in L^2(\mathbb{P})$ nice U-statistic of order k. Then

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq c \max_{***}\{\|f_i \star_r^l f_i\|, \|f_i \star_r^l f_j\|, \|f_i\|^2\}$$

with $N \sim \mathcal{N}(0,1)$.

Lachieze-Rey, Peccati

4th moment theorem

Central limit theorem

$F \in L^2(\mathbb{P})$ a U-statistic of order k with $f \geq 0$. Then

$$d_W\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq C\sqrt{\mathbb{E}\left(\frac{F-\mathbb{E}F}{\sqrt{\mathbb{V}F}}\right)^4 - 3}$$

with $N \sim \mathcal{N}(0,1)$.

Lachieze-Rey, Peccati

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Lachieze-Rey, Peccati

...and much more: Bourguin, Lachieze-Rey, Last, Peccati, Penrose, Schulte, Thäle, ...

Poisson U-statistic

Poisson point process with intensity measure $\mu = t\mu_0$, $t \geq 1$.

$F = \sum f$ is a **geometric** U-statistic if

$$f(x_1, \ldots, x_k) \text{ is independent of } t.$$

Poisson U-statistic

Poisson point process with intensity measure $\mu = t\mu_0$, $t \geq 1$.

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Central limit theorem

$F \in L^2(\mathbb{P})$ a geometric U-statistic of order k with $\|f_1\| > 0$. Then

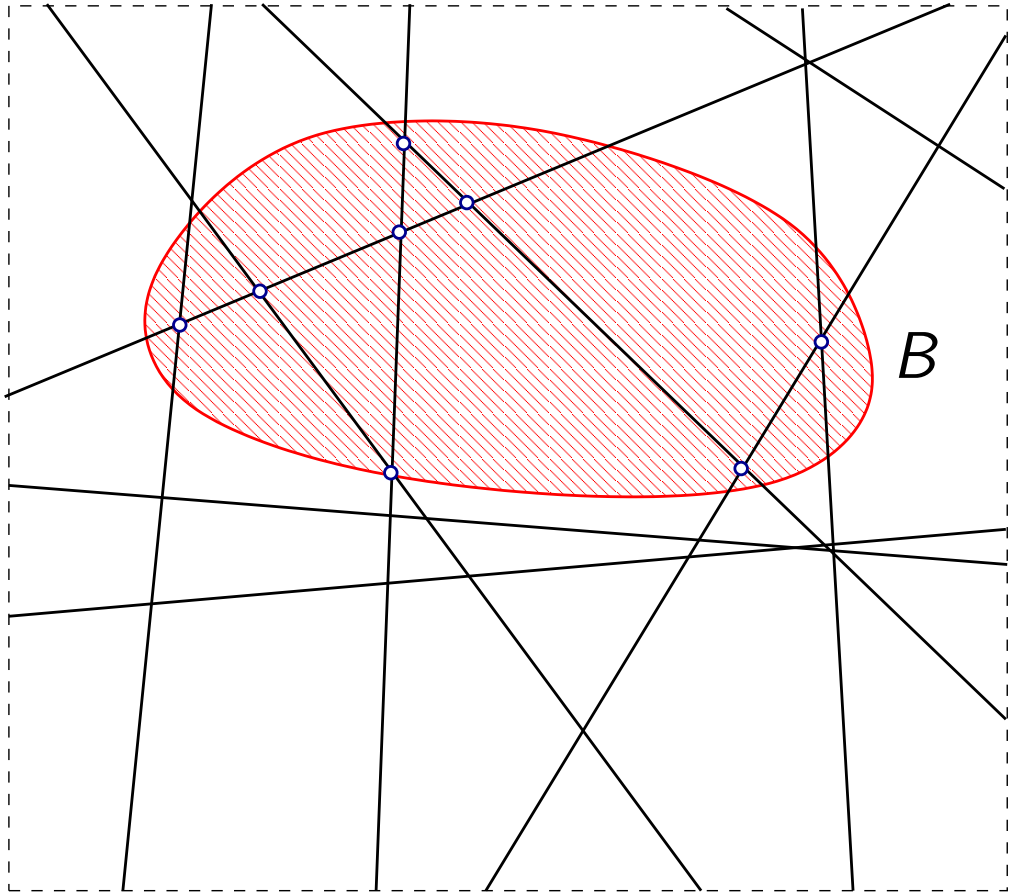
$$\mathbb{V}F \approx \|f_1\|^2 \approx c_f t^{2k-1}$$

