

Stein's Method and Characteristic Functions

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Workshop “New Directions in Stein’s method”

Topics

- ▶ Central Limit Theorems for Sum of Independent and Dependent Random variables
- ▶ CLT for Maximal Sum
- ▶ CLT for Multivariate Random Variables
- ▶ CLT for Hilbert Space Valued Random Variables
- ▶ Semi circular Law for Random Matrices
- ▶ CLT for Linear Statistics of Eigenvalues of Random Matrices

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Stein Comments

The abstract normal approximation theorem of Section 2 is elementary in the sense that it uses only the basic properties of conditional expectation and elements of analysis, including the solution of a first order linear differential equation. It also direct, in the sense the expectation of a fairly arbitrary function of the random variable W in question is approximated directly *without going through characteristic function.*

Stein Comments. Continue

Because of the clumsiness of the technique used on this point , it is likely that slightly better results could be obtained by using the basic identity, Lemma 2.1, to approximate the characteristic function of W and by standard procedures, to go from this to an approximation for the distribution of W . *However, I believe that, in the long run, better results will be obtained by direct methods.* **Ch. Stein, 1972**

The Notation

Let $\{\Omega, \mathfrak{M}, \Pr\}$ be probability space and let, for any $n \geq 1$,

$\mathfrak{F}_n \subset \mathfrak{M}$ be sub σ -algebra of \mathfrak{M} s.t. $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$.

Let $(G_n)_{n=1}^{\infty}$ be a sequence of r.v.'s with $E|G_n| < \infty$.

Let $W_n := E\{G_n | \mathfrak{F}_n\}$.

For any sequence of r.v.'s W_n^* with $E|W_n^*| < \infty$ define the following quantities:

$$D_n := W_n - W_n^*, \quad (1)$$

The Notation. Continue

$$\delta_n(t) := \mathbb{E} G_n \left(\frac{e^{D_n} - 1}{it} \right) - 1,$$

$$\varepsilon_{n1}(t) := \mathbb{E} G_n e^{itW_n^*},$$

$$\varepsilon_{n2}(t) := \mathbb{E} \left| \mathbb{E} \left\{ (G_n(e^{D_n} - 1) - \mathbb{E} G_n(e^{D_n} - 1)) \middle| \mathfrak{F}_n \right\} \right|,$$

$$\varepsilon_n(t) := \varepsilon_{n1}(t) + \varepsilon_{n2}(t), \quad \tilde{\delta}_n(T) := \sup_{t: |t| \leq T} |\delta_n(t)|.$$

The main Lemma

Lemma

For any $0 < \gamma < 1$ and for any $T > 0$ s.t. $\tilde{\delta}_n(T) \leq \gamma$, there exists an absolute constant $C > 0$ s.t. the following inequality holds

$$\begin{aligned} \sup_x |\Pr\{W_n \leq x\} - \Phi(x)| &\leq \frac{C}{T} + \frac{\tilde{\delta}_n(T)}{1-\gamma} \\ &\quad + \frac{1}{1-\gamma} \int_0^T \frac{e^{-\frac{t^2}{2}}}{t} \int_0^t |\varepsilon_n(t)| e^{\frac{u^2}{2}} du dt, \end{aligned}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du.$$

The proof of Main Lemma

Let $f_n(t) := E e^{itW_n}$. Hence $E |W_n| < \infty$ and $E |G_n| < \infty$ we may write

$$f'_n(t) = i E W_n e^{itW_n} = i E G_n e^{itW_n}. \quad (2)$$

Continuing, we get

$$\begin{aligned} f'_n(t) &= i E G_n e^{itW_n^*} + i f_n(t) E G_n (1 - e^{-itD_n}) \\ &\quad + E \left[E \left\{ G_n (1 - e^{-itD_n}) - E G_n (1 - e^{-itD_n}) \middle| \mathcal{F}_n \right\} \right] e^{itW_n}. \end{aligned}$$

The Continuing

- ▶ Let $\theta(\cdot)$ (with an index or without) denote any function s.t.
$$|\theta(\cdot)| \leq 1 ,$$
- ▶ By symbol C we shall denote an absolute constant, which can be changed from row to row.

