Introduction to
Collapsible Pushdown Automata
and
Higher-Order Recursion Schemes

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University of Warsaw

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Higher-order pushdown automata [Maslov 1974] - definition

A 1-stack is an ordinary stack. A 2-stack (resp. $(n+1)$-stack) is a stack of 1-stacks (resp. n-stack).

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Operations on 2-stacks: s_i are 1-stacks. Top of stack is on right.

$$push_2 \quad : \quad [s_1 \ldots s_{i-1} s_i] \quad \rightarrow \quad [s_1 \ldots s_{i-1} s_i s_i]$$

$$pop_2 \quad : \quad [s_1 \ldots s_{i-1} s_i] \quad \rightarrow \quad [s_1 \ldots s_{i-1}]$$

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$$pop_2 : [s_1 \ldots s_{i-1} s_i] \rightarrow [s_1 \ldots s_{i-1}]$$

$$push_1 x : [s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1} a_j]] \rightarrow [s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1} a_j x]]$$
$$pop_1 : [s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1} a_j]] \rightarrow [s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1}]]$$

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$$pop_1 : [s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1} a_j]] \rightarrow [s_1 \ldots s_{i-1} [a_1 \ldots a_{j-1}]]$$

An **order-n PDA** has an order-n stack, and has $push_i$ and pop_i for each $1 \leq i \leq n$.

The next operation depends on the topmost stack symbol, the state, and the next letter on the input.

Higher-order pushdown automata - example

Language: $\{b^{2^k} : k \in \mathbb{N}\}$

- order 2
- 3 stack symbols: $\perp$, x, #

$$(\_, q_1) \xrightarrow{\varepsilon} (q_1, push_1(x))$$

any stack symbol

q_1 $\boxed{\perp}$

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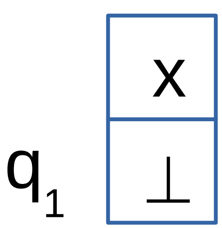

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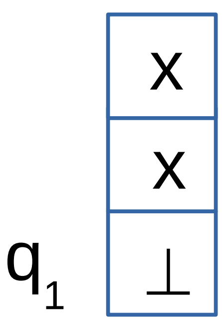

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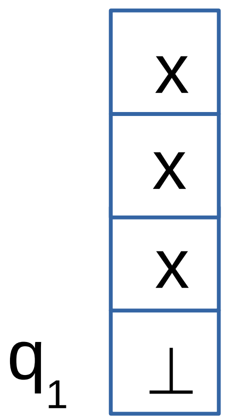

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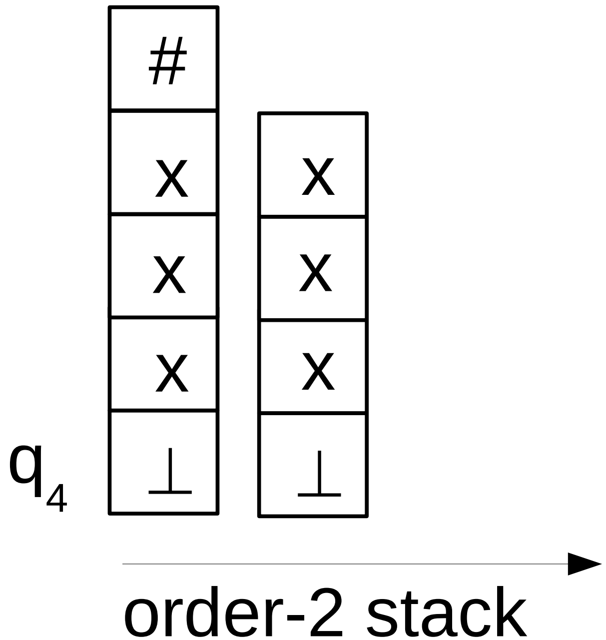

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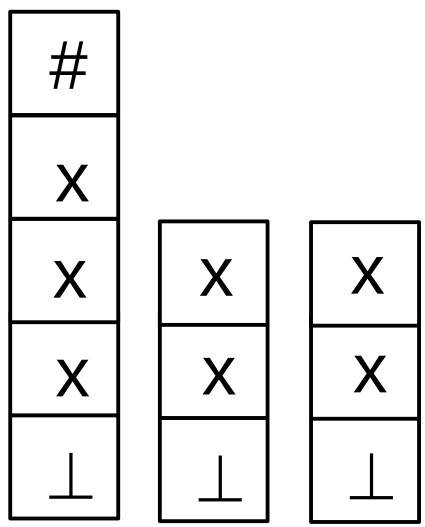

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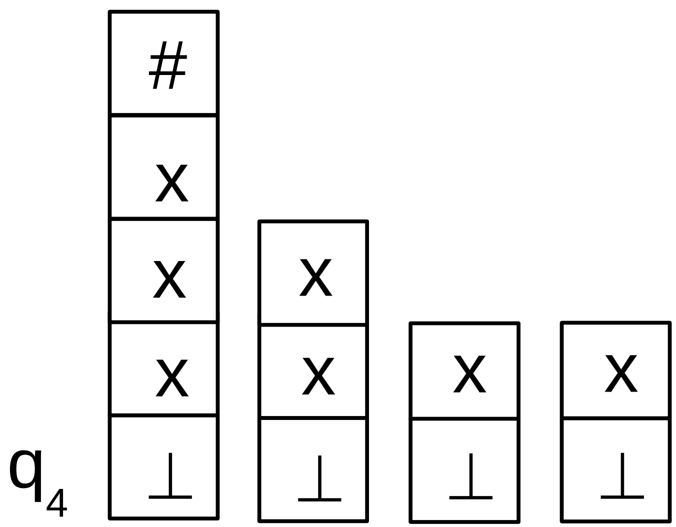

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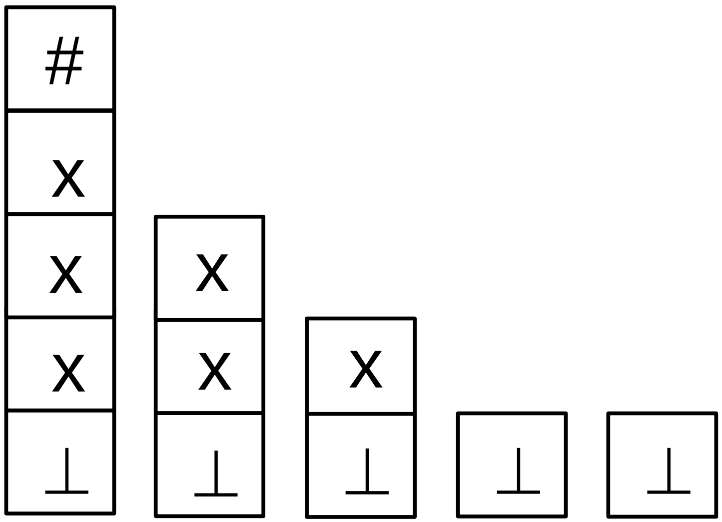

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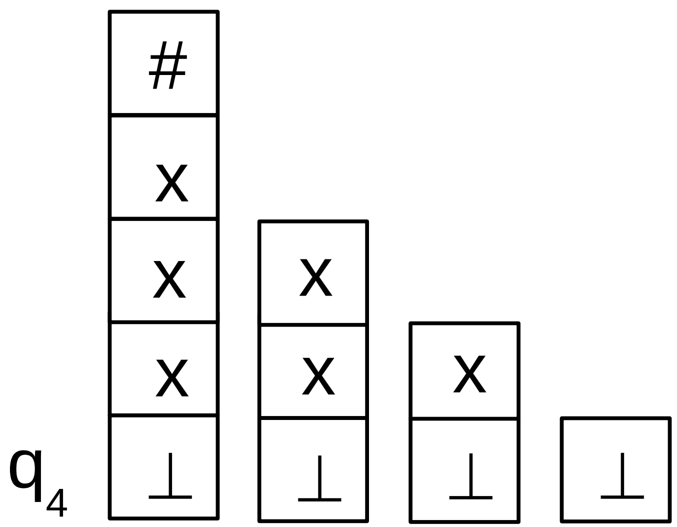

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Input: b

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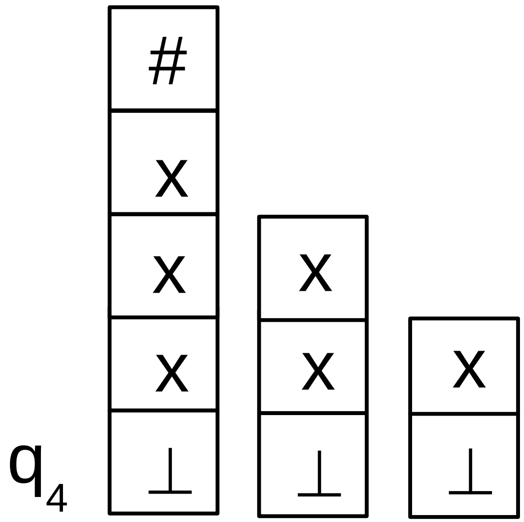

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Input: b b

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- 3 stack symbols: $\perp$, x, #

	#
	x
	x
	x
q_4	$\perp$

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$(\#,q_4) \xrightarrow{\varepsilon} (q_{acc},id)$

Input: b b b b b b b b

Higher-order pushdown automata

"Traditional" view:

- a nondeterministic HOPDA recognizing a language of words, as on previous slides

"Modern" view:

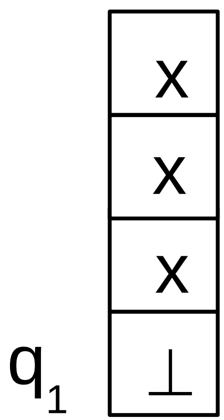
- a deterministic HOPDA generating a single tree (node-labeled, ranked, ordered, usually infinite)

One can also consider configuration graphs of HOPDA – not in this talk.

