From the Nualart-Peccati criterion to the Gaussian product conjecture

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Workshop on Stein's method

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The Nualart-Peccati criterion

Nualart-Peccati 2005

Let $F_n = I_p(f_n) = \int_0^\infty \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{p-1}} f_n(t_1, \cdots, t_p) dW_{t_1} \cdots dW_{t_p}$ a sequence of mutiple Wiener-Itô integrals such that $\mathbb{E}(F_n^2) = 1$ and $\mathbb{E}(F_n^4) \to 3$ then

$$F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1).$$

Nourdin-Peccati 2008

$$\text{“Malliavin-Stein method”} \Rightarrow d_{TV}(F_n, N) \leq \frac{2}{\sqrt{3}} \sqrt{\mathbb{E}(F_n^4) - 3}.$$

Question : What are the properties of multiple integrals responsible for the fourth moment phenomenon ?

E. Azmoodeh, S. Campese, G.P (2013)

Let L be a Markov diffusive operator on some probability space $(E, \mathcal{F}, \mu)$. Assume that :

- $L^2(\mu) = \bigoplus_{k=0}^{\infty} \mathbf{Ker}(L + \lambda_k \mathbf{Id})$,
- $F \in \mathbf{Ker}(L + \lambda_p \mathbf{Id})$,
- $F^2 \in \bigoplus_{\lambda_k \leq 2\lambda_p} \mathbf{Ker}(L + \lambda_k \mathbf{Id})$, **(Stability property)**

"Dirichlet-Stein's method" $\Rightarrow d_{TV}(F, N) \leq \frac{2}{\sqrt{3}} \sqrt{\mathbb{E}(F^4) - 3}$.

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Expository papers :

- L.H.Y. Chen : Stein meets Malliavin in normal approximation. Acta Mathematica Vietnamica (2015)

- L.H.Y Chen, G.P : Stein's method, Malliavin calculus, Dirichlet forms and the fourth moment theorem. Festschrift Masatoshi Fukushima. (2014)

Question : What is special with the number 4 in the Nualart-Peccati criterion ?

E. Azmoodeh, D. Malicet, G. Mijoule, G.P. (2014)

if $LF = -\lambda F$ and $\forall p \geq 1$,

$$F^p \in \bigoplus_{\alpha \in sp(-L) \cap [0, p\lambda]} \mathbf{Ker}(L + \alpha \mathbf{Id}), \quad \textbf{(Strong stability)},$$

then using "Dirichlet-Stein method" :

$$d_{TV}(F,N) \leq \frac{4}{\sqrt{2p(p-1)\int_0^1 \left(\frac{1+t^2}{2}\right)^{p-2} dt}} \sqrt{\frac{\mathbb{E}(F^{2p})}{(2p-1)!!} - 1}$$

Generalization 2 of the Nualart-Peccati criterion

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The polarization constant problem

The polarization constant

Let $y_1, \cdots, y_d$ be unitary vectors of $\mathbb{R}^d$. We define the polarization constant S by

$$S = \sup_{u \in \mathcal{S}^{d-1}} \left| < u, y_1 > \cdots < u, y_d > \right|.$$

$\rightarrow$ It is conjectured that $S \geq \frac{1}{\sqrt{d^d}}$.

- Problem introduced by Benitez-Sarantopoulos-Tonge (1998).
- Case of **complex** vectors solved by Arias de Reyna (1998) and Ball (2001).
- Pappas, Revesz (2003) case $d \leq 5$.
- Frenkel (2007) produced the best (non optimal) bounds for S.

The Gaussian product conjecture

Gaussian product conjecture

Let $(X_1, \cdots, X_d)$ be a Gaussian vector. It is conjectured that for all $p \geq 1$:

$$\mathbb{E}\left(X_1^{2p} \cdots X_d^{2p}\right) \geq \mathbb{E}(X_1^{2p}) \cdots \mathbb{E}(X_d^{2p}).$$

- case $p = 1$ solved by Frenkel (2007) by using exclusively tools of linear algebra like **Hafnians, Pfaffians**.
- case $p = 2$ remains unsolved but supported by computer simulations
- the case of complex Gaussian solved by Arias de Reyna (1998)

