Malliavin-Stein method for Variance-Gamma approximation on Wiener space

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Singapore, New Directions in Stein's Method, May 2015

Menu

- **why Variance-Gamma ?**
- **a 6 moment theorem via Malliavin-Stein**
- **extensions**

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- **extensions**

joint work with Christoph Thäle (RUB)

LINDEBERG-method

(more than) 90 years LINDEBERG-method: **replacement trick/ swapping trick**

renaissance in random matrix theory and for other models
(CHATTERJEE, TAO, VU,...)

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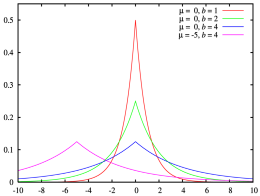
survey article with M. LÖWE, 2013
in progress: use of the method for moderate deviations questions...

A. TODA, 2012: $(X_j)_j$ independent, $\mathbb{E}(X_j) = 0$, $\mathsf{V}(X_j) = 1$
ν_p geometric random variable with mean $1/p$, independent of X_j's

$$p^{1/2} \sum_{j=1}^{\nu_p} X_j \to L(0, 1/\sqrt{2}) \quad (p \to 0)$$

Laplace-distribution



density of the Laplace-distribution $L(0, b)$ with parameter b:

$$p_b(x) = \frac{1}{2b} \exp\left(-\frac{|x|}{b}\right), \quad 2b^2 = \sigma^2$$

Stein-**characterization**:

$$\mathbb{E}\big[b^2 X\, f''(X) + 2b^2\, f'(X) - X\, f(X)\big] = 0 \quad \ldots$$

LAPLACE-distribution, double exponential

- weak limit of the geometric sum of independent but not identically distributed random variables
- **fatter tails** than the normal distribution
- the difference between two independent identically distributed **exponential** (λ)-random variables is $L(0, \lambda^{-1})$
- consider $\lambda = \frac{1}{2}$: **exponential** $(\frac{1}{2}) = \chi_2^2 = \Gamma_{1/2,1}$

$$\mathbf{L(0,2) = G_1^2 + G_2^2 - G_3^2 - G_4^2}$$

U-statistics

$$U_n(h_n) := \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} h_n(X_i, X_j)$$

examples: **homogenous sums**, quadratic forms:

$$Q_n(f, X) := \sum_{1 \le i,j \le n} f_n(i,j) X_i X_j$$

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HOEFFDING-decomposition:

$$\frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} h(X_i, X_j) = \frac{2}{n} \sum_{i=1}^{n} g(X_i) + \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \eta(X_i, X_j)$$

$$g(x) := \mathbb{E}[h(x, X_2)] \quad \eta(x, y) := h(x, y) - g(x) - g(y) + \mathbb{E}[h(X_1, X_2)]$$

U-statistics, LAPLACE-distribution

degenerated case: $g(x) = 0$

example: $h(x, y) = x\,y$

$$n\,U_n(h) \xrightarrow{d} \sum_j \lambda_j(h)(\mathbf{G}_j^2 - 1)$$

SERFLING, 1980; RUBIN, VITALE, 1980

the LAPLACE-distribution appears as a limit of degenerate U-statistics (**!**)

the question

does the LAPLACE-distribution appear as the limit of a sequence of WIENER-ITÔ **integrals** $I_q(f_n)$ of fixed order q ?

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Double WIENER-ITÔ integrals, $q = 2$:
$(B_t)_{t\in\mathbb{R}_+}$ Brownian motion

$$\int_0^\infty f(t)dB_t =: I_1(f), \quad f \in L^2(\mathbb{R}_+)$$

$$I_2(f) = \int_{[0,\infty)^2} f(t,s)\,dB_t dB_s = 2\int_0^\infty dB_t \int_0^t dB_s\, f(t,s), \quad f \in L^2_s(\mathbb{R}^2_+)$$

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fact:

$$I_2(f) = \sum_{j=1}^\infty \lambda_j(f)(\mathbf{G}_j^2 - 1) \quad \text{in law}$$

$(\mathbf{G}_j)_j$ independent $N(0,1)$, convergence in $L^2(\Omega)$ and a.s.

convergence in law: double integrals

NOURDIN, POLY, 2012:

Let $(F_n)_n$ be a sequence of double WIENER-ITÔ integrals that converges in law to F_∞.

