

ON STEIN OPERATORS FOR DISCRETE APPROXIMATIONS

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OUTLINE

1 STEIN OPERATORS VIA PGF

- The General Idea
- Convolutions of Measures

2 PERTURBED SOLUTIONS TO THE STEIN EQUATION

3 APPROXIMATION TO SUMS OF INDICATOR VARIABLES

STEIN METHOD ON $\mathbb{Z}_+$ -VALUED RVS

Let Y be a $\mathbb{Z}_+$ -valued rv with $\mathbb{E}(|Y|) < \infty$ and

$\mathcal{F} := \{f|f : \mathbb{Z}_+ \rightarrow \mathbb{R} \text{ is bounded}\}.$

We want to estimate $\mathbb{E}f(Z) - \mathbb{E}f(Y)$ for some rv Z concentrated on $\mathbb{Z}_+$ and $f \in \mathcal{F}$. Stein's method is then realized in three consecutive steps.

- For any bounded function $g : \mathbb{Z}_+ \rightarrow \mathbb{R}$, a linear operator $\mathcal{A}$ satisfying $\mathbb{E}(\mathcal{A}g)(Y) = 0$ is established and is called a Stein operator.

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METHOD CONTD.

- In the next step, the so-called Stein equation

$$(\mathcal{A}g)(j) = f(j) - \mathbb{E}f(Y), \quad j \in \mathbb{Z}_+, f \in \mathcal{F} \quad (1)$$

is solved with respect to $g(j)$ in terms of f and is referred to as solution to Stein equation (1).

As a rule, solutions to the Stein equations have useful properties, such as

$\|\Delta g\| := \sup_{j \in \mathbb{Z}_+} |\Delta g(j)|$ is small, where $\Delta g(j) := g(j+1) - g(j)$ denotes the first forward difference.

Note that the properties of Δg depend on the form of $\mathcal{A}$ and properties of Y .

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- Finally, taking expectations on both sides of (1), we get

$$\mathbb{E}f(Z) - \mathbb{E}f(Y) = \mathbb{E}(\mathcal{A}g)(Z) \quad (2)$$

and estimates for $\mathbb{E}(\mathcal{A}g)(Z)$ are established through the bounds for Δg and $\Delta^{k+1}g(j) := \Delta^k(g(j+1) - g(j))$, $k = 1, 2, \dots$

For more detailed account on the procedure for Stein's method under more general setup, we refer the readers to Goldstein and Reinert (2005, 2013), Ley, Reinert and Swan (2014), Barbour and Chen (2014) and references therein.

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For some standard distributions, a Stein operator can be established easily.

Indeed, let $\mu_j := P(Y = j) > 0, j \in \mathbb{Z}_+$. Then

$\sum_{j=0}^{\infty} \mu_j \left(\frac{(j+1)\mu_{j+1}}{\mu_j} g(j+1) - jg(j) \right) = 0$. Therefore,

$$(\mathcal{A}g)(j) = \frac{(j+1)\mu_{j+1}}{\mu_j} g(j+1) - jg(j), \quad j \in \mathbb{Z}_+, \quad (3)$$

and it can be easily verified the $\mathbb{E}(\mathcal{A}g)(Y) = 0$.

Some well-known examples are listed below.

EXAMPLE 1 (POISSON DISTRIBUTION)

For $\alpha > 0$, let Y_1 be a Poisson $P(\alpha)$ rv with $\mu_j = \alpha^j e^{-\alpha} / j!$. Then

$$(\mathcal{A}g)(j) = \alpha g(j+1) - jg(j), \quad j \in \mathbb{Z}_+. \quad (4)$$

EXAMPLE 2 (PSEUDO-BINOMIAL DISTRIBUTION)