Higher-order pushdown automata - example

nondeterminism – what to do next?

- order 2
- 3 stack symbols: $\perp$, x, #



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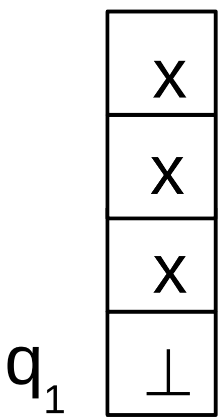
Higher-order pushdown automata - example

- order 2
- 3 stack symbols: ⊥, x, #



letter a of rank 2

$(\_,q_1) \xrightarrow{a} (q_1,push_1(x))$
$\phantom{(\_,q_1) \xrightarrow{a}} (q_2,push_1(\#))$

$(\#,q_2) \xrightarrow{\varepsilon} (q_3,push_2)$

$(\#,q_3) \xrightarrow{\varepsilon} (q_4,pop_1)$

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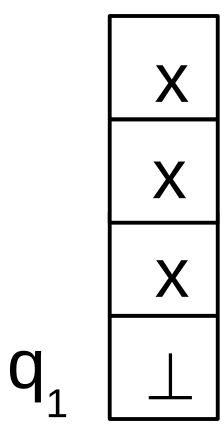

$$(\_,q_1) \xrightarrow{a} \begin{cases} (q_1,push_1(x)) \\ (q_2,push_1(\#)) \end{cases}$$

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$$(\#,q_4) \xrightarrow{c}$$

letter c of rank 0, instead of an accepting state

Higher-order pushdown automata - example

- order 2
- 3 stack symbols: $\perp$, x, #

Tree-generating HOPDA - definition

From every pair of stack symbol & state there is either:

- one ε-transition
- one transition reading a letter of rank k, resulting in k (ordered) pairs of state & operation.

$(\_,q_1)$ — a → $(q_1,push_1(x))$, $(q_2,push_1(\#))$

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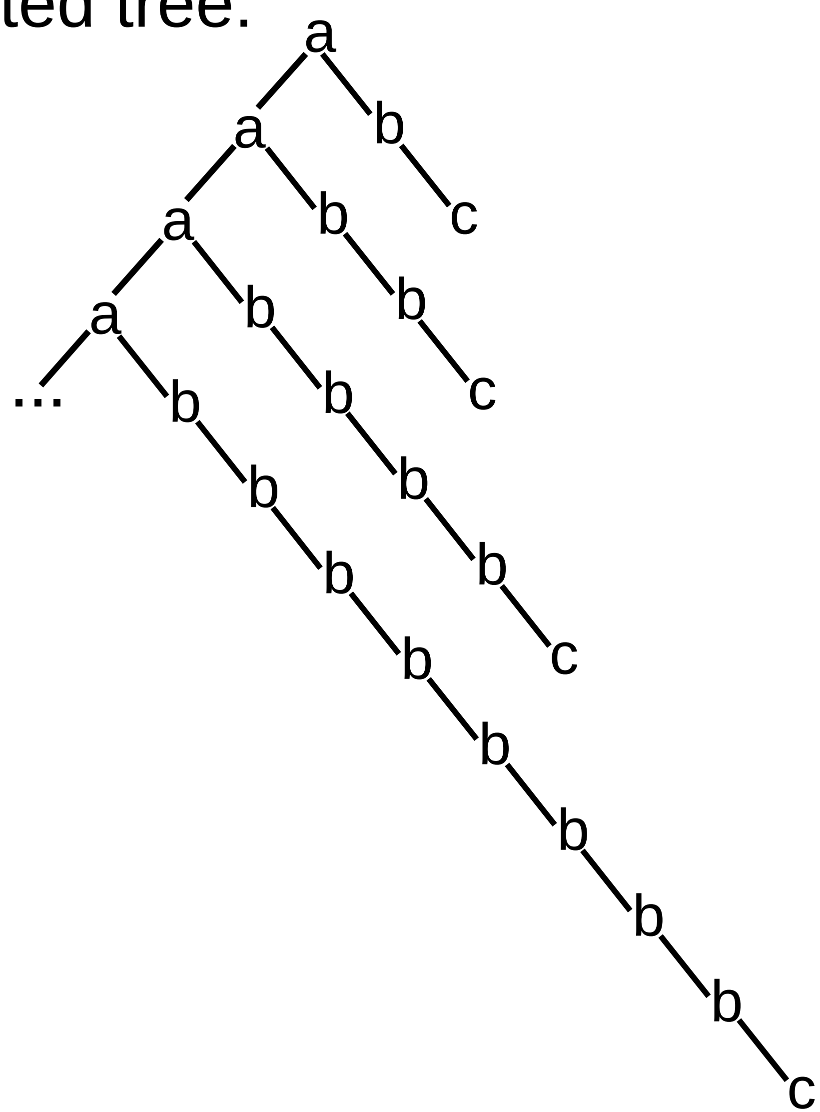
Higher-order pushdown automata - example

- order 2
- 3 stack symbols: $\perp$, x, #

Generated tree:



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Higher-order recursion schemes

pushdown automata $\xrightarrow{\text{generalization}}$ higher-order pushdown automata

context-free grammars $\xrightarrow{\text{generalization}}$ higher-order recursion schemes

Higher-order recursion schemes - definition

Nonterminals may take arguments, that can be then used on the right side of productions.

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Every nonterminal (every argument) has assigned some type.

Types:

$$\alpha ::= o \mid \alpha \to \beta$$

- o – type of a tree
- $o \to o$ – type of a function that takes a tree, and produces a tree
- $o \to (o \to o) \to o$ – type of a function that takes a tree and a function of type $o \to o$, and produces a tree

abbreviation of $o \to ((o \to o) \to o)$

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Order:

$$\mathrm{ord}(o) = 0$$

$$\mathrm{ord}(\alpha_1 \to \ldots \to \alpha_k \to o) = 1+\max(\mathrm{ord}(\alpha_1), \ldots, \mathrm{ord}(\alpha_k))$$

- $\mathrm{ord}(o) = 0$,
- $\mathrm{ord}(o \to o) = \mathrm{ord}(o \to o \to o) = 1$,
- $\mathrm{ord}(o \to (o \to o) \to o) = 2$

Higher-order recursion schemes – example

Ranked alphabet:

$a^{o\to o\to o}$ of rank 2, $b^{o\to o}$ of rank 1, c^{o} of rank 0

Nonterminals:

S^{o} (starting), $A^{(o\to o)\to o}$, $D^{(o\to o)\to o\to o}$

Higher-order recursion schemes – example (of order 2)

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order 0 order 2 order 2

Order of a HORS = maximal order of (a type of) its nonterminal

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Rules:

$S \to A\ b$

$A\ f \to a\ (A\ (D\ f))\ (f\ c)$

$D\ f\ x \to f\ (f\ x)$

It is required that:

1) types are respected

e.g. D of type $(o\to o)\to o\to o$ is applied to f of type $o\to o$,

A of type $(o\to o)\to o$ is applied to $D\ f$ of type $o\to o$, etc.