- Then there exists $\lambda_0 \in \mathbb{R}$, $f \in L^2_s(\mathbb{R}^2_+)$ such that

$$F_\infty = N(0, \lambda_0) + I_2(f)$$

-

$$F_n \to L(0, b) \text{ in law } \Leftrightarrow \kappa_i(F_n) \to \kappa_i(L(0, b)) \text{ for } i = 2, 4, 6$$

  $\kappa_i(X)$: i'th **cumulant**

- their result is much more general...

convergence in law: multiple integrals

a nice observation by KUSUOKA, TUDOR, 2013:

within the PEARSON class of probability distributions:

the only possible limits of sequences of multiple integrals $(I_q(f_n))_n$ are the **Gaussian law** and the **Gamma law**

$$\text{density} \quad p : \quad \frac{p'(x)}{p(x)} = \frac{a + x}{b_2 x^2 + b_1 x + b_0}$$

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LAPLACE-approximation for $I_q(f)$? Error bounds ?

Gaussian analysis

- $\mathcal{H}_q$: closed linear subspace of $L^2(\Omega)$ generated by $H_q(I_q(f))$
  H_q: q'th HERMITE polynomial

- $$L^2(\Omega) = \bigoplus_{q\geq 0} \mathcal{H}_q \qquad L^2(\Omega) \ni F = \mathbb{E}(F) + \sum_{q\geq 0} I_q(f_q)$$
  $$I_q(f^{\otimes q}) = H_q\Big(\int_0^\infty f\, dB_t\Big)$$

- more general: $X = \{X(h)\}_{h\in\mathcal{H}}$: **isonormal Gaussian process** over $\mathcal{H}$
  $$\mathbb{E}\big(X(f)\,X(g)\big) = \langle f, g\rangle_{\mathcal{H}}, \qquad \text{covariance}$$
  $$\mathbb{E}\big(I_p(f)\,I_q(g)\big) = p!\langle f, g\rangle_{\mathcal{H}} 1_{\{p=q\}}, \qquad \textbf{isometry property}$$

4 moment theorem

Nourdin, Peccati, 2009: let $\mathbb{E}(I_q(f)^2) = 1$

$$d_K(I_q(f), N) \leq C\left(\mathbb{E}\left(\frac{1}{q}\|DI_q\|^2 - 1\right)^2\right)^{1/2} \leq \left(\frac{q-1}{3q}|\mathbb{E}(I_q(f)^4) - 3|\right)^{1/2}$$

$D_t I_q(f) := qI_{q-1}(f(\cdot, t))$ **Malliavin**-derivative

Stein: $F := I_q(f)$

$$\mathbb{E}(Ff(F)) = \mathbb{E}\big(f'(F)\frac{1}{q}\langle DF, DF\rangle\big) \qquad \text{integration-by-parts}$$

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product-formula and **isometry property**:

$$\mathbb{E}\left(\frac{1}{q}\|DF\|^2 - 1\right)^2 = \sum_{r=1}^{q-1} c(r,q)\|f\widetilde{\otimes}_r f\|^2$$

Stein for Laplace

$$\frac{p_b'(x)}{p_b(x)} = \frac{1}{b}\,\mathrm{sign}(x)$$

Pike, Ren, 2013: $\mathbb{E}(f''(X)) = \frac{1}{b^2}\mathbb{E}(f(X) - f(0))$
application: geometric sums, see also Döbler, 2013

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Malliavin: $f(x) \to xf(x)$:

$$\mathbb{E}\big[b^2 X f''(X) + 2b^2 f'(X) - Xf(X)\big] = 0 \qquad \textbf{Stein}\text{-characterization}$$

this is exactly the characterization in Gaunt, 2014

integration-by-parts

$F := I_q(f)$

$$\left|\mathbb{E}\left[b^2 F f''(F) + 2b^2 f'(F) - F f(F)\right]\right| \leq \textbf{?}$$

apply

$$\mathbb{E}\left[H\,G\right] = \mathbb{E}\left[H\right]\mathbb{E}\left[G\right] + \mathbb{E}\left[\langle DH, -DL^{-1}G\rangle\right]$$

$$\begin{aligned}\mathbb{E}\left[F f(F)\right] &= \mathbb{E}\left[f'(F)\Gamma_2(F)\right] \\ &= \mathbb{E}\left[f'(F)\right]\mathbb{E}\left[\Gamma_2(F)\right] + \mathbb{E}\left[f''(F)\Gamma_3(F)\right]\end{aligned}$$

with $\Gamma_2(F) := \langle DF, -DL^{-1}F\rangle = \frac{1}{q}\langle DF, DF\rangle$ and $\Gamma_3(F) := \langle DF, -DL^{-1}\Gamma_2(F)\rangle$