Let $0 < p < 1$, $q = 1 - p$, $\tilde{M} > 1$, and Y_2 have pseudo-binomial distribution (see Čekanavičius and Roos (2004), p. 370) so that

$$\mu_j = \frac{1}{\tilde{C}} \binom{\tilde{M}}{j} p^j q^{\tilde{M}-j}, \quad j \in \{0, 1, \dots, \lfloor \tilde{M} \rfloor\},$$

where $\tilde{C} = \sum_{j=0}^{\lfloor \tilde{M} \rfloor} \binom{\tilde{M}}{j} p^j q^{\tilde{M}-j}$, $\lfloor \tilde{M} \rfloor$ denotes integer part of $\tilde{M}$ and $\binom{\tilde{M}}{j} = \frac{\tilde{M}(\tilde{M}-1)\dots(\tilde{M}-j+1)}{j!}$. If $\tilde{M}$ is an integer, then Y_2 is a binomial rv. Suppose now $g(0) = 0$ and $g(\lfloor \tilde{M} \rfloor + 1) = g(\lfloor \tilde{M} \rfloor + 2) = \dots = 0$. Then, from (3)

$$(\mathcal{A}g)(j) = \frac{(\tilde{M} - j)p}{q} g(j+1) - jg(j), \quad j = 0, 1, \dots, \lfloor \tilde{M} \rfloor.$$

EXAMPLE 2 CONTD.

Multiplying the above expression by q , we can get the following Stein operator:

$$(\mathcal{A}g)(j) = (\tilde{M} - j)pg(j+1) - jqg(j), \quad j = 0, 1, \dots, \lfloor \tilde{M} \rfloor. \quad (5)$$

EXAMPLE 3 (NEGATIVE BINOMIAL DISTRIBUTION)

Let $Y_3 \sim \text{NB}(r, \bar{p})$, $0 < \bar{p} < 1$, be negative binomial rv with

$\mu_j = \Gamma(r+j)/(\Gamma(r)j!) \bar{p}^r \bar{q}^j$, for $j \in \mathbb{Z}_+$, $r > 0$ and $\bar{q} = 1 - \bar{p}$. Then (3) reduces to

$$(\mathcal{A}g)(j) := \bar{q}(r+j)g(j+1) - jg(j), \quad j \in \mathbb{Z}_+. \quad (6)$$

REMARK

Observe that (3) is not very useful if we do not have simple expressions for μ_j .

For example, if we consider compound distribution or convolution of two or more distributions, then μ_j 's are usually expressed through sums or converging series of probabilities.

Therefore, some other refined approaches for obtaining Stein operator(s) are needed.

PGF APPROACH: GENERAL IDEA

Let N be a $\mathbb{Z}_+$ -valued rv with $\mu_k = P(N = k)$ and finite mean. Then its *pgf*

$$G_N(z) = \sum_{k=0}^{\infty} \mu_k z^k \quad (7)$$

satisfies

$$G'_N(z) = \frac{d}{dz} G_N(z) = \sum_{k=1}^{\infty} k \mu_k z^{k-1} = \sum_{k=0}^{\infty} (k+1) \mu_{k+1} z^k. \quad (8)$$

If we can express $G'_N(z)$ through $G_N(z)$ then, by collecting factors of z^k , the recursion follows.

COMPOUND POISSON DISTRIBUTION

Let $\{X_j\}$ be an iid sequence of random variables with $P(X_j = k) = p_k$, $k = 0, 1, 2, \dots$. Also, let $N \sim P(\lambda)$ and N be independent of the $\{X_j\}$. Then the distribution of $Y_4 := \sum_{j=1}^N X_j$ is known as compound Poisson distribution with *pgf*

$$G_{cp}(z) = \exp\left\{\sum_{j=1}^{\infty} \lambda_j (z^j - 1)\right\}, \quad (9)$$

where $\lambda_j = \lambda p_j$ and $\sum_{j=1}^{\infty} j|\lambda_j| < \infty$.

COMPOUND POISSON DISTRIBUTION CONTD.

Then

$$\begin{aligned}
 G'_{cp}(z) &= G_{cp}(z) \sum_{j=1}^{\infty} j \lambda_j z^{j-1} \\
 &= \sum_{k=0}^{\infty} \mu_k z^k \sum_{j=1}^{\infty} j \lambda_j z^{j-1} = \sum_{k=0}^{\infty} z^k \sum_{m=0}^k \mu_m (k - m + 1) \lambda_{k-m+1}.
 \end{aligned}$$

Comparing the last expression to the right-hand side of (8), we obtain recursive relation, for all $k \in \mathbb{Z}_+$, as

$$\sum_{m=0}^k \mu_m (k - m + 1) \lambda_{k-m+1} - (k + 1) \mu_{k+1} = 0.$$

COMPOUND POISSON DISTRIBUTION CONTD.