Gaussian product conjecture ⇒ polarization constant problem

For some vectors $y_1, \cdots, y_d$ of $\mathbb{R}^d$:

$$\begin{aligned}
\mathbb{E}\left(X_1^{2p}\cdots X_d^{2p}\right) &= \int_{\mathbb{R}^d}\Big(\langle x,y_1\rangle\cdots\langle x,y_d\rangle\Big)^{2p} e^{-\frac{|x|^2}{2}}\frac{dx}{\sqrt{2\pi}^d} \\
&= \int_{\mathcal{S}^{d-1}}\Big(\langle u,y_1\rangle\cdots\langle u,y_d\rangle\Big)^{2p} du \\
&\times \int_0^\infty r^{2pd} r^{d-1} e^{\frac{-r^2}{2}}\frac{dr}{\sqrt{2\pi}^d}
\end{aligned}$$

We use that

$$\int_{\mathcal{S}^{d-1}}\left(\Big(\langle u,y_1\rangle\cdots\langle u,y_d\rangle\Big)^{2p} du\right)^{\frac{1}{2p}} \xrightarrow[p\to\infty]{} S.$$

Arias de Reyna strategy for the case $\mathbb{C}^d$

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Arias de Reyna

Let $(X_1, \cdots, X_d)$ be a **complex** Gaussian vector. Then, for all $p \geq 1$:

$$\mathbb{E}\left(|X_1|^{2p} \cdots |X_d|^{2p}\right) \geq \mathbb{E}(|X_1|^{2p}) \cdots \mathbb{E}(|X_d|^{2p}).$$

- Complex Gaussian can be seen as complex linear forms on $\mathbb{C}^d$ under the probability measure $\gamma(dz) = \frac{1}{(2\pi)^d} e^{-\frac{\|x\|^2+\|y\|^2}{2}} dx_1 \cdots dx_d dy_1 \cdots dy_d$.

- Simple computations show that $z \to z^i$ are orthogonal polynomials with respect to the complex Gaussian measure, when $i \neq j$:
$$\int_{\mathbb{R}^2} z^i \bar{z}^j \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy = 0.$$

Arias de Reyna strategy for the case $\mathbb{C}^d$

- For any systems of vectors $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ of $\mathbb{C}^d$ we have (using orthogonality of $z \to z^i$) :

$$\int_{\mathbb{C}^d} \prod_{i=1}^{n} < a_i, z > \prod_{i=1}^{n} < z, b_i > d\gamma(z) = 2^n \mathrm{Per}(< a_i, b_i >)$$

- We can hence give an exact formula for $\mathbb{E}\left(\prod_{i=1}^{d} |X_i|^{2p}\right)$ in terms of the permanent of some Hermitian matrix
- one is only reduced to use the next **Lieb inequality** for a positive Hermitian matrix :

$$A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix} \Rightarrow \mathrm{Per}(A) \geq \mathrm{Per}(B)\mathrm{Per}(C).$$

Main issue for extending this approach to the real case

- The stability of $z \to z^i$ via product is crucially used in the exact computation of $\mathbb{E}\left(\prod_{i=1}^{d} |X_i|^{2p}\right)$ in terms of permanents
- In the real case, the orthogonal polynomials with respect to the Gaussian measure are Hermite polynomials :

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}\left(e^{-\frac{x^2}{2}}\right) e^{\frac{x^2}{2}}$$

- $H_k(x)H_l(x)$ is not an Hermite polynomial ! More precisely we have

$$H_k(x)H_l(x) = \sum_{r=O}^{\min(k,l)} r! \binom{l}{r}\binom{k}{r} H_{k+l-2r}(x).$$

Main Theorem

D. Malicet, I. Nourdin, G. Peccato, G.P (2015)

Let $X_1, \cdots, X_d$ be a Gaussian vector. Then, for all $p \geq 1$:

$$\mathbb{E}\left(H_p(X_1)^2 \cdots H_p(X_d)^2\right) \geq \mathbb{E}(H_p(X_1)^2) \cdots \mathbb{E}(H_p(X_d)^2).$$

- When $p = 1$, we recover Frenkel inequality
- Strong analogy with the complex case et Arias de Reyna strategy
- we get closer to the cases $p = 2, p = 3$ since

$$\mathbb{E}\left((X_1^4 + 1) \cdots (X_d^4 + 1)\right) \geq \mathbb{E}(X_1^4 + 1) \cdots \mathbb{E}(X_d^4 + 1).$$

$$\mathbb{E}\left((X_1^6 + 3) \cdots (X_d^6 + 3)\right) \geq \mathbb{E}(X_1^6 + 3) \cdots \mathbb{E}(X_d^6 + 3).$$

Step 1 of the proof :

$\rightarrow$ We want to associate to the Hermite polynomial a differential operator and do "integrations by parts".