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$$\left|\mathbb{E}\left[\underline{b^2 F f''(F)} + 2b^2 f'(F) - \underline{F f(F)}\right]\right| \leq \textbf{?}$$

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$\Rightarrow$ try to bound $\mathbb{E}(b^2 F - \Gamma_3(F))^2$

Variance-Gamma distributions

remark: $\kappa_3(F) = 2\mathbb{E}\big[\Gamma_3(F)\big]$

Laplace-distribution is a member of the three-parameter family of Variance-Gamma distributions:

Gaunt, 2014

$$\Big|\mathbb{E}\big[\sigma^2(F+r\theta)f''(F) + (\sigma^2 r + 2\theta(F+r\theta))f'(F) - Ff(F)\big]\Big| \leq \textbf{?}$$

$$\theta V + \sigma\sqrt{V}U$$

with $U \sim N(0,1)$, $V \sim \Gamma(r/2, 1/2)$, $r > 0, \sigma > 0$ and $\theta \in \mathbb{R}$

conditioning on V the random variable is $N(0, \sigma^2 V)$

examples

- symmetrized Gamma distribution
- product of two normal distributed random variables, correlated
- difference of two correlated Gamma distributions

results

remark: Azmoodeh, Peccati, Poly, 2014; finite linear combination of centered χ^2-distributions

Theorem

Let Y be **Variance-Gamma** *distributed, $F = I_q(f)$*

$$d_W(F, Y) \leq C_1 \mathbb{E}\left|\sigma^2(F + r\theta) + 2\theta\Gamma_2(F) - \Gamma_3(F)\right| + C_2\left|r\sigma^2 + 2r\theta^2 - \mathbb{E}(\Gamma_2(F))\right|$$

Theorem

Y **symmetrized Gamma-distributed**, $\frac{\lambda^r}{2\Gamma(r)}|x|^{r-1}e^{-\lambda|x|}$, *and q is even:*

$$\mathbb{E}\left(\frac{1}{\lambda^2}F - \Gamma_3(F)\right)^2 = q!\left\|\frac{1}{\lambda^2}f - \sum_{r=1}^{q-1} c_q(r, q-r)((f\widetilde{\otimes}_r f)\widetilde{\otimes}_{q-r} f)\right\|^2_{\mathcal{H}^{\otimes q}}$$

$$+ \sum_{k=0, k\neq q/2}^{\frac{3q}{2}-2} (2k)!\left\|\sum_{r\in C_{2k}} c_q(r, 3q/2 - k - r)((f\widetilde{\otimes}_r f)\widetilde{\otimes}_{\frac{3q}{2}-k-r} f)\right\|^2_{\mathcal{H}^{\otimes 2k}}$$

compare with free probability

compare with Deya, Nourdin, 2012:

Wigner-integrals

the tetilla law (symmetrized Marchenko-Pastur law) is the *free* Laplace-distribution

all $c_q(\cdot, \cdot)$ are 1 (**!**): there is a 6 moment theorem for every $I_q(f)$

results for double integrals

Theorem

Let Y be a **symmetrized Gamma-distributed** *r.v. with $r, \lambda > 0$ and suppose that $\mathbb{E}[F_n^2] = 2r/\lambda^2$. Then, as $n \to \infty$, following assertions are equivalent:*

(a) $F_n = I_2(f_n)$ *converges in distribution to* Y,

(b) $\mathbb{E}[F_n^4] \to \mathbb{E}[Y^4]$ *and* $\mathbb{E}[F_n^6] \to \mathbb{E}[Y^6]$,

(c) $\|4((f_n \widetilde{\otimes}_1 f_n) \widetilde{\otimes}_1 f_n) - \frac{1}{\lambda^2} f_n\|_{\mathcal{H}^{\otimes 2}} \to 0$ *and* $\|((f_n \widetilde{\otimes}_1 f_n) \widetilde{\otimes}_2 f_n)\|^2 \to 0$.