Then, we have

$$\begin{aligned}
 0 &= \sum_{k=0}^{\infty} g(k+1) \left[\sum_{m=0}^k \mu_m (k-m+1) \lambda_{k-m+1} - (k+1) \mu_{k+1} \right] \\
 &= \sum_{m=0}^{\infty} \mu_m \left[\sum_{k=m}^{\infty} g(k+1) (k-m+1) \lambda_{k-m+1} - mg(m) \right] \\
 &= \sum_{m=0}^{\infty} \mu_m \left[\sum_{j=1}^{\infty} j \lambda_j g(j+m) - mg(m) \right].
 \end{aligned}$$

COMPOUND POISSON DISTRIBUTION CONTD.

Therefore, a Stein operator for the compound Poisson distribution, defined in (9), is

$$\begin{aligned}
 (\mathcal{A}g)(j) &= \sum_{l=1}^{\infty} l\lambda_l g(j+l) - jg(j) \\
 &= \sum_{l=1}^{\infty} l\lambda_l g(j+1) - jg(j) + \sum_{m=2}^{\infty} m\lambda_m \sum_{l=1}^{m-1} \Delta g(j+l), \quad j \in \mathbb{Z}_+, \quad (10)
 \end{aligned}$$

since $\mathbb{E}(\mathcal{A}g)(Y_4) = 0$. This operator coincides with the one from Barbour, Chen and Loh (1992a).

CONVOLUTIONS AND STEIN OPERATORS

Recall that $Y_1 \sim P(\alpha)$ ($\alpha > 0$), $Y_2 \sim Bi(M, p)$ ($M \in \mathbb{N}$, $0 < p < 1$), $Y_3 \sim NB(r, \bar{p})$ ($0 < \bar{p} < 1$, $r > 0$) and Y_4 follows the compound Poisson distribution defined in (9). We assume that Y_1, Y_2, Y_3 and Y_4 are independent. Then the *pgf*'s of $Y_1 + Y_2$, Y_2 and Y_3 are given by

$$G_{12}(z) = (q + pz)^M \exp\{\alpha(z-1)\}, \quad G_2(z) = (q + pz)^M, \quad G_3(z) = \left(\frac{\bar{p}}{1 - \bar{q}z} \right)^r, \quad (11)$$

respectively. Here $\bar{q} = 1 - \bar{p}$ and $q = 1 - p$.

THEOREM 1.1

Let $G_{cp}(z)$ be the pgf of Y_4 , $g : \mathbb{Z}_+ \rightarrow \mathbb{R}$ be a bounded function and

$\lambda = \sum_{m=1}^{\infty} j\lambda_j$. Then we have the following results:

(i) The rv $Y_{24} = Y_2 + Y_4$ has the pgf $G_2(z)G_{cp}(z)$ and its Stein operator is

$$\begin{aligned} (\mathcal{A}g)(j) &= \left(M + \frac{\lambda}{p} - j\right)pg(j+1) - qjg(j) \\ &\quad + \sum_{m=2}^{\infty} (qm\lambda_m + p(m-1)\lambda_{m-1}) \sum_{l=1}^{m-1} \Delta g(j+l). \end{aligned} \quad (12)$$

(ii) The rv $Y_{34} = Y_3 + Y_4$ has the pgf $G_3(z)G_{cp}(z)$ and a corresponding Stein operator is

$$\begin{aligned} (\mathcal{A}g)(j) &= \left(\frac{\lambda\bar{p}}{\bar{q}} + r + j\right)\bar{q}g(j+1) - jg(j) \\ &\quad + \sum_{m=2}^{\infty} (m\lambda_m - \bar{q}(m-1)\lambda_{m-1}) \sum_{l=1}^{m-1} \Delta g(j+l). \end{aligned} \quad (13)$$