and

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq C_f t^{-\frac{1}{2}}$$

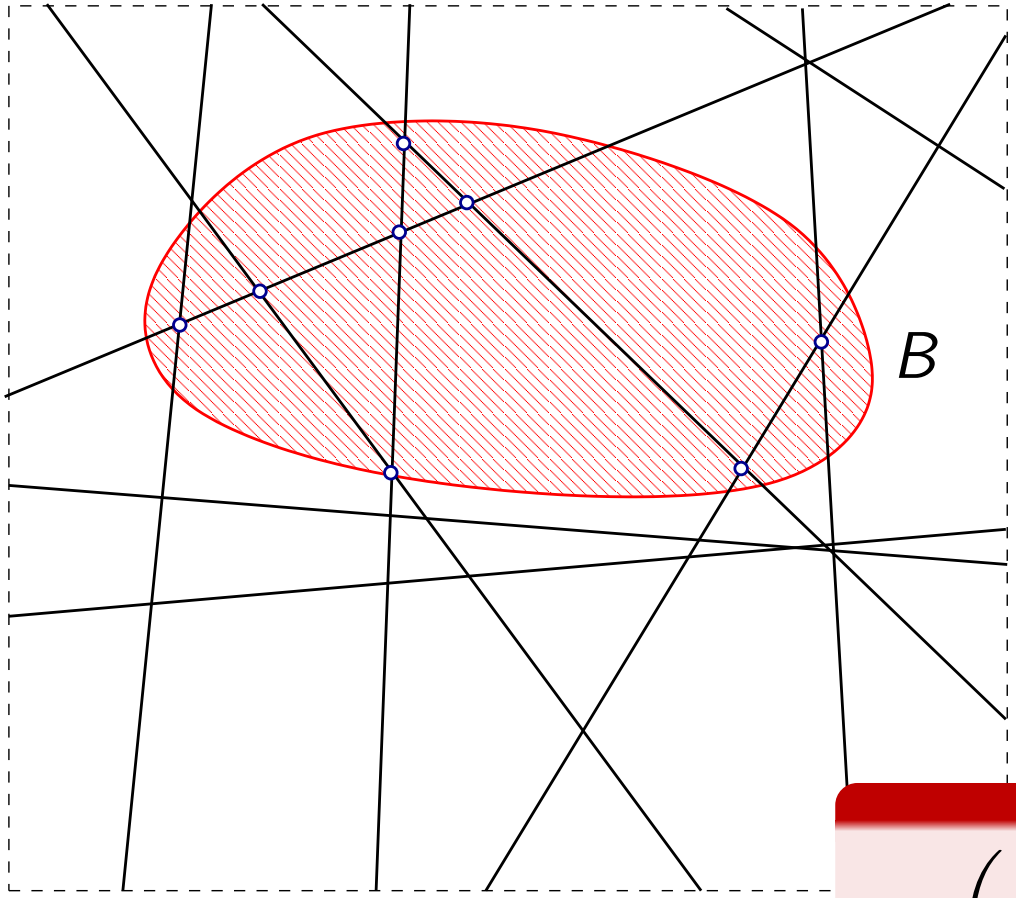
with $N \sim \mathcal{N}(0, 1)$.