We may rewrite now the previous equation for ch. f. in the form

$$f'_n(t) = -t(1 - \delta_n(-t))f_n(t) + \theta_n(t)\varepsilon_n(t).$$

Solving this equation with initial condition $f_n(0) = 1$, we get

$$\begin{aligned} f_n(t) &= \exp \left\{ -\frac{t^2}{2} + \int_0^t u \delta_n(u) du \right\} \\ &\quad \times \left(1 + \int_0^t \theta_n(u) \varepsilon_n(u) \exp \left\{ \frac{u^2}{2} - \int_0^u v \delta_n(v) dv \right\} du \right). \end{aligned}$$

Furthermore, note that

$$\left| \int_u^t v \delta_n(v) dv \right| \leq \frac{\gamma^2}{2} (t^2 - u^2), \text{ for } |u| \leq |t| \leq T. \quad (3)$$

The last relation immediately implies that, for $|t| \leq T$,

$$\begin{aligned} f_n(t) &= \exp \left\{ -\frac{t^2}{2}(1 + \theta_{n1}(t)\tilde{\delta}_n(T)) \right\} \\ &\quad + \theta_{n2}(t) \int_0^t |\varepsilon_n(u)| \exp \left\{ -\frac{(t^2 - u^2)(1 - \gamma)}{2} \right\} du. \end{aligned}$$

From here it follows that

$$|f_n(t) - e^{-\frac{t^2}{2}}| \leq \tilde{\delta}_n(T)t^2 e^{-\frac{(1-\gamma)t^2}{2}} + \int_0^t |\varepsilon_n(u)| e^{\frac{(1-\gamma)(t^2-u^2)}{2}} du. \quad (4)$$

Applying the famous Esseen's inequality, we get

$$\begin{aligned} \sup_x |\Pr\{W_n \leq x\} - \Phi(x)| &\leq \frac{C}{T} + \frac{\tilde{\delta}_n(T)}{1-\gamma} \\ &+ \frac{1}{1-\gamma} \int_0^T \frac{e^{-(1-\gamma)t^2/2}}{t} \left(\int_0^t |\varepsilon(u)| e^{(1-\gamma)u^2/2} du \right) dt. \end{aligned} \tag{5}$$

Corollary

Corollary

Assume that $\delta_n(t) \rightarrow 0$ and $\varepsilon_n(t) \rightarrow 0$ as $n \rightarrow \infty$ uniformly in all bounded interval $[0, T]$. Then

$$\lim_{n \rightarrow \infty} \sup_x |\Pr\{W_n \leq x\} - \Phi(x)| = 0. \quad (6)$$

Applications of Main Lemma. The Sums of Weakly Dependent R.V.'s

Let X_1, X_2, \dots be a stationary sequence of random variables with $E X_1 = 0$ and $E X_1^2 = 1$. Let \mathfrak{M}_a^b denote σ -algebra generated by X_j , when $j \in [a, b]$. We denote the strong mixing coefficient of the sequence $(X_j)_{j=1}^\infty$ by the equality

$$\alpha(m) := \sup_{A \in \mathfrak{M}_1^k, B \in \mathfrak{M}_{k+m}^\infty} |\Pr(AB) - \Pr(A)\Pr(B)| \quad (7)$$

Theorem

Let the following conditions hold for some $\delta > 0$,

$$\sum_{m=1}^{\infty} [\alpha(m)]^{\frac{\delta}{2+\delta}} < \infty,$$

$$\mathbb{E}|X_1|^{2+\delta} < \infty,$$

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}(\sum_{j=1}^n X_j)^2 = \sigma^2 > 0.$$

Then

$$\lim_{n \rightarrow \infty} \sup_x |\Pr\{B_n^{-1} \sum_{j=1}^n X_j \leq x\} - \Phi(x)| = 0 \quad (8)$$

To prove this result we use our Lemma. Let I be uniformly distributed on the set $\{1, \dots, n\}$ random variable. Put $G = nB_n^{-1}X_I$. Then $W = E\{G|\mathfrak{F}\}$. Put $W^* = B_n^{-1} \sum_{j:|j-I|>m} X_j$. Then we have the estimation

$$|\varepsilon_{n1}(t)| = |E G e^{itW^*}| \leq Cn B_n^{-1} E^{\frac{\delta}{2+\delta}} |X_1|^{2+\delta} [\alpha(m)]^{\frac{1+\delta}{2+\delta}}. \quad (9)$$