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BINOMIAL-POISSON

THEOREM 1.2

Let $Y_{12} = Y_1 + Y_2$ have pgf $G_{12}(z)$ as defined in (11). Then, for all $j \in \mathbb{Z}_+$ and bounded functions $g : \mathbb{Z}_+ \rightarrow \mathbb{R}$, a Stein operator for Y_{12} is

$$(\mathcal{A}g)(j) = (Mp + \alpha - jp)g(j+1) - jqg(j) + p\alpha\Delta g(j+1). \quad (14)$$

If, in addition $p/q < 1$, then

$$(\mathcal{A}g)(j) = (\alpha + Mp)g(j+1) - jg(j) + M \sum_{l=2}^{\infty} (-1)^{l+1} \left(\frac{p}{q}\right)^l \sum_{k=1}^{l-1} \Delta g(j+k). \quad (15)$$

REMARK 1.1

(i) As is known in the literature (see Goldstein and Reinert (2005), Ley, Reinert and Swan (2014)), we have two significantly different Stein operators (see (14) and (15)) for the approximation problem.

(ii) Observe that, the operator given in (14) is similar to the operator given in (5), where $\tilde{M}$ is replaced by $M + \alpha/p$, except for the last term, and hence is known as a binomial perturbation.

(iii) Similarly, the operator given in (15) is similar to the operator given in (4), where α is replaced by $Mp + \alpha$, except for the last sum, leading to a Poisson perturbation.

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BINOMIAL-NEGATIVE BINOMIAL

We consider convolution of negative binomial and binomial distributions. It is logical to use the binomial approximation for sums of rv's with variances smaller than their means and the negative binomial approximation if variances are larger than means. Therefore, one can expect that the sum of binomial and negative binomial rv as universal discrete approximation.

BINOMIAL-NEGATIVE BINOMIAL

THEOREM 1.3

Let $Y_{23} = Y_2 + Y_3$ have pgf $G_{23}(z) = G_2(z)G_3(z)$ and $(p/q) < 1$. Then, for $j \in \mathbb{Z}_+$ and $g \in \mathcal{F}$, the rv Y_{23} has the following Stein operators:

$$\begin{aligned} (\mathcal{A}_1 g)(j) &= (Mp + rq\bar{q} - pj + q\bar{q}j)g(j+1) \\ &\quad + (r\bar{q}p - Mp\bar{q} + p\bar{q}j)g(j+2) - qjg(j), \end{aligned} \quad (16)$$

$$\begin{aligned} (\mathcal{A}_2 g)(j) &= p\left(\frac{r\bar{q}}{p\bar{p}} + M - j\right)g(j+1) - qjg(j) \\ &\quad + r(q\bar{q} + p) \sum_{m=2}^{\infty} \bar{q}^{m-1} \sum_{l=1}^{m-1} \Delta g(j+l), \end{aligned} \quad (17)$$

BINOMIAL-NEGATIVE BINOMIAL

$$\begin{aligned}
 (\mathcal{A}_3 g)(j) &= \bar{q} \left(\frac{Mp\bar{p}}{\bar{q}} + r + j \right) g(j+1) - jg(j) \\
 &\quad + M \left(\frac{p}{q} + \bar{q} \right) \sum_{m=2}^{\infty} (-1)^{m+1} \left(\frac{p}{q} \right)^{m-1} \sum_{l=1}^{m-1} \Delta g(j+l), \quad (18)
 \end{aligned}$$

$$\begin{aligned}
 (\mathcal{A}_4 g)(j) &= \left(Mp + \frac{r\bar{q}}{\bar{p}} \right) g(j+1) - jg(j) \\
 &\quad + \sum_{m=2}^{\infty} \left(M(-1)^{m+1} \left(\frac{p}{q} \right)^m + r\bar{q}^m \right) \sum_{l=1}^{m-1} \Delta g(j+l). \quad (19)
 \end{aligned}$$

BINOMIAL-NEGATIVE BINOMIAL

REMARK 1.2

As discussed earlier, the operators $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ are binomial, negative binomial and Poisson perturbations, respectively. Note, however, $\mathcal{A}_1$ can not be seen as a perturbation operator.

