Combinatorics of transformation semigroups and synchronization

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Abstract

A large body of recent work arises in questions about finite transformation semigroups. I will discuss some of this work.

The first topic is synchronization. A finite deterministic automaton is *synchronizing* if there is a word w (called a *reset word*) with the property that, after reading w, the automaton is in a known state, independent of its starting state. Much research here has been driven by the (still unsolved) *Černý conjecture*, which states that if an n-state automaton is synchronizing, then it has a reset word of length at most $(n-1)^2$: this bound is best possible. This is a question about transformation semigroups because the transitions of an automaton generate a semigroup of transformations of the set of states, and the automaton is synchronizing if and only if the semigroup contains a transformation of rank 1. It turns out that there is a single obstruction to synchronization: a semigroup is non-synchronizing if and only if it is contained in the endomorphism monoid of an undirected graph with clique number equal to chromatic number. Much of the research in this area has focussed on finding the permutation groups G which have the property that $\langle G, t\rangle$ is synchronizing for every non-permutation t: this class of groups lies between primitive and 2-homogeneous.

The other area of interest involves the study of "classical" properties of transformation semigroups of the form $\langle G, t\rangle$ for a permutation group G. I will discuss in particular a recent example which is a conjectured classification of the primitive permutation groups G for which, whenever t is a transformation of rank 2, the semigroup $\langle G, t\rangle$ is idempotent-generated.