Furthermore,

$$|\delta_n(t)| \leq \frac{1}{B_n} \left| \sum_{j=1}^n E X_j \frac{e^{itD_{nj}} - 1 - it}{it} \right| \leq \frac{C|t|^{1+\delta} m^{1+\delta}}{\sqrt{n}} E |X_1|^{2+\delta}, \quad (10)$$

where $D_{nj} = B_n^{-1} \sum_{k:|j-k|\leq m} X_k$.

For $\varepsilon_{n2}(t)$ we have the following estimation

$$|\varepsilon_{n2}| = \mathsf{E} \left| \frac{1}{B_n} \sum_{j=1}^n X_j (e^{itD_{nj}} - 1) - \mathsf{E} X_j (e^{itD_{nj}} - 1) \right| \leq C |t|^\delta \frac{m^\delta}{n^{\frac{\delta}{2}}}. \quad (11)$$

CLT for Quadratic forms

Let X_1, X_2, \dots be a sequence of r.v.'s s.t. $\mathbb{E} X_k = 0$, $\mathbb{E} X_k^2 = 1$
and let the sequence be uniformly integrated, i.e.

$$\sup_{k \geq 1} \mathbb{E} X_k^2 \mathbb{I}\{|X_k| > M\} \rightarrow 0, \text{ as } M \rightarrow 0. \quad (12)$$

We consider the quadratic form

$$Q = \sum_{j,k=1}^n a_{jk}^{(n)} X_j X_k,$$

where the matrix of coefficients $\mathbf{A}_n = (a_{jk}^{(n)})_{j,k=1}^n$ is supposed
symmetric and diagonal entries of \mathbf{A} are equal zero, $a_{kk}^{(n)} = 0$.

Let $\|\mathbf{A}\|$ denote the operator norm of matrix \mathbf{A} and let $\|\mathbf{A}\|_2$ denote the Hilbert – Schmidt norm of matrix \mathbf{A} . We shall assume that

$$\lim_{n \rightarrow \infty} \frac{\|\mathbf{A}_n\|}{\|\mathbf{A}_n\|_2} = 0. \quad (13)$$

This condition is equivalent to

$$\lim_{n \rightarrow \infty} \frac{|\lambda_1^{(n)}|}{\sigma_n} = 0, \quad (14)$$

where $\Lambda_1^{(n)}$ is the maximum modulus eigenvalue of \mathbf{A}_n and $\sigma_n^2 = \mathbb{E} Q_n^2 = 2\|\mathbf{A}_n\|_2^2$.

Theorem

Assume conditions (13) and (12). Then

$$\sup_x |\Pr\{\sigma_n^{-1} Q_n \leq x\} - \Phi(x)| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (15)$$

To apply main Lemma, we denote by $\mathbb{T}_j := \{1, \dots, n\} \setminus \{j\}$ and introduce r.v.'s $\xi_j = X_j \sum_{k \in \mathbb{T}_j} a_{jk} X_k$. Note that $Q = \sum_{j=1}^n \xi_j$. We put $G = n\xi_I$. We introduce $W^* := Q^{(I)}$, where

$Q^{(I)} := \sum_{k,l \in \mathbb{T}_I} a_{kl} X_k X_l$. Note that

$$W - W^* = 2\xi_I$$

First we assume that $|X_j| \leq M$ for all $j = 1, \dots, n$. We may estimate now $\delta_n(t)$ and $\varepsilon_n(t)$. For instance

$$\begin{aligned} |\varepsilon_{n1}| &\leq \sum_{l=1}^n (it)^{-1} |\mathsf{E} \xi_I (e^{2it\xi_l} - 12 - it\xi_l)| \leq C|t|^2 \sum_{l=1}^n \mathsf{E} |\xi_l|^3 \\ &\leq Ct^2 M^2 \sum_{l=1}^n \left(\sum_{k \in \mathbb{T}_I} |a_{kl}|^2 \right)^{\frac{3}{2}} \leq C|t|^2 |\lambda_n|. \end{aligned} \tag{16}$$