results for double integrals

Theorem

Let Y be a **Variance-Gamma distributed** *random variable and suppose that $\mathbb{E}[F_n^2] = r(\sigma^2 + 2\theta^2)$. Then, as $n \to \infty$, following assertions are equivalent:*

(a) $F_n = I_2(f_n)$ *converges in distribution to Y,*

(b) $\mathbb{E}[F_n^j] \to \mathbb{E}[Y^j]$ *for all* $j = 3, 4, 5, 6$,

(c) $\|4((f_n \widetilde{\otimes}_1 f_n) \widetilde{\otimes}_1 f_n) - 2\theta\,(f_n \widetilde{\otimes}_1 f_n) \ - \sigma^2 f_n\|_{\mathcal{H}^{\otimes 2}} \to 0$ *and*

$\|((f_n \widetilde{\otimes}_1 f_n) \widetilde{\otimes}_2 f_n)\|_{\mathcal{H}^{\otimes 2}} \to \frac{3}{4} r\theta\sigma^2 + r\theta^3.$

results for double integrals

Let $F_n = I_2(f_n)$ and Y be a **symmetrized Gamma-distributed** r.v.

Theorem

Assume that $\mathbb{E}[F_n^2] \to \frac{2r}{\lambda^2}$. *Then there are constants* $C_1 = C_1(\lambda, r) > 0$ *and* $C_2 = C_2(\lambda, r) > 0$ *such that*

$$\begin{aligned} d_W(F_n, Y) \quad &\leq \quad C_1 \Big(\frac{1}{120} \kappa_6(F_n) - \frac{1}{6r} \kappa_4(F_n) \kappa_2(F_n) + \frac{1}{4r^2} \kappa_2(F_n)^3 + \frac{1}{6} \kappa_3(F_n)^2 \Big)^{1/2} \\ &\qquad + C_2 \Big| \frac{2r}{\lambda^2} - \kappa_2(F_n) \Big|. \end{aligned}$$

the third moment of F_n converges to zero automatically (**!**)

homogeneous sums

$$H_n(X,q) := \sum_{1\le i_1,\dots,i_q\le n} h_n(i_1,\dots,i_q)\, X_{i_1}\cdots X_{i_q}, \qquad (X_i)_i \text{ independent}$$

Theorem

Suppose that $\mathbb{E}[H_n(G,q)^2] = r(\sigma^2+2\theta^2)$, let Y be a **Variance-Gamma** *distributed random variable Then, as $n\to\infty$, the following assertions are equivalent:*

(a) $H_n(X,q)$ converges in distribution to Y, for every independent $(X_i)_i$.

(b) $H_n(G,q)$ converges in distribution to Y.

If $q=2$ then (a) and (b) are equivalent to $\mathbb{E}[H_n(G,2)^j]\to\mathbb{E}[Y^j]$ for $j=3,4,5,6$.

see Nourdin, Peccati, Reinert, 2010

multivariate extensions

$F_n := (F_{n,1}, \dots, F_{n,d})$ and put $Y := (Y_1, \dots, Y_d)$.

$$A_n(j) := \mathbb{E}\Big|\sigma_j^2(F_{n,j} + r_j\theta_j) - 2\theta_j\Gamma_2(F_{n,j}) - \Gamma_3(F_{n,j})\Big| + \Big|r_j\sigma_j^2 + 2r_j\theta_j^2 - \mathbb{E}[\Gamma_2(F_{n,j})]\Big|,$$

and for $j \neq i$ define

$$B_n(i,j) := \mathbb{E}\Big|\langle DF_{n,i}, -DL^{-1}F_{n,j}\rangle\Big|.$$

Theorem

There are constants $C_1 > 0$ and $C_2 > 0$ only depending on d and the parameters r_j, θ_j and σ_j, $j = 1, \dots, d$, such that

$$\mathrm{dist}(F_n, Y) \leq C_1 \sum_{j=1}^{d} A_n(j) + C_2 \sum_{\substack{i,j=1 \\ i \neq j}}^{d} B_n(i,j).$$

see Bourguin, Peccati, 2012

new research group

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Thank you for your attention!