Hyperovals in $\mathbb{P}^2(\mathbb{F}_q)$

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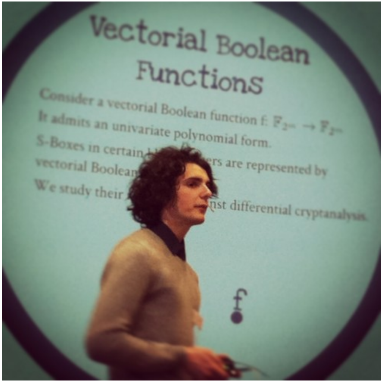
Germany



Florian Caullery

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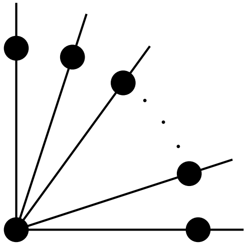

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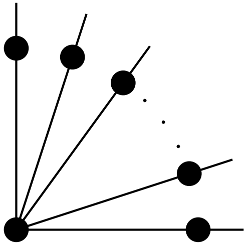
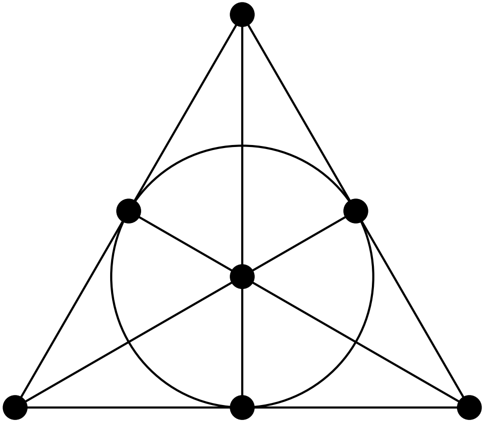


In case of equality, the arc is a called a hyperoval.

A hyperoval in the Fano plane



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