SOLUTION TO STEIN EQUATION

Let us discuss some known facts and explore the properties of exact and approximate solutions to the Stein equation.

Assume that Y and Z are rvs concentrated on $\mathbb{Z}_+$ and $f, g \in \mathcal{F}$

Henceforth, $\|f\| = \sup_k |f(k)|$. The second step in Stein's method is solving the equation (1).

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Suppose a Stein operator is given by

$$(\mathcal{A}g)(j) = \alpha_j g(j+1) - \beta_j g(j), \quad (20)$$

where $\beta_0 = 0$ and $\alpha_k - \alpha_{k-1} \leq \beta_k - \beta_{k-1}$ ($k = 1, 2, \dots$), then the solution to (1) satisfies

$$|\Delta g(j)| \leq 2\|f\| \min \left\{ \frac{1}{\alpha_j}, \frac{1}{\beta_j} \right\}. \quad j \in \mathbb{Z}_+, f \in \mathcal{F}. \quad (21)$$

This can be seen as follows.

Let g_i be a solution to (1) with the choice $f(j) = I(j = i)$, where $I(A)$ denotes the indicator function of A . Then, from (2.18) and Theorem 2.10 of Brown and Xia (2001), we have

$$|\Delta g(i)| = \left| \sum_{j=0}^{\infty} f(j) \Delta g_j(i) \right| \leq \sup_{j \geq 0} f(j) |\Delta g_j(i)| \leq \sup_{j \geq 0} f(j) \min\{\alpha_i^{-1}, \beta_i^{-1}\},$$

for any nonnegative functions f .

The proof of (21) can now be completed by following steps similar to that of Lemma 2.2 from Barbour (1987).

If $f : \mathbb{Z}_+ \rightarrow [0, 1]$, then $2\|f\|$ in (21) should be replaced by 1.

Note that different choices of f lead to different probabilistic metrics.

In this talk, we consider total variation norm which is twice the total variation metric. That is,

$$\begin{aligned}\|\mathcal{L}(Y) - \mathcal{L}(Z)\|_{TV} &= \sum_{j=0}^{\infty} |P(Y=j) - P(Z=j)| = \sup_{\|f\| \leq 1} |\mathbb{E}f(Y) - \mathbb{E}f(Z)| \\ &= 2 \sup_{f \in \mathcal{F}_1} |\mathbb{E}f(Y) - \mathbb{E}f(Z)| = 2 \sup_A |P(Y \in A) - P(Z \in A)|,\end{aligned}$$

where $\mathcal{F}_1 = \{f|f : \mathbb{Z}_+ \rightarrow [0, 1]\}$, and the supremum is taken over all Borel sets in the last equality.

Let g be the solution to (1) for Poisson or negative binomial or pseudo-binomial rv with the Stein operator given by (4) or (6) or (5), respectively. Then the corresponding bounds are given respectively as

$$\|\Delta g\| \leq \frac{2\|f\|}{\max(1, \lambda)}, \quad \|\Delta g\| \leq \frac{2\|f\|}{r\bar{q}}, \quad \|\Delta g\| \leq \frac{2\|f\|}{[\tilde{N}]pq}. \quad (22)$$

The first two estimates follow directly from (21). Observe that for pseudo-binomial distribution, the assumptions of (21) are not always satisfied. The last estimate of (22) follows from Lemma 9.2.1 in Barbour, Holst and Janson (1992b), and using similar arguments as above.

If a Stein operator has a form different from (20), then solving (1) and checking for properties similar to (21) becomes rather dubious.

In such situations, one can try perturbation technique introduced in Barbour and Xia (1999) and further developed in Barbour and Čekanavičius (2002) and Barbour, Čekanavičius and Xia (2007).

Roughly, the basic idea of perturbation can be summarized in the following way: good properties of the solution of (1) can be carried over to solutions of Stein operators in similar forms.

Next, we formulate a partial case of Lemma 2.3 and Theorem 2.4 from Barbour, Čekanavičius and Xia (2007) under the following setup.