Poisson hyperplane process in $\mathbb{R}^d$



$$Y = \{\bigcap_{i=1}^{d} H_i \cap B : H_i \in \eta\}$$
$$= \sum_{\eta^d_{\neq}} \mathbb{1}(\bigcap H_i \cap B)$$

Poisson hyperplane process in $\mathbb{R}^d$



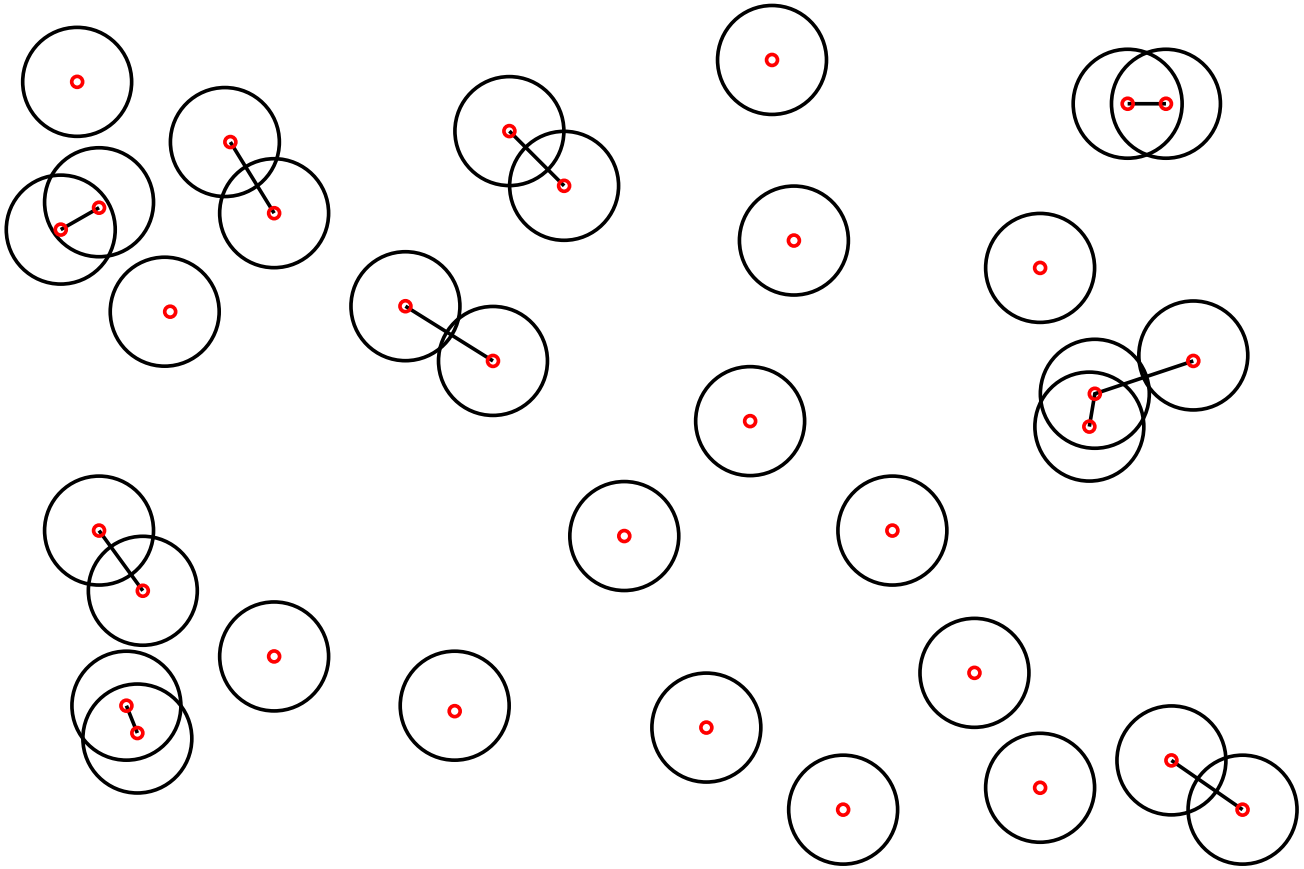
$$Y = \{\bigcap_{i=1}^{d} H_i \cap B : H_i \in \eta\}$$
$$= \sum_{\eta^d_{\neq}} \mathbb{1}(\bigcap H_i \cap B)$$

$$d_W\left(\frac{Y - \mathbb{E}Y}{\sqrt{\mathbb{V}Y}}, N\right) \leq C\, t^{-\frac{1}{2}}$$

with $\mathbb{V}Y \approx c_Y(B) t^{2d-1}$

Random geometric graph

geometric graph $(\mathcal{V}, \mathcal{E}) = \left(\eta, \{(x, y) \in \eta^2_{\neq} : \ \|x - y\| \leq \delta_t\}\right)$



Poisson U-statistic

Poisson point process with intensity measure $\mu = t\mu_0$, $t \geq 1$.