Asymptotic distribution of quadratic forms

Consider the quadratic forms

$$Q = \sum_{1 \leq j \neq k \leq n} a_{jk} X_j X_k + \sum_{j=1}^n a_{jj} (X_j^2 - 1)$$

Let $V^2 = \sum_{j=1}^n a_{jj}^2$, $\mathcal{L}_j^2 = \sum_{k=1}^n a_{jk}^2$, $\|\mathbf{A}\|_2^2 = \sum_{j,k=1}^n a_{jk}^2$.

Let $\mu_k = \mathbb{E} X_1^k$. Let $\mathbf{A}^{(0)} = (a_{jk})_{j,k=1}^n$ with $A_{jj}^{(0)} = 0$, and let $\mathbf{d} = (a_{11}, \dots, a_{nn})$. Let \mathbf{Y} and $\bar{\mathbf{Y}}$ be two independent copies of standard Gaussian vector in \mathbb{R}^n . Let

$$G := \mu_2 < \mathbf{A}^{(0)} \mathbf{Y}, \mathbf{Y} > + \mu_3 \mu_2^{-\frac{1}{2}} < \mathbf{d}, \mathbf{Y} > + \sqrt{\mu_4 - \mu_2^2} < \mathbf{d}, \bar{\mathbf{Y}} >.$$

Characterized equation

Denote by \mathbf{R}_t the resolvent matrix of $\mathbf{A}_0.$,

$$\mathbf{R}_t = (\mathbf{I} - 2it\mu_2 \mathbf{A}^{(0)})^{-1}$$

Introduce the function

$$\mathcal{G}(t) = \mu_2 \text{Tr}(\mathbf{R}_t \mathbf{A}^{(0)}) - t^2 \mu_3^2 < \mathbf{d}, \mathbf{R}_t^2 \mathbf{A}^{(0)} \mathbf{d} > + it(\mu_4^4 - \mu_2^2) < \mathbf{d}, \mathbf{d} >. \quad (17)$$

The characteristic function $f_n(t) = \mathbb{E} \exp\{itQ_n\}$ satisfy the equation

$$f'_n(t) = i\mathcal{G}(t)f_n(t) + \varepsilon_n(t), \quad (18)$$

where the function $\varepsilon_n(t)$ convergence to 0 uniformly in all bounded intervals.

CLT for Generalized U -statistics

Consider statistics of type $U_2 = \sum_{1 \leq l \neq j \leq n} g_{lj}(X_l, X_k)$. Denote by $\sigma_{kj}^2 := \mathbb{E} g_{kj}^2(X_k, X_j) < \infty$, $\sigma_n^2 = 2 \sum_{k,j} \sigma_{kj}^2$. About the functions $g_{k,l}(x, y)$ we shall assume

- ▶ (a) $g_{lj}(x, y) = g_{jl}(y, x)$;
- ▶ (b) $\mathbb{E} g_{kj}(X_k, X_j) | X_j = 0$;
- ▶ (c) $\lim_{M \rightarrow \infty} \max_{1 \leq k \neq j \leq k_n} \sigma_{kj}^{-2} \mathbb{E} g_{kj}^2(X_k, X_j) \mathbb{I}\{|g_{kj}(X_k, X_j)| \geq M\} = 0$;
- ▶ (d) $\lim_{n \rightarrow \infty} \sigma_n^{-2} \sum_{l \neq j} \mathbb{E} \left| \sum_k g_{lk}(X_l, X_k) g_{jk}(X_j, X_k) \right|^2 = 0$;
- ▶ (e) $\lim_{n \rightarrow \infty} \sigma_n^{-2} \sum_k \sigma_{jk}^2 = 0$.