Let $\mathcal{A}_0$ be a Stein operator for some distribution with support $\{0, 1, 2, \dots, T\}$ ($T = \infty$ is allowed) and

g_0 be the solution of (1) with $\mathcal{A}$ replaced by $\mathcal{A}_0$.

Also, let there exist $\omega_1, \gamma > 0$ such that

$$\|\Delta g_0\| \leq \omega_1 \|f\| \min(1, \gamma^{-1}).$$

LEMMA 2.1

Let H be a (signed) measure with Stein operator $\mathcal{A} = \mathcal{A}_0 + U$, W be any rv, both concentrated on $\mathbb{Z}_+$ and $H(\mathbb{Z}_+) = 1$. For $g \in \mathcal{F}$, there exist $\omega_2, \varepsilon > 0$ such that

$$\|Ug\| \leq \omega_2 \|\Delta g\|, \quad |\mathbb{E}(\mathcal{A}g)(W)| \leq \varepsilon \|\Delta g\|$$

and $\omega_1 \omega_2 < \gamma$, then

$$\|\mathcal{L}(W) - H\|_{TV} \leq \frac{\gamma}{\gamma - \omega_1 \omega_2} \left(\varepsilon \omega_1 \min(1, \gamma^{-1}) + 2\eta + 2P(W > T) \right),$$

where $\eta = \sum_{j=T+1}^{\infty} |H\{j\}|$.

Next, using the assumptions of Lemma 2.1 and (22), we evaluate the values of ω_1 , ω_2 and γ to the various Stein operators derived in Section 1. Our observations are as follows:

COMPOUND POISSON

If a Stein operator is given by

$$\begin{aligned} (\mathcal{A}g)(j) &= \sum_{l=1}^{\infty} l\lambda_l g(j+l) - jg(j) \\ &= \sum_{l=1}^{\infty} l\lambda_l g(j+1) - jg(j) + \sum_{m=2}^{\infty} m\lambda_m \sum_{l=1}^{m-1} \Delta g(j+l), \quad j \in \mathbb{Z}_+, \end{aligned}$$

then we have the Poisson perturbation, $\omega_1 = 2$, $\gamma = \sum_{m=1}^{\infty} m\lambda_m$,

$$\|Ug\| \leq \|\Delta g\| \sum_{m=2}^{\infty} m(m-1)|\lambda_m| = \|\Delta g\|_{\omega_2}$$

and $\omega_1\omega_2 < \gamma$, provided $\{\lambda_m\}_{m \geq 2}$ is sufficiently small.

For a general description of the problem, see Barbour, Chen and Loh (1992a).

BINOMIAL-POISSON

For a Stein operator given by

$$(\mathcal{A}g)(j) = (Mp + \alpha - jp)g(j+1) - jqg(j) + p\alpha\Delta g(j+1). \quad (23)$$

we have pseudo-binomial perturbation, $\omega_1 = 2/pq$, $\gamma = \lfloor M + \alpha/p \rfloor$, $\omega_2 = p\alpha$ and $\omega_1\omega_2 < \gamma$, if p is sufficiently small.

BINOMIAL-POISSON

Consider the Stein operator given by

$$(\mathcal{A}g)(j) = (\alpha + Mp)g(j+1) - jg(j) + M \sum_{l=2}^{\infty} (-1)^{l+1} \left(\frac{p}{q}\right)^l \sum_{k=1}^{l-1} \Delta g(j+k). \quad (24)$$

(15), then we have Poisson perturbation, $\omega_1 = 2$, $\gamma = Mp + \alpha$,

$\omega_2 = Mp^2/(q-p)^2$ and $\omega_1\omega_2 < \gamma$, whenever p is sufficiently small (see

Theorem 3.1).

BINOMIAL-NEGATIVE BINOMIAL

For the Stein operator given by

$$\begin{aligned}
 (\mathcal{A}_2 g)(j) &= p \left(\frac{r\bar{q}}{p\bar{p}} + M - j \right) g(j+1) - qjg(j) \\
 &\quad + r(q\bar{q} + p) \sum_{m=2}^{\infty} \bar{q}^{m-1} \sum_{l=1}^{m-1} \Delta g(j+l),
 \end{aligned}$$

(17), we have pseudo-binomial perturbation, $\omega_1 = 2$, $\gamma = \lfloor M + r\bar{q}/(p\bar{p}) \rfloor pq$ and $\omega_2 = \frac{r\bar{q}(q\bar{q}+p)}{\bar{p}^2}$. The condition $\omega_1\omega_2 < \gamma$ is satisfied if p and $\bar{q}$ are sufficiently small.