2) right side of every rule is of type o

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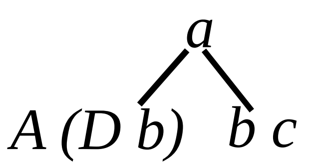

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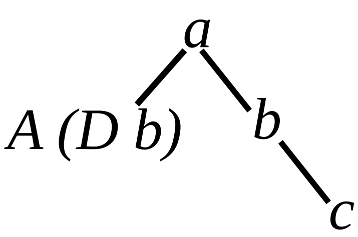

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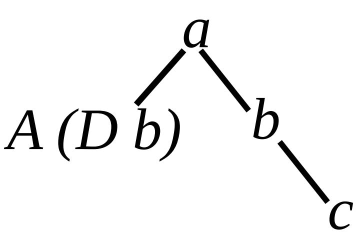

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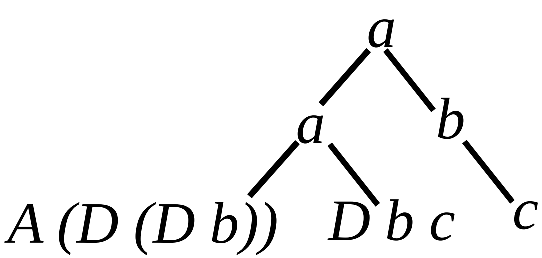

Rules:

$$
\begin{aligned}
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A\ f &\to a\ (A\ (D\ f))\ (f\ c) \\
D\ f\ x &\to f\ (f\ x)
\end{aligned}
$$

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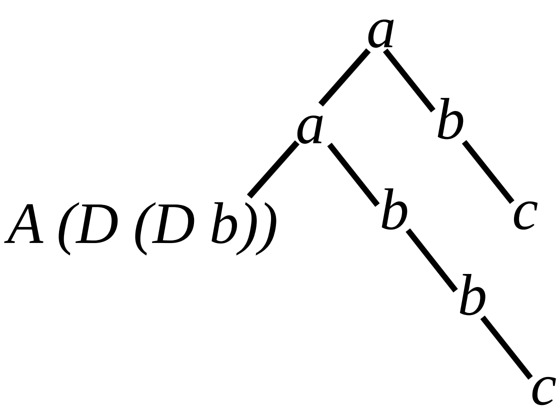

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$$D\, b\, c \to b\, (b\, c)$$

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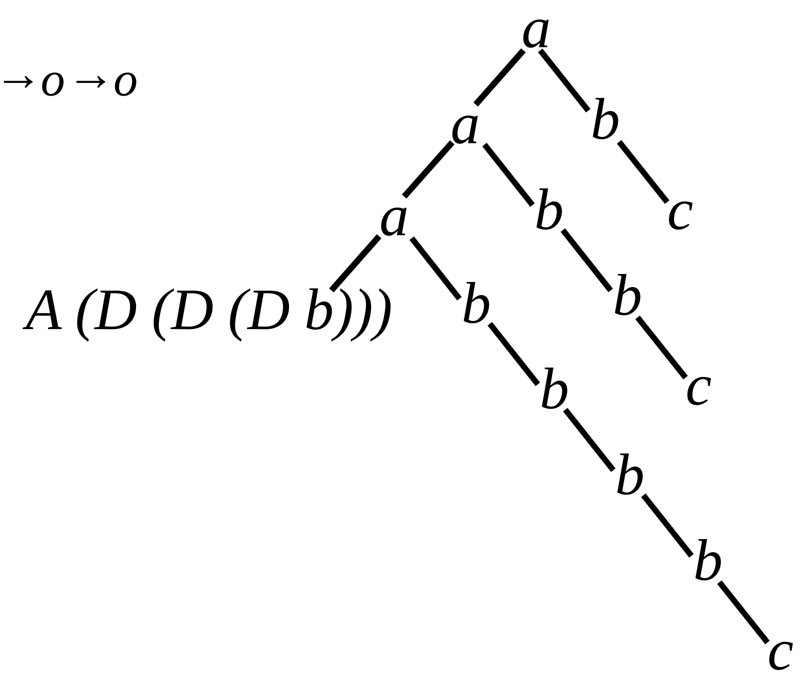

$S \to A\ b \to a\ (A\ (D\ b))\ (b\ c)$

$A\ (D\ b) \to a\ (A\ (D\ (D\ b)))\ (D\ b\ c)$

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$D\ (D\ b)\ c \to D\ b\ (D\ b\ c) \to b\ (b\ (D\ b\ c))$

Higher-order recursion schemes – example (of order 2)

Ranked alphabet:

$a^{o\to o\to o}$ of rank 2, $b^{o\to o}$ of rank 1, c^{o} of rank 0

Nonterminals:

S^{o} (starting), $A^{(o\to o)\to o}$, $D^{(o\to o)\to o\to o}$

Rules:

$$S \to A\ b$$
$$A\ f \to a\ (A\ (D\ f))\ (f\ c)$$
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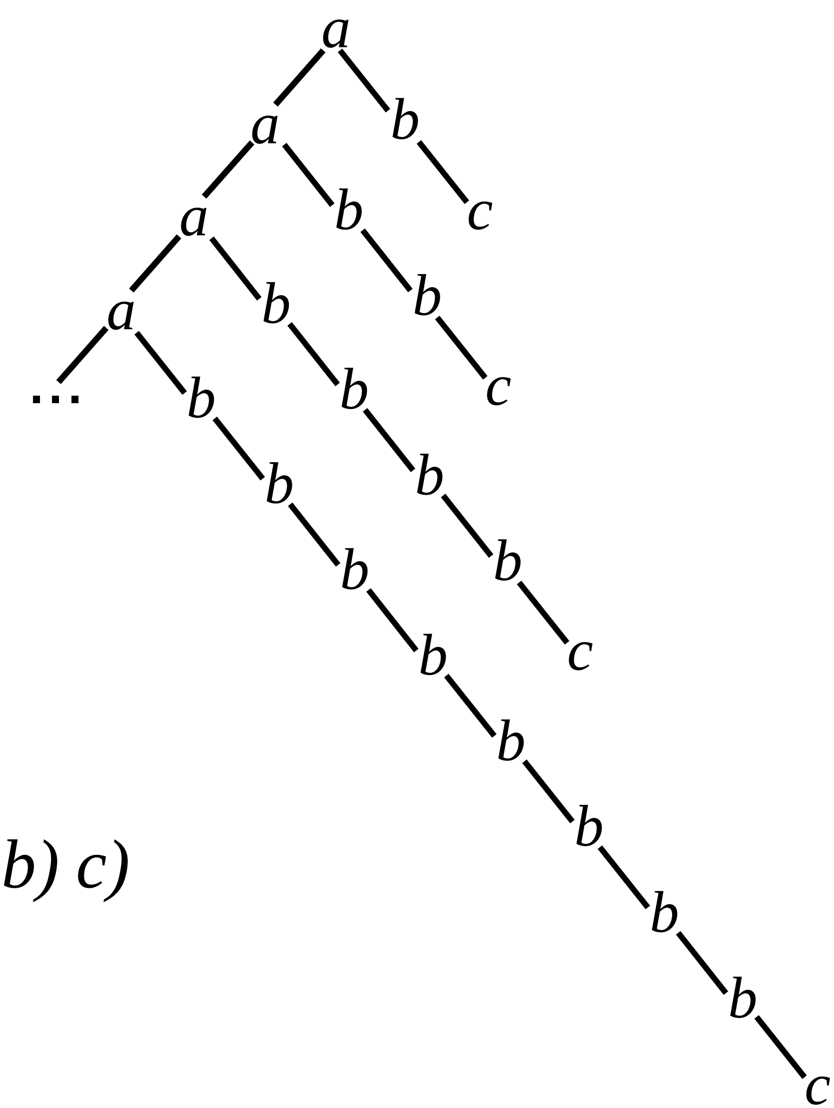

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Higher-order recursion schemes

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end of word marker

Example:

Alphabet: ~~a of rank 2~~, b of rank 1, c of rank 0

Nonterminals: S^{o} (starting), $A^{(o\to o)\to o}$, $D^{(o\to o)\to o\to o}$

Rules: $S \to A\, b$

$\cancel{A\, f \to a\,(A\,(D\,f))\,(f\,c)}$ $A\, f \to A\,(D\,f)$

$A\, f \to f\, c$

$D\, f\, x \to f\,(f\,x)$

Recognized language: $\{b^{2^k} : k \in \mathbb{N}\}$

HOPDA vs HORS

Are these two formalisms equivalent?

higher-order pushdown automata $\overset{?}{\Longleftrightarrow}$ higher-order recursion schemes