$F = \sum f$ is a **local** U-statistic if

$$f(x_1, \ldots, x_k) = 0 \quad \text{if } \|x_i - x_j\| \geq \delta_t$$

for some $i, j \in \{1, \ldots, k\}$

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Central limit theorem

$F \in L^2(\mathbb{P})$ a local U-statistic of order k with $\|f_1\| > 0$. Then

$$\mathbb{V}F \approx \|f_1\|^2 \approx c_f t^{2k-1}, \quad \text{and}$$

$$d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\mathbb{V}F}}, N\right) \leq C_f t^{-\frac{1}{2}}$$

with $N \sim \mathcal{N}(0, 1)$.

Random geometric graph

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with $t\delta_t^d = 1$

$$N_t \ = \ \sum_{\eta^2_{\neq}} \mathbb{1}(x, y \in B, \ \|x - y\| \leq \delta_t)$$

Random geometric graph

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$$d_W\left(\frac{N_t - \mathbb{E}N_t}{\sqrt{\mathbb{V}N_t}}, N\right) \leq C\, t^{-\frac{1}{2}}$$

with $\mathbb{V}Y \approx c(B)t$

Random geometric graph

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...and much more: Lachieze-Rey, Peccati, Penrose, Schulte, Thäle, ...

Concentration

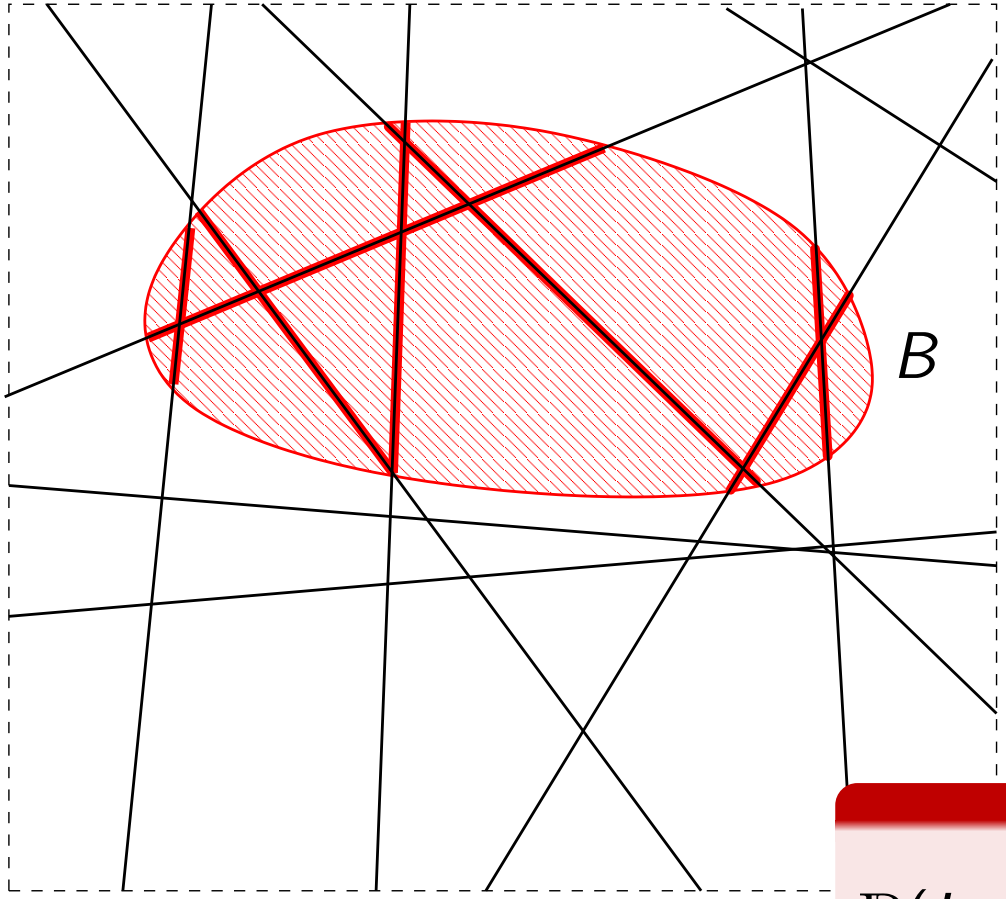
. . . for first order U-statistics with $f \geq 0$ . . .