- $\gamma_d = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{\|\mathbf{x}\|^2}{2}} d\mathbf{x},$

- $L[\phi] := \Delta\phi - \mathbf{x} \cdot \nabla\phi,$

- $\Gamma[\phi, \psi] = \nabla\phi \cdot \nabla\psi.$

Main properties of the Ornstein-Uhlenbeck operator

- $\int_{\mathbb{R}^d} \phi\, L[\psi] \mathrm{d}\gamma_d = \int_{\mathbb{R}^d} L[\phi]\, \psi \mathrm{d}\gamma_d$ (Symmetry)
- $\int_{\mathbb{R}^d} \Gamma[\phi, \psi] \mathrm{d}\gamma_d = -\int_{\mathbb{R}^d} \phi\, L[\psi] \mathrm{d}\gamma_d$ (Integration by parts)
- $\Gamma[f(\phi), \psi] = f'(\phi)\Gamma[\phi, \psi]$ (chain rule)
- $L[\phi\psi] = 2\Gamma[\phi, \psi] + \phi L[\psi] + \psi L[\phi]$ (Link between L and Γ)
- $\mathrm{Sp}(-L) = \mathbb{N}$ and

  $$\mathrm{Ker}(L + kId) = \mathrm{Vect}\left\{ H_{k_1}(x_1) \cdots H_{k_d}(x_d) \,\middle|\, k_1 + \cdots + k_d = k \right\}.$$

Step 2 of the proof :

$\rightarrow$ We want to use the polynomial structure of the eigenspaces of L to produce "Bernstein"-type inequalities.

Let $F_1, \cdots, F_d$ be eigenfunctions of L of orders $k_1, \cdots, k_d$, then :

$$F_1 \cdots F_d \in \bigoplus_{\alpha \leq k_1 + \cdots + k_d} \mathrm{Ker}(L + \alpha Id),$$

$$\Downarrow$$

$$\mathbb{E}\Big(F_1 \cdots F_d (L + (k_1 + \cdots + k_d) Id)[F_1 \cdots F_d]\Big) \geq 0.$$

Using the relation between L and Γ one has

$$(L+(k_1+\cdots+k_d)Id)[F_1\cdots F_d]=\sum_{i\neq j}\left(\prod_{k\neq i,j}F_k\right)\ \Gamma[F_i,F_j]$$

Hence :

$$\mathbb{E}\left(\prod_i F_i \sum_{i\neq j}\left(\prod_{k\neq i,j}F_k\right)\ \Gamma[F_i,F_j]\right)\geq 0.$$

Equivalently, by integration by parts :

$$\sum_{i=1}^{d}\mathbb{E}\Big(L[F_i^2]\prod_{j\neq i}F_j^2\Big)\leq 0.$$

Step 3 of the proof (first method) : We set

$$f(t) = \mathbb{E}\left(\prod_{i=1}^{d} P_t[F_i^2]\right),$$

$$f'(t) = \sum_{j=1}^{d} \mathbb{E}\left(L[P_t F_j^2] \prod_{i \neq i} P_t[F_i^2]\right) \leq 0.$$

Hence,

$$f(0) = \mathbb{E}\left(\prod_{i=1}^{d} F_i^2\right) \geq \lim_{t \to \infty} f(t) = \prod_{i=1}^{d} \mathbb{E}\left(F_i^2\right).$$

Step 3 of the proof (second method) :

$\rightarrow$ We want to prove by induction on d and the orders k_i that

$$\mathbb{E}\left(F_1^2 \cdots F_d^2\right) \geq \mathbb{E}(F_1^2) \cdots \mathbb{E}(F_d^2).$$

Using again relation between L and Γ : $L[F_i^2] = 2\Gamma[F_i, F_i] - 2k_i F_i^2$

$$\begin{aligned}
(k_1 + k_2 + \cdots + k_d)\mathbb{E}\left(F_1^2 \cdots F_d^2\right) &= \sum_{i=1}^{d} \mathbb{E}\left(\Gamma[F_i, F_i] \prod_{j \neq i} F_j^2\right) \\
&- \frac{1}{2} \sum_{i=1}^{d} \mathbb{E}\left(L[F_i^2] \prod_{j \neq i} F_j^2\right)
\end{aligned}$$

$$(k_1 + k_2 + \cdots + k_d)\mathbb{E}\left(F_1^2 \cdots F_d^2\right) \geq \sum_{i=1}^{d} \mathbb{E}\left(\Gamma[F_i, F_i] \prod_{j \neq i} F_j^2\right).$$