HOPDA vs HORS

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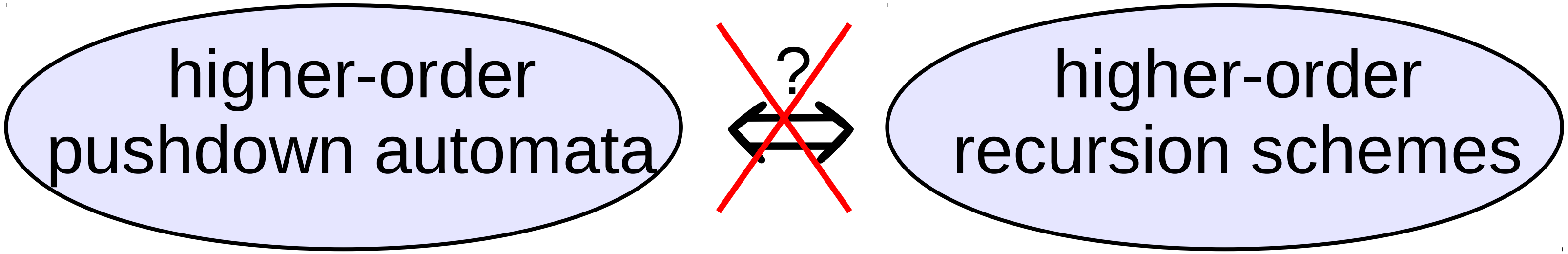
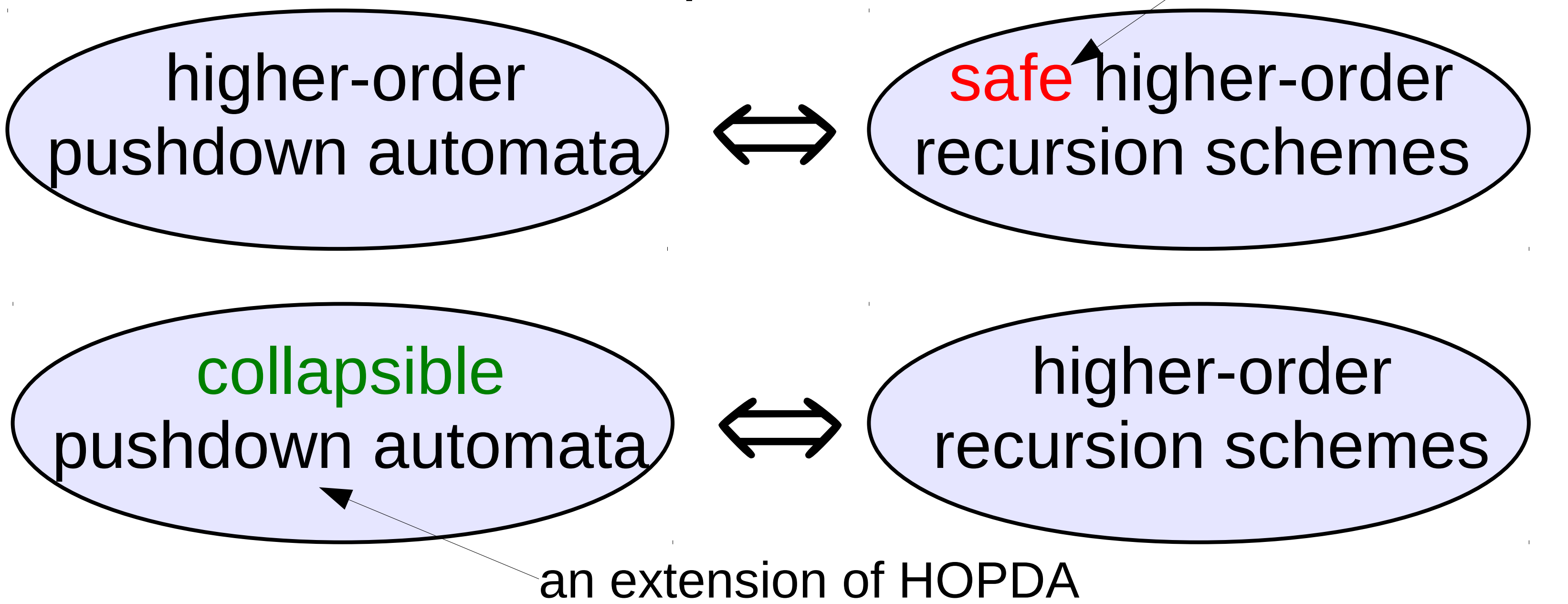


Not exactly!

HOPDA vs HORS

Are these two formalisms equivalent?



Theorem [Knapik, Niwiński, Urzyczyn 2002 & earlier results]
For every n, HOPDA of order n and safe HORSes of order n generate the same trees (recognize the same word languages);
[Caucal 2002] these are trees from the Caucal hierarchy, defined by iterating MSO interpretations and unfolding of graphs into trees.

Theorem [Hague, Murawski, Ong, Serre 2008]
For every n, collapsible HOPDA of order n and HORSes of order n generate the same trees (recognize the same word languages).

What is safety?

Restriction on terms appearing on right sides of rules:

- unrestricted terms:

$$M ::= a \mid x \mid A \mid M\ N$$

- safe terms:

$$M ::= a \mid x \mid A \mid M\ N_1 \dots N_k$$

only if $ord(M\ N_1 \dots N_k) \leq ord(N_i)$ for all i

In other words: if we apply an argument of some order k, then we have to apply also all arguments of order $\geq k$

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✓safe

$ord(D\ f) = 1 \leq 1 = ord(f)$ → OK

All other subterms are of order 0 → OK

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In other words: if we apply an argument of some order k, then we have to apply also all arguments of order $\geq k$

Example: Unsafe HORS (generating ”Urzyczyn's tree” U):
Types: $a^{o\to o\to o}$, $b^{o\to o}$, $c^{o\to o}$, d^{o}, e^{o}, S^{o}, $F^{(o\to o)\to o\to o\to o}$
Rules: $S \to F\; b\; d\; e$

$$F\; f\; x\; y \to a\; (F\; (F\; f\; x)\; y\; (c\; y))\; (a\; (f\; y)\; x)$$

✘ unsafe
(and not equivalent to any safe HORS)

$ord(F\; f\; x) = 1 > 0 = ord(x)$
(F expects two order-0 arguments; we have applied one (x), but not the other)

Why safety helps?

Theorem [Knapik, Niwiński, Urzyczyn 2002; Blum, Ong 2007]
Substitution (hence β-reduction) in safe λ-calculus can be implemented without renaming bound variables.

Bad example: when you substitute $(\lambda x. y\ x)\ [a\ x\ x\ /\ y]$, it is neccessary to change the first two x to some other variable name

Collapsible pushdown automata

- Every stack symbol has an identifier.
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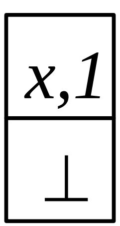

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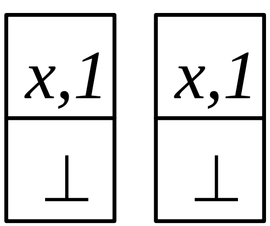
$push_1 x$

$push_2$



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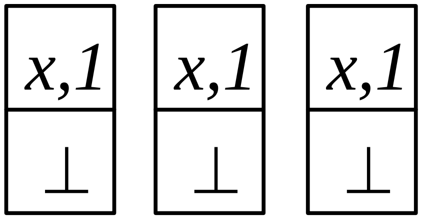

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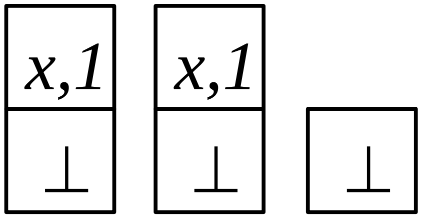
$push_2$
pop_1



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$push_2$

pop_1

$push_1 x$

$x,1$	$x,1$	$x,2$
$\perp$	$\perp$	$\perp$

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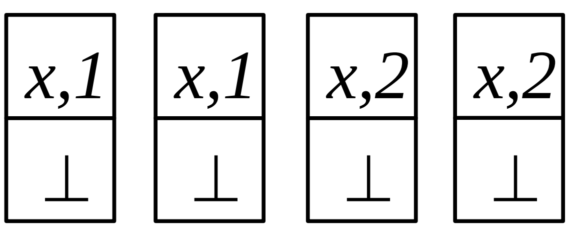

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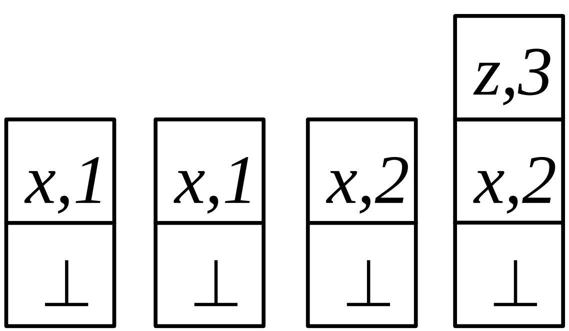