BINOMIAL-NEGATIVE BINOMIAL

If the Stein operator is given by

$$\begin{aligned}
 (\mathcal{A}_3 g)(j) &= \bar{q} \left(\frac{Mp\bar{p}}{\bar{q}} + r + j \right) g(j+1) - jg(j) \\
 &\quad + M \left(\frac{p}{q} + \bar{q} \right) \sum_{m=2}^{\infty} (-1)^{m+1} \left(\frac{p}{q} \right)^{m-1} \sum_{l=1}^{m-1} \Delta g(j+l),
 \end{aligned}$$

(18), then we have negative binomial perturbation, $\omega_1 = 2$, $\gamma = Mp\bar{p} + r\bar{q}$, $\omega_2 = Mpq(p/q + \bar{q})(q-p)^{-2}$ and $\omega_1\omega_2 < \gamma$, provided p and $\bar{q}$ are sufficiently small.

BINOMIAL-NEGATIVE BINOMIAL

Finally, consider the Stein operator given by

$$\begin{aligned}
 (\mathcal{A}_4 g)(j) &= \left(Mp + \frac{r\bar{q}}{\bar{p}} \right) g(j+1) - jg(j) \\
 &\quad + \sum_{m=2}^{\infty} \left(M(-1)^{m+1} \left(\frac{p}{q} \right)^m + r\bar{q}^m \right) \sum_{l=1}^{m-1} \Delta g(j+l).
 \end{aligned}$$

(19), then we have Poisson perturbation, $\omega_1 = 2$, $\gamma = Mp + r\bar{q}/\bar{p}$,

$\omega_2 = Mp^2/(q-p)^2 + r\bar{q}^2/\bar{p}^2$ and $\omega_1\omega_2 < \gamma$, whenever p and $\bar{q}$ are sufficiently small.

REMARKS

- (i) Note that, for the Stein operator in (16), no perturbation seems to be available.
- (ii) We also remark here that once a Stein operator is derived, the properties of the associated exact “solution to the Stein equation” must be derived and this can be quite difficult.

The perturbation approach, as discussed in some examples above, can be useful to get an upper bound on approximate solution to the Stein equation.

APPROXIMATION TO SUMS OF INDICATOR VARIABLES

Let us investigate the effect of different forms of Stein operator on its estimates.

SETUP

Consider the sum $W = \sum_{i=1}^n \mathbb{I}_i$ of possibly dependent indicator variables and let $W^{(i)} = W - \mathbb{I}_i$, $P(\mathbb{I}_i = 1) = p_i = 1 - P(\mathbb{I}_i = 0) = 1 - q_i$ ($i = 1, 2, \dots, n$). Assume also $\tilde{W}^{(i)}$ satisfy $P(\tilde{W}^{(i)} = k) = P(W^{(i)} = k | \mathbb{I}_i = 1)$, for all k .

We choose $Y_{12} = Y_1 + Y_2$ as the approximating variable, where $Y_1 \sim P(\alpha)$, $Y_2 \sim Bi(M, p)$ and are independent. Denote its distribution by BCP whose *pgf* is given in (11).

CHOICE OF PARAMETERS.

Note that BCP is a three-parametric distribution.

We choose the parameters p , M and α to ensure the almost matching of the first three moments of W .