End of the proof

$$\Gamma[F,F] = \nabla F \cdot \nabla F \quad = \quad \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}^2$$

If $F \in Ker(L + kId)$ then $\forall i, \frac{\partial F}{\partial x_i} \in Ker(L + (k-1)Id)$ and by induction :

$$\begin{aligned}
\mathbb{E}\left(\Gamma[F_i, F_i] \prod_{j \neq i} F_j^2\right) \quad &= \quad \sum_{k=1}^{d} \mathbb{E}\left((\frac{\partial F_i}{\partial x_k})^2 \prod_{j \neq i} F_j^2 \right) \\
&\geq \quad \mathbb{E}((\frac{\partial F_i}{\partial x_k})^2) \prod_{j \neq i} \mathbb{E}(F_j^2) \\
&= \quad \mathbb{E}\left(\Gamma[F_i, F_i]\right) \prod_{j \neq i} \mathbb{E}(F_j^2)
\end{aligned}$$

By integration by parts, $\mathbb{E}\left(\Gamma[F_i, F_i]\right) = k_i \mathbb{E}(F_i^2)$.

Comments on the proof

- The result proved is actually stronger than the product of squares of Hermite since it holds for product of Wiener chaos.
- We can give bounds for norm of product of polylinear forms in the case when the F_i are homogeneous polynomials.
- Focusing on the Gaussian case, we slightly improve Frenkel inequality since we have :

$$\mathbb{E}\left(\prod_{i=1}^{d} X_i^2\right) \geq \frac{1}{d}\sum_{j=1}^{d}\mathbb{E}(X_j^2)\mathbb{E}(\prod_{i\neq j} X_i^2).$$

- it leads to a natural strategy (so far unsuccessful) to attack the Gaussian product conjecture by expanding x^{2p} into the basis of H_k^2 and studying the signs of the coefficients

Comments on the proof

- The method by induction actually gives the equality case :

$$\mathbb{E}\left(\prod_{i=1}^{d} F_i^2\right) = \prod_{i=1}^{d} \mathbb{E}\left(F_i^2\right) \Leftrightarrow (F_1, \cdots, F_d) \text{ jointly independent.}$$

- It is possible to give an asymptotic version for sequence of chaotic vectors $(F_{1,n}, \cdots, F_{d,n})$ converging in law towards $(Z_1, \cdots, Z_d)$.

$$E\left(\prod_{i=1}^{d} Z_i^2\right) = \prod_{i=1}^{d} \mathbb{E}\left(Z_i^2\right) \Rightarrow (Z_1, \cdots, Z_d) \text{ jointly independent.}$$

Some corollary : an exact Hadamard inequality

The celebrated Mehler formula asserts that $z \in]-1,1[$ that

$$\sum_{n=0}^{\infty} z^n \frac{H_n(x)^2}{2^n n\,!} = \frac{1}{\sqrt{1-z^2}} \exp\left(\frac{2x^2 z}{1+z}\right).$$

Then, for any Gaussian vector with any covariance Σ :

$$\mathbb{E}\left(e^{\sum_{i=1}^{d} t_i X_i^2}\right) = \sum_{n_1,\cdots n_d=0}^{\infty} t_1^{n_1}\cdots t_d^{n_d} \frac{1}{2^{n_1+\cdots n_d} n_1!\cdots n_d!} \mathbb{E}\left(\prod_{i=1}^{d} H_{n_i}(X_i)^2\right)$$

we compute and we get...

Exact Hadamard inequality

An Hadamard "equality"

Let $S = (S_{ij})$ be a symmetric positive definite matrix. Set I_d the identity matrix and Z the diagonal part of S. If $Z < I_d$ and $Z + S < 2I_d$:

$$\Sigma = I_d - \frac{1}{2}(I_d - Z)^{-\frac{1}{2}}(S - Z)(I_d - Z)^{-\frac{1}{2}},$$

is symmetric, positive definite and satisfies $\Sigma_{ii} = 1$ for each i. For $(X_1, \ldots, X_d)$ a centered Gaussian vector of covariance Σ,

$$\det S = \left(\sum_{k_1,\ldots,k_d=0}^{\infty} \frac{E[H_{k_1}(X_1)^2 \ldots H_{k_d}(X_d)^2]}{k_1! \ldots k_d!} \prod_{i=1}^{d} \sqrt{S_{ii}}(1 - S_{ii})^{k_i} \right)^{-2}. \tag{1}$$

We recover in particular the classical Hadamard inequality : $\det S \leq \prod_{i=1}^{d} S_{ii}$.