$push_1 x$

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$push_1 x$

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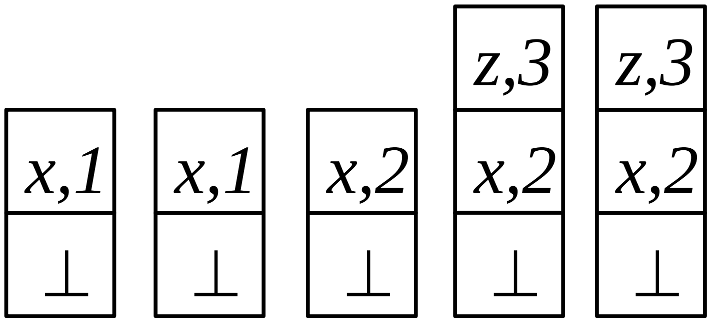

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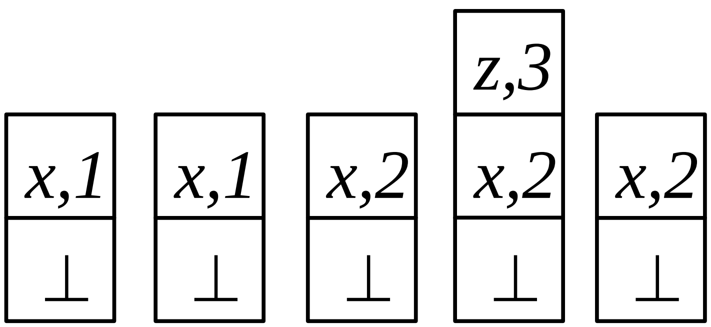

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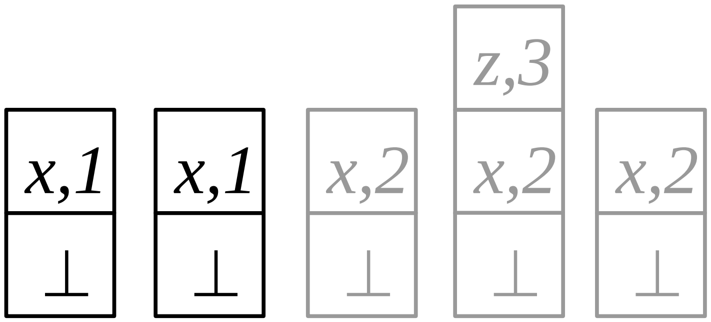

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$push_2$
pop_1
$push_1 x$
$push_2$
$push_1 z$
$push_2$
pop_1
$collapse_2$

Collapsible pushdown automata

How collapse can be useful? – Urzyczyn's language U
(≈ branches in the Urzyczyn's tree)

alphabet: [,], $*$
U contains words of the form:

$$\underbrace{[\,[\,]\,[}_{A}\,\underbrace{[\,]\,[\,[\,]\,]}_{B}\,\underbrace{****}_{C}$$

- segment A forms a prefix of a well-bracketed word that ends in [not matched in the entire word
- segment B forms a well-bracketed word
- the number of stars in C equals the number of brackets in A

Collapsible pushdown automata

How collapse can be useful? – Urzyczyn's language U

(≈ branches in the Urzyczyn's tree)

Words in U:

A) a prefix of a well-bracketed word

B) a well-bracketed word

C) as many stars as brackets in part A

- one stack symbol
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[[ ] [[ ] [[ ]] ****

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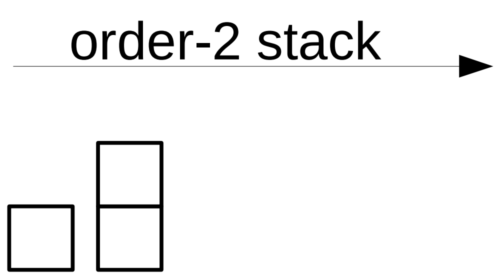

[[ ] [[ ] [[ ]] * * * *

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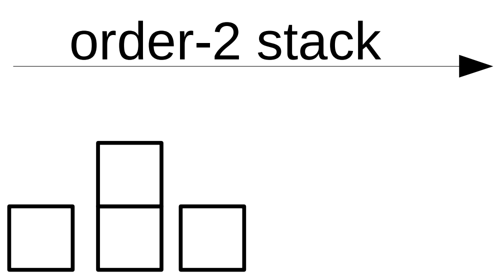

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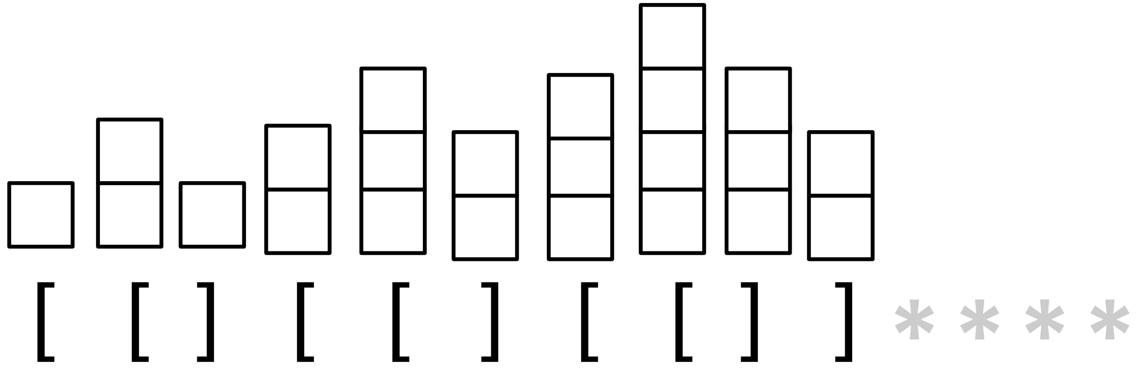

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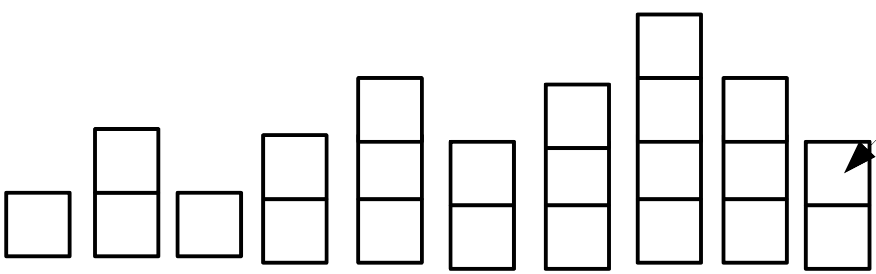
- count the number of stacks

Collapse = remove all stacks on which the topmost symbol is present



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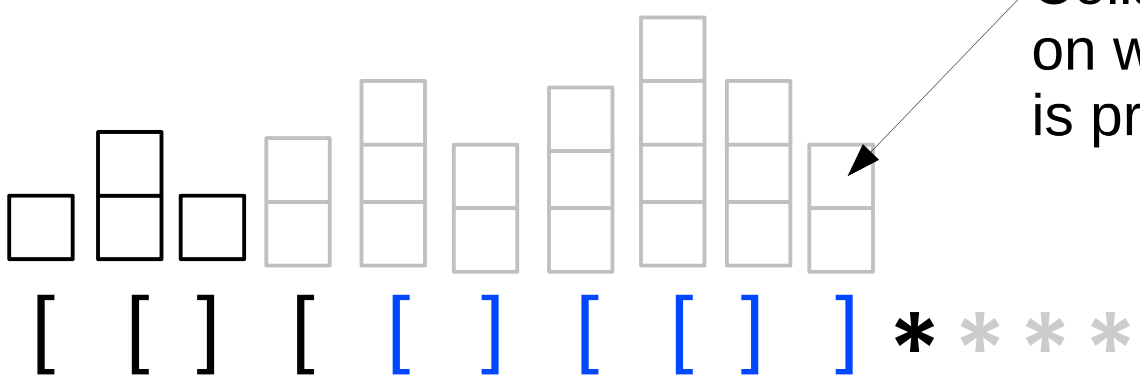

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Remark:
A nondeterministic order-2 PDA without collapse can recognize U,
as it can guess when is the beginning of the "B" part.
But not a deterministic HOPDA without collapse, of any order!
(This means that the Urzyczyn's tree cannot be generated by a HOPDA)

Expressivity questions

Tree(n)= trees generated by HORSes (CPDA) of order n
SafeTree(n) = trees generated by safe HORSes (HOPDA) of order n

$$\begin{array}{ccccccccc}
\text{SafeTree}(0) & \subseteq & \text{SafeTree}(1) & \subseteq & \text{SafeTree}(2) & \subseteq & \text{SafeTree}(3) & \subseteq & \dots \\
\vert\!\cap & & \vert\!\cap & & \vert\!\cap & & \vert\!\cap & & \\
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\| & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \\
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\| & & & & & & & & \\
\text{regular trees} & & & & & & & &
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\| & & \| & & & & & & \\
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\end{array}$$

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Expressivity questions

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Theorem [Engelfriet 1991]
For every n, SafeLang(n) $\neq$ SafeLang($n+1$),
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Separating language: correct sequences of operations of order-($n+1$) HOPDA
(including the topmost stack symbol after every step).
Proof: "Simple trick" using the fact that reachability for order-n HOPDA is in (n-1)-EXPTIME, while reachability for order-($n+1$) HOPDA is n-EXPTIME-hard.