Denoting as before the integral part by $\lfloor \cdot \rfloor$, we define

$$M := \left\lfloor \left(\sum_{i=1}^n p_i^2 \right)^3 \left(\sum_{i=1}^n p_i^3 \right)^{-2} \right\rfloor, \quad (25)$$

$$\delta := \left(\sum_{i=1}^n p_i^2 \right)^3 \left(\sum_{i=1}^n p_i^3 \right)^{-2} - M, \quad 0 \leq \delta < 1, \quad (26)$$

$$p := \left(\sum_{i=1}^n p_i^3 \right) \left(\sum_{i=1}^n p_i^2 \right)^{-1}; \quad \alpha := \sum_{i=1}^n p_i - Mp. \quad (27)$$

Then the following relations hold:

$$Mp^2 = \sum_{i=1}^n p_i^2 - \delta p^2, \quad Mp^3 = \sum_{i=1}^n p_i^3 - \delta p^3. \quad (28)$$

Observe that

$$\left(\sum_{i=1}^n p_i^2 \right)^2 \leq \sum_{i=1}^n p_i \sum_{i=1}^n p_i^3.$$

Therefore, for $\alpha > 0$, the BCP is not a signed measure, but a distribution.

Observe that α and Mp can be of the same order. Indeed, let n be even and

$p_1 = p_2 = \dots = p_{n/2} = 1/6$, $p_{n/2+1} = \dots = p_n = 1/12$, we have

$Mp = O(n) = \alpha$.

POISSON PERTURBATION.

Let I_1 and I denote the degenerate distributions concentrated at 1 and 0, respectively. The convolution operator is denoted by $*$. Also, let

$$d := \left\| \mathcal{L}(W) * (I_1 - I)^{*2} \right\|_{TV} = \sum_{k=0}^n |\Delta^2 P(W = k)|, \quad (29)$$

$$d_1 := \max_i \left\| \mathcal{L}(W^{(i)}) * (I_1 - I)^{*2} \right\|_{TV} = \max_i \sum_{k=0}^n |\Delta^2 P(W^{(i)} = k)|, \quad (30)$$

$$\hat{\lambda} = \sum_{i=1}^n p_i, \quad \sigma^2 = \sum_{i=1}^n p_i q_i, \quad \tau = \max_i p_i q_i,$$

$$\eta_1 := \sum_{i=1}^n p_i (1 + 2p_i + 4p_i^2) \mathbb{E} |\tilde{W}^{(i)} - W^{(i)}|,$$

$$\theta_1 := \frac{Mp^2}{(1-2p)^2(Mp + \alpha)} = \frac{\sum_{i=1}^n p_i^2 - \delta p^2}{(1-2p)^2 \sum_{i=1}^n p_i}. \quad (31)$$

THEOREM 3.1

Let $\max(p, \theta_1) < 1/2$. Then

$$\|\mathcal{L}(W) - \text{BCP}\|_{TV} \leq \frac{2}{(1 - 2\theta_1)\hat{\lambda}} \left\{ d_1 \sum_{i=1}^n p_i^4 + \frac{dMp^4}{(1 - 2p)^2} + (1 + 2p)\delta p^2 + \eta_1 \right\}.$$

COROLLARY 3.2

Let W be the sum of n independent Bernoulli rvs, $\max(p, \theta_1) < 1/2$ and $\sigma^2 > 3\tau$. Then

$$\begin{aligned} \|\mathcal{L}(W) - \text{BCP}\|_{TV} \\ \leq \frac{2}{(1 - 2\theta_1)\hat{\lambda}} \left\{ \frac{2 \sum_{i=1}^n p_i^4}{\sqrt{(\sigma^2 - \tau)(\sigma^2 - 3\tau)}} + \frac{2Mp^4}{(1 - 2p)^2 \sigma \sqrt{\sigma^2 - \tau}} + (1 + 2p)\delta p^2 \right\}. \end{aligned} \quad (32)$$

REMARK 3.1

(i) Observe that $\theta_1 < p(1 - 2p)^{-2} \leq \max_i p_i(1 - 2 \max_i p_i)^{-2}$. Therefore, a sufficient condition for $\max(p, \theta_1) < 1/2$ is

$$\max_i p_i < (3 - \sqrt{5})/4 = 0.19098 \dots$$

(ii) If all $p_i \asymp C$, then the order of accuracy in (32) is $O(n^{-1})$.

(iii) Also, one can compare (32) with the classical Poisson approximation result (see, Chen and Röllin (2013) eq. (1.1)-(1.2)), where for $p_i \asymp C$ and the order of accuracy is $O(1)$.

Thank You

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