Theorem

Assuming conditions (a) – (e), we have

$$\sup_x |\Pr\{\sigma_n^{-1} U_2 \leq x\} - \Phi(x)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Moreover, there exist an absolute positive constant $C > 0$ s.t.

$$\begin{aligned} \sup_x |\Pr\{\sigma_n^{-1} U_2 \leq x\} - \Phi(x)| &\leq C \left(\sigma_n^{-3} \sum_{j,k=1}^{k_n} \mathbb{E} |g_{jk}(X_j, X_k)|^3 \right. \\ &\quad \left. + \sigma_n^{-4} \sum_{j,k=1}^{k_n} \mathbb{E} |g_{jk}(X_j, X_k)|^4 + \sigma_n^{-4} \sum_{j,k=1}^{k_n} \mathbb{E} \left| \sum_{l=1}^{k_n} g_{kl}(X_k, X_l) g_{jl}(X_j, X_l) \right|^2 \right. \\ &\quad \left. + \sigma_n^{-4} \sum_j \mathbb{E} \left(\sum_{k=1}^{k_n} \mathbb{E} \{g_{jk}^2(X_j, X_k) | X_j\} \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Maximal sum

It is well-known that the limit distribution of the maximal sum of independent random variables is

$$G(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-u^2/2} du \mathbb{I}\{x \geq 0\}, \quad (19)$$

i.e. the distribution of module of standard Gaussian r.v. Note that $\Phi(x) = (1 - G(-x) + G(x))/2$ and $f(t) = \operatorname{Re} g(t)$, where $f(t), g(t)$ are characteristic functions of $\Phi(x)$ and $G(x)$ respectively. That mean we may proof CLT for the symmetrization of maximal sum using our Lemma.

CLT for Hilbert-valued r.v.'s

Let X_1, X_2, \dots be a sequence of random elements of the Hilbert space H with zero mean, $E X_j = O$, and covariation operators $\mathbf{A}_j = \text{cov}(X_j)$. Suppose the sequence $\{X_j : j = 1\dots\}$ satisfies the uniformly strong mixing condition with coefficient $\varphi(m)$. Let

$$S_n = \sum_{j=1}^n X_j, \quad \mathbf{B}_n = \text{cov}(S_n), \quad Z_n = \frac{S_n}{\sqrt{\Delta \mathbf{B}_n}}. \quad (20)$$

The well-known equations for the characteristic function

$$f(t) = \mathbb{E} \exp\{it\|\xi + a\|^2\}:$$

$$f'(t) = \mathbf{i} \left(\text{Tr } \mathbf{B} (\mathbf{I} - 2it\mathbf{B})^{-1} \right) f(t) + \langle a, (\mathbf{I} - 2\mathbf{B})^{-2} a \rangle f(t).$$

The main result is the following theorem:

Theorem

Let $E \|X_j\|^3 \leq L < \infty$ and constants $K, \beta > 0$ exist such that $\varphi(m) < K \exp\{-\beta m\}$, for all $m \geq 1$. Let also the following conditions hold:

$$\lim_{n \rightarrow \infty} \frac{\mathbf{A}_1 + \cdots + \mathbf{A}_n}{n} = \mathbf{A} \text{ and } \varphi(m) < 1. \quad (21)$$

Then there exists a constant c depending on the first eigenvalues of the operator \mathbf{A} such that

$$\sup_r |\Pr\{\|Z_n+a\| < r\} - \Pr\{\|Y_n+a\| < r\}| \leq c L n^{-1/2} \log\{n(1+\|a\|)\},$$

where Y_n , is a Gaussian r. v. with covariance operator equal to

Semi Circular Law for Random Matrices

Let $p(x)$ denote the density of semi-circular law,

$p(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{I}_{[-2,2]}(x)$, and let $f(t)$ be the characteristic function of semicircular law, $f(t) = \int_{-\infty}^{\infty} e^{itx} p(x) dx$.