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The same proof works for CPDA.
Thus Tree(n) ≠ Tree($n+1$) & Lang(n) ≠ Lang($n+1$).

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Another separator:

T_n = tree with branches $a^k b^{exp_n(k)} c$, where $exp_n(k) = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^k}}}}}}_{n}$

We have SafeTree($n+1$) $\ni T_n \notin$ Tree(n).

pumping lemma [Kartzow, P. 2012]

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pumping lemma [Kartzow, P. 2012]

For languages we do not know:

$$\mathrm{SafeLang}(n+1) \ni \{b^{exp_n(k)} : k \in \mathbb{N}\} \overset{?}{\notin} \mathrm{Lang}(n).$$

Open problem: a pumping lemma
for nondeterministic HORSes.

Expressivity questions

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Is safety really a restriction?

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Is safety really a restriction?

For trees – yes.

Example: Urzyczyn's tree U

Tree($\mathit{2}$) $\ni U \notin$ SafeTree(n) for every n [P. 2012]

For word languages – open problem (e.g. SafeLang(3) $\overset{?}{\neq}$ Lang(3))

Expressivity questions

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Are these languages context-sensitive?

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CSens = context-sensitive languages (type-1 in the Chomsky hierarchy)

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Lang(n) $\overset{?}{\subseteq}$ CSens for $n \geq 4$ – open problem

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- Consider the following algorithm: starting from the initial nonterminal, follow nondeterministically rules of the HORS, trying to derive the input word.
- It works well if all intermediate terms are smaller than the derived word (= input word).
- The "only difficulty": describe/eliminate nonterminals that are "not productive", i.e., that do not increase the size of the derived word.

Algorithmic questions

Problem: MSO model-checking

Input: MSO formula ϕ, HORS S

Output: does ϕ hold in the tree generated by S?

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Theorem: MSO model-checking is decidable.

[Knapik, Niwiński, Urzyczyn 2002] – safe schemes only
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- polynomial when n, ϕ, and maximal arity of a nonterminal fixed
- despite high complexity, solvable in practice *(see next talk)*

Algorithmic questions

Theorem: MSO model-checking is decidable.

Idea of a proof

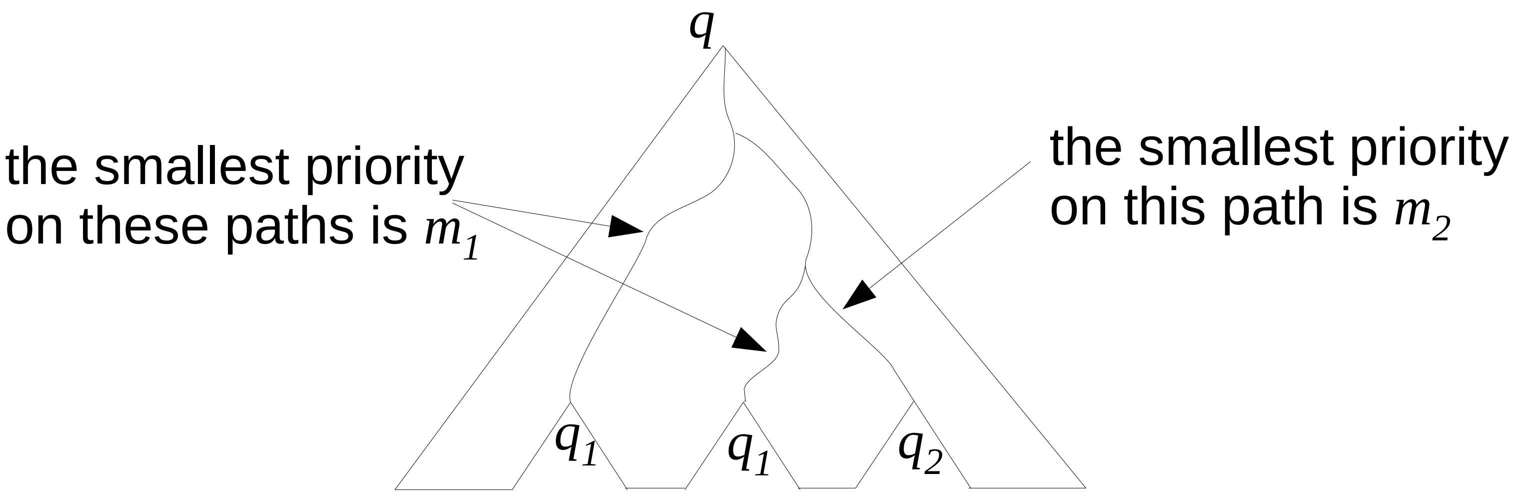
Input: alternating parity automaton A, HORS S
Question: does A accept the tree generated by S?

We refine simple types into intersection types of the form:

o is refined to $q \in Q$ (a state)

$\alpha \to \beta$ is refined to $\{(\tau_1, m_1), \ldots, (\tau_k, m_k)\} \to \tau$ where τ_i refines α, τ refines β, m_i is a priority

Intuition: a function with type $\{(q_1,m_1),(q_2,m_2)\} \to q$ (refining $o \to o$):



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MSO reflection

Input: MSO formula $\phi(x)$, HORS S

Output: HORS S' generating the same tree as S, where the nodes x in which $\phi(x)$ holds are marked.

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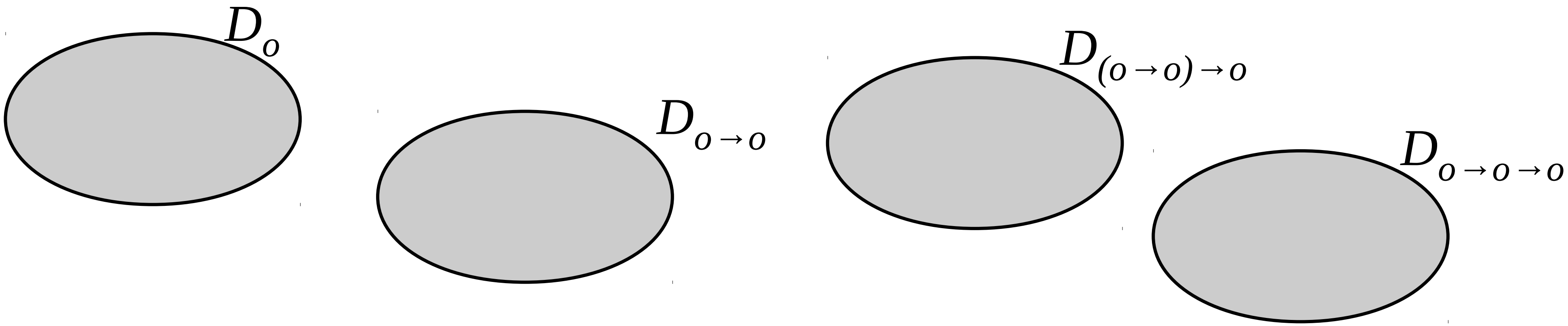
What is a model?