$$\mathbb{P}(F - \mathbb{E}F \geq u) \leq e^{-\frac{\mathbb{E}F}{\|f\|_\infty} g\left(\frac{u}{\mathbb{E}F}\right)}$$

with $g(u) = (1+u)\ln(1+u) - u, \;\; u \geq 0.$

Houdre and Privault, Ane and Ledoux
Reynaud-Bouret, Breton, et al.

Poisson hyperplane process in $\mathbb{R}^d$



$$L = \lambda_{d-1}(\eta \cap B)$$

$$= \sum_{\eta} \lambda_{d-1}(H \cap B)$$

$$\mathbb{P}(L - \mathbb{E}L \geq u) \leq e^{-\frac{\mathbb{E}L}{\|f\|_\infty} g\left(\frac{u}{\mathbb{E}L}\right)}$$

with

$$g(u) = (1+u)\ln(1+u) - u, \;\; u \geq 0.$$

Concentration

...for higher order U-statistics with $f \geq 0$?

$$d_T(\eta, A) = \max_{\|u\|_{2,\eta} \leq 1} \min_{\zeta \in A} \int u \; d(\eta \backslash \zeta)$$

(Talagrand's convex distance)

$$\mathbb{P}(A)\mathbb{P}\left(d_T(\eta, A) \geq s\right) \leq e^{-\frac{s^2}{4}}.$$

Reitzner

Concentration

. . . for local U-statistics with $f \geq 0$

$$d_T(\eta, A_m) \geq \frac{F(\eta) - \mathbb{M}F}{\sqrt{F(\eta)}} \cdot \frac{1}{\sqrt{B}} \mathbb{1}(F(x : \eta) \leq B \ \forall x \in \eta) .$$

Concentration

...for local U-statistics with $f \geq 0$

$$d_T(\eta, A_m) \geq \frac{F(\eta) - \mathbb{M}F}{\sqrt{F(\eta)}} \cdot \frac{1}{\sqrt{B}} \mathbb{1}(F(x:\eta) \leq B \ \forall x \in \eta)\,.$$

$$\mathbb{P}(|F - \mathbb{M}F| \geq u) \leq 4\mu(\Omega) e^{-\frac{1}{4}\|f\|_\infty^{-\frac{1}{k}} \left(\frac{u^2}{(u+\mathbb{M}F)}\right)^{\frac{1}{k}}}$$

for $u \geq U(f, \mu)$.

...Lachieze-Rey, Reitzner, Schulte, Thäle

Random geometric graph

F a nice local U-statistic with $f \geq 0$. Then for all $r \geq 0$,

$$\mathbb{P}(F \geq \mathbb{E}F + r) \leq e^{-\frac{((\mathbb{E}F+r)^{1/(2k)} - (\mathbb{E}F)^{1/(2k)})^2}{2k^2 c_d}}$$

$$\mathbb{P}(F \leq \mathbb{E}F - r) \leq e^{-\frac{r^2}{2k\mathbb{V}F}}$$

$$\mathbb{P}(F > \mathbb{M}F + r) \leq 2e^{-\frac{r^2}{4k^2 c_d (r+\mathbb{M}F)^{2-1/k}}}$$

$$\mathbb{P}(F < \mathbb{M}F - r) \leq 2e^{-\frac{r^2}{4k^2 c_d (\mathbb{M}F)^{2-1/k}}}$$

...Bachmann, Peccati, Reitzner

Thank you!