Lemma

The characteristic function of semi-circular law is the unit solution of the equation

$$f'(t) = \int_0^t f(u)f(t-u)du, \quad f(0) = 1. \quad (22)$$

Recall that the moments $\{\alpha_k\}$ of semi-circular law satisfies the relation

$$\alpha_{k+1} = \sum_{\nu=0}^{k-1} \alpha_\nu \alpha_{k-\nu}. \quad (23)$$

We have representation

$$f'(t) = i \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \alpha_{k+1}$$

From the other hand side

$$\begin{aligned} \int_0^t f(u)f(t-u)du &= \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \alpha_\nu \alpha_\mu \int_0^t \frac{(iu)^\mu (i(t-u))^\nu}{\nu! \mu!} du \\ &= \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \alpha_\nu \alpha_\mu \frac{i^{\mu+\nu} t^{\mu+\nu+1}}{(\mu+\nu+1)!} \end{aligned}$$

Changing the order of summing, we get

$$\int_0^t f(u)f(t-u)du = \sum_{s=0}^{\infty} \frac{i^s t^{s+1}}{(s+1)!} \sum_{\nu=0}^s \alpha_{\nu} \alpha_{s-\nu}.$$

Continuing this equality and applying relation (23), we obtain

$$\int_0^t f(u)f(t-u)du = \sum_{s=0}^{\infty} \frac{i^s t^{s+1}}{(s+1)!} \alpha_{s+2} = -i \sum_{s=1}^{\infty} \frac{(it)^s}{s!} \alpha_{s+1} = -f'(t).$$

Lemma

Let for the characteristic functions $f_n(t)$ the following representation holds

$$f'_n(t) = - \int_0^t f_n(s) f_n(t-s) ds + \varepsilon_n(t), \quad (24)$$

where $\varepsilon_n(t) \rightarrow 0$ as $n \rightarrow \infty$ uniformly for all bounded intervals.

Then

$$\lim_{n \rightarrow \infty} f_n(t) = f(t), \quad (25)$$

where $f(t)$ is characteristic functions of semi-circular law.

We represent $f_n(t)$ in the form

$$f_n(t) = 1 + \int_0^t \varepsilon_0^s f_n(u) f_n(s-u) du ds + \int_0^t \varepsilon_n(s) ds. \quad (26)$$

If we put now $g_n(t) = g(t)$ then we get

$$g_n(t) = \int_0^t \int_0^s g_n(u) g_n(s-u) + \int_0^t \varepsilon_n(s) ds. \quad (27)$$

Let $\bar{g}_n(t) = \sup_{s \in [0,t]} |g_n(s)|$ and $\bar{\varepsilon}_n(t) = \sup_{s \in [0,t]} |\varepsilon_n(s)|$. It is straightforward to check that, for any $k \geq 1$

$$\bar{g}_n(t) \leq 2 \frac{|t|^k}{k!} \bar{g}_n(t) + e^{|t|} \bar{\varepsilon}_n(t). \quad (28)$$

The last inequality implies the claim.

How we may used the last Lemma?

- ▶ Let $\mathbf{W} = \frac{1}{\sqrt{n}}(X_{jk})_{j,k=1}^n$ be $n \times n$ Hermitian matrix with independent entries (up to symmetry).
- ▶ Let $\mathbf{U}(t) = e^{it\mathbf{W}}$. Then $f_n(t) = \frac{1}{n} \text{Tr } \mathbf{U}(t)$.
- ▶ We have $E f'_n(t) = \frac{i}{n} E \text{Tr } \mathbf{W} \mathbf{U}(t) = \frac{i}{n\sqrt{n}} \sum_{j,k=1}^n E X_{jk} U_{kj}(t)$.
- ▶ It is well-known

$$\frac{\partial U_{pq}(t)}{\partial X_{jk}} = i(1 + \delta_{jk})^{-1} (U_{jp} * U_{kq}(t) + U_{pk} * U_{jq}(t)), \quad (29)$$

where $f * g(t) := \int_0^t f(s)g(t-s)ds$.

Using the last relations, we get

$$\mathbb{E} f'_n(t) = - \int_0^t \mathbb{E} \left(\frac{1}{n^2} \sum_{j,k=1}^n (U_{jj}(s) U_{kk}(t-s)) \right) + \varepsilon_n(t) = - \int_0^t \mathbb{E} f_n(s) f_n(t-s) ds + \varepsilon_n(t)$$

(30)

For the Marchenko – Pastur Law we may used representation
for characteristic function of symmetrizing law

$$f'(t) = -y \int_0^t f(s)f(t-s)ds - (1-y) \int_0^t f(s)ds. \quad (31)$$

Thank you for your attention!