Input: MSO formula ϕ, HORS S

Output: a finite set D_α for every sort α



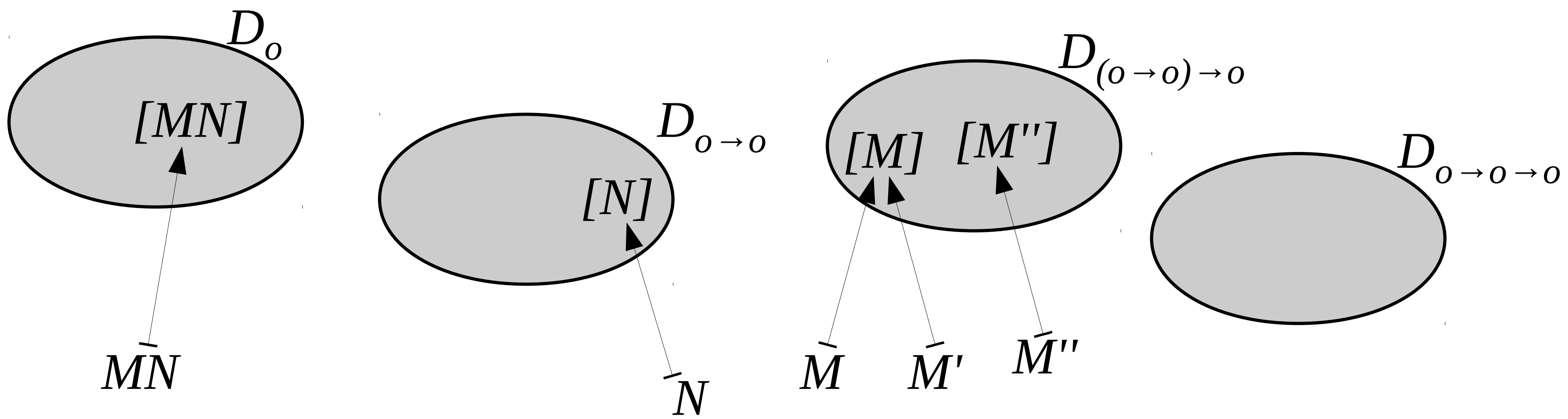
What is a model?

$[M]$ depends on valuation of free variables of M

Input: MSO formula ϕ, HORS S

Output: a finite set D_α for every sort α,

a value $[M]\in D_\alpha$ for every term M sort α



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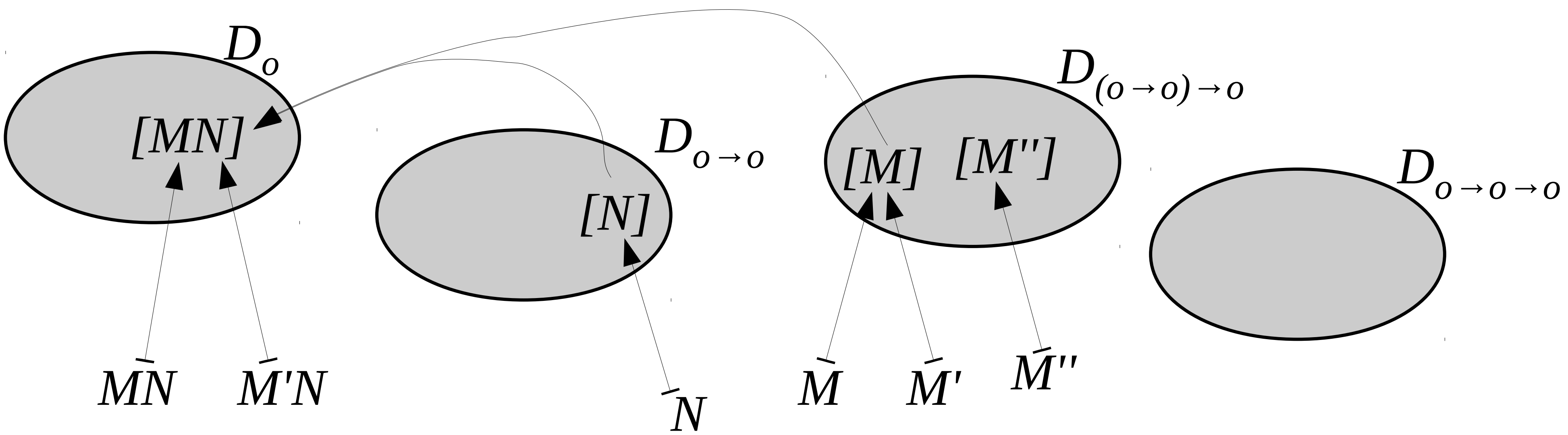

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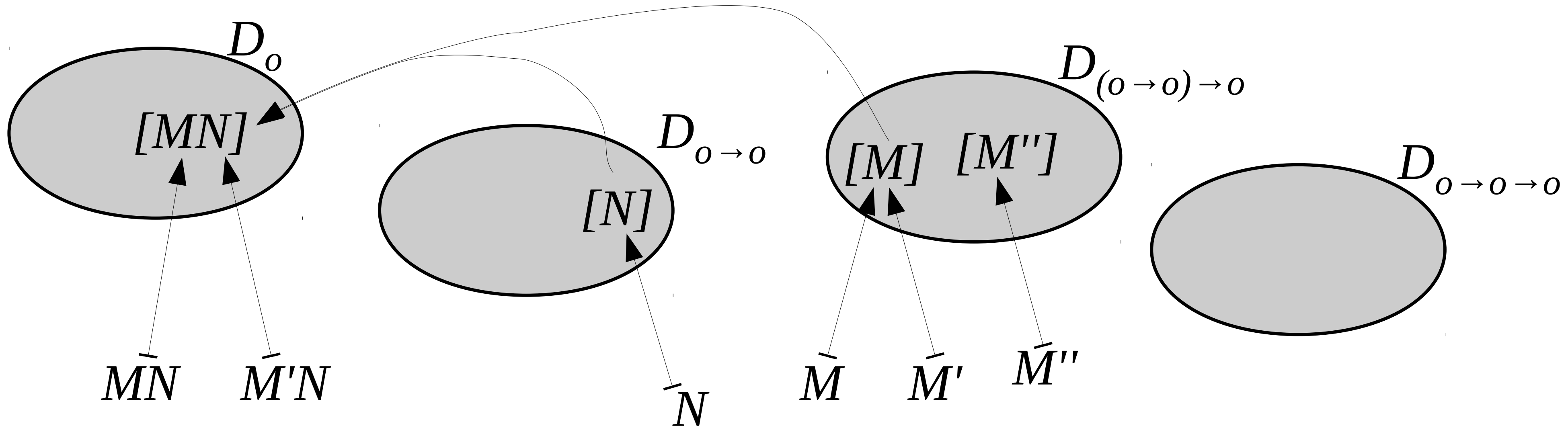

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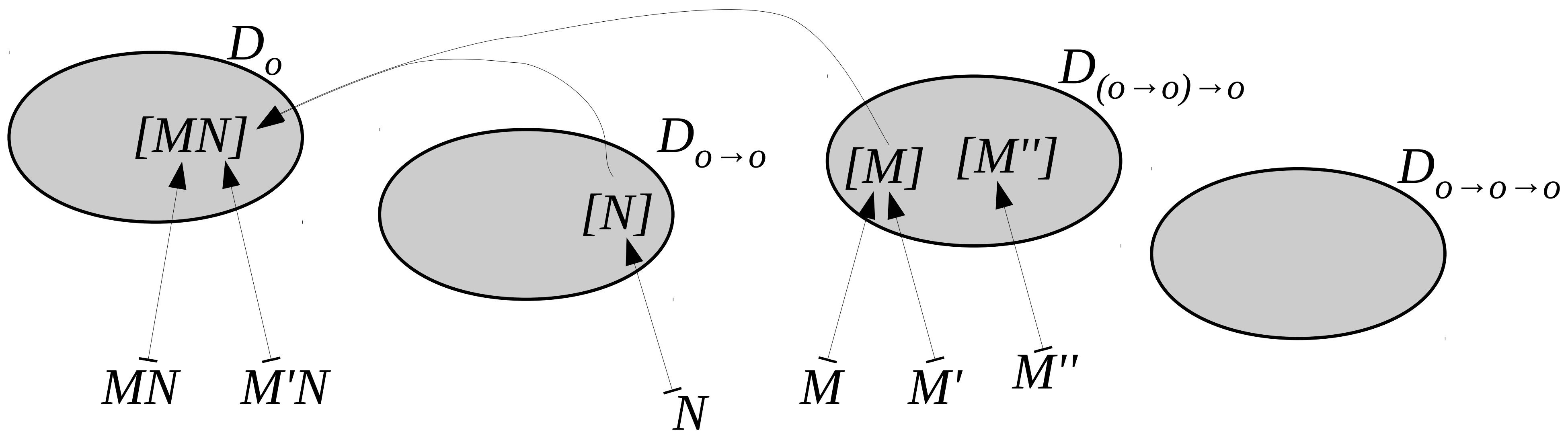

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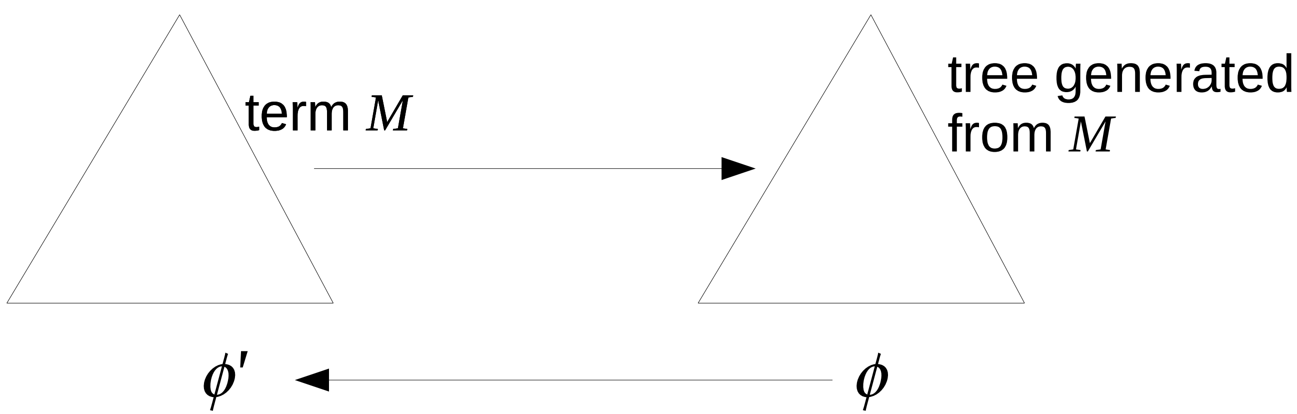
$\Downarrow$ transfer theorem

Basing on ϕ one can construct ϕ' such that ϕ' holds in a closed term M of sort o iff ϕ holds in the tree generated from M.



(special case: M = starting nonterminal)

Beyond MSO?

Problem: WMSO+U model-checking
Input: WMSO+U formula ϕ, HORS S
Output: does ϕ hold in the tree generated by S?

Ongoing work: WMSO+U model-checking is decidable.

MSO+U = Weak MSO (set quantifiers range over finite sets only)
+ new quantifier U

where: U$X.\phi$ means that ϕ holds for some arbitrarily large finite sets X

Downward closure

Let L be a set of words. Its downward closure $L\downarrow$ contains all words that can be obtained from words in L by removing some letters.

E.g. $L=\{abc\}$, $L\downarrow=\{e,a,b,c,ab,bc,ac,abc\}$

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Theorem [Zetzsche 2015, Hague, Kochems, Ong 2016, Clemente, P., Salvati, Walukiewicz 2016]
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Some ideas:

- For every regular language K we check whether $L{\downarrow}=K$.

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- $L{\downarrow}$ (so K as well) is necessarily a finite union of languages of the form $S_i=A_0^*a_1^?A_1^*a_2^?\ldots A_{k-1}^*a_k^?A_k^*$. It remains to check whether $S_i\subseteq L{\downarrow}$ for all i.

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- By transforming the scheme, this reduces to the diagonal problem:

Input: a scheme S recognizing $L\subseteq a_1^*a_2^*\dots a_k^*$ (with different letters)

Question: does $L\downarrow=a_1^*a_2^*\dots a_k^*$?

(in other words: is it the case that for every n we have in L words with more than n appearances of every letter?)

This is the actual problem to be solved.

The diagonal problem

Input: a scheme S recognizing $L \subseteq a_1^* a_2^* \dots a_k^*$ (with different letters)

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How to solve it?

a scheme S of order n with a word written on a branch $\xrightarrow{\text{step 1}}$ a scheme S of order n-1 with this word written in leaves

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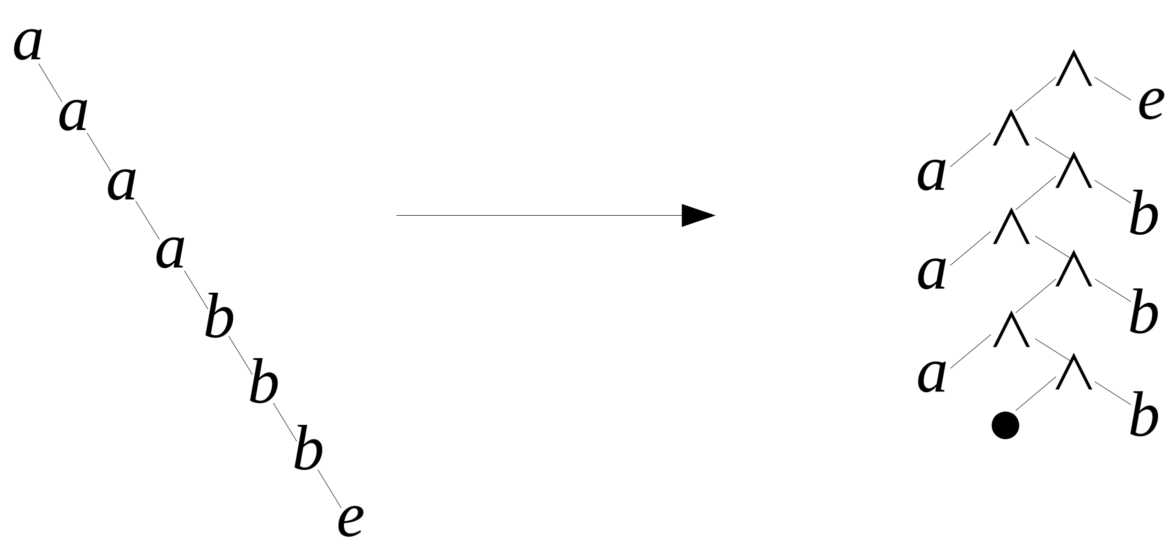
Example:

$$S \to A\, e \qquad\qquad S \to \wedge\, A\, e$$
$$A\, x \to a\, (A\, (b\, x)) \quad \longrightarrow \quad A \to \wedge\, a\, (\wedge\, A\, b))$$
$$A\, x \to x \qquad\qquad A \to \bullet$$

(rank 1: a, b; rank 0: e) (rank 2: $\wedge$; rank 0: a, b, e, $\bullet$)



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2) All arguments of type o are dropped ($\Rightarrow$ order decreases).

3) Every subterm $M\,N$ with N of type o can be replaced

a) either by $\wedge\, M\, N$ (when the argument is used in M),

b) or by M (when the argument is ignored in M).

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4) Additional work is required to choose correctly a) or b).

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1) Choose (nondeterministically) only one branch.

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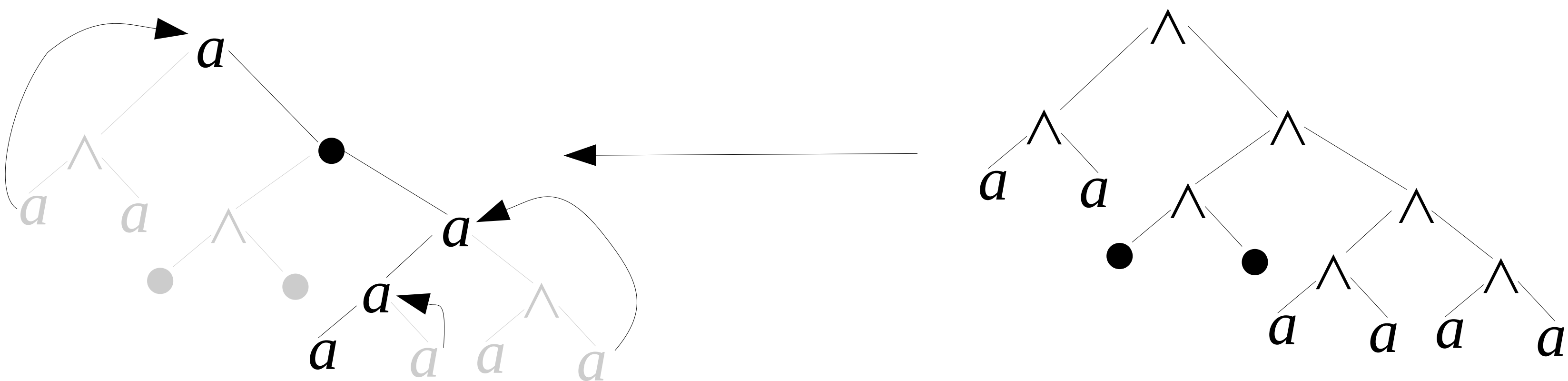

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Repeat these steps until the order drops down to 0,
and solve the diagonal problem for a regular language.

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3. ~~Consider a system with one leader and some (unspecified) number of contributors, that communicate via common register (read or write, without any locks). The reachability problem in such system reduces to computation of the downward closure [La Torre, Muscholl, Walukiewicz 2015].~~ *(Yesterday's talk – downward closure no longer needed)*

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Another open problem: computation of downward closure for schemes recognizing languages of trees.

(By Kruskal's tree theorem the downward closure of any language of trees is a regular language.)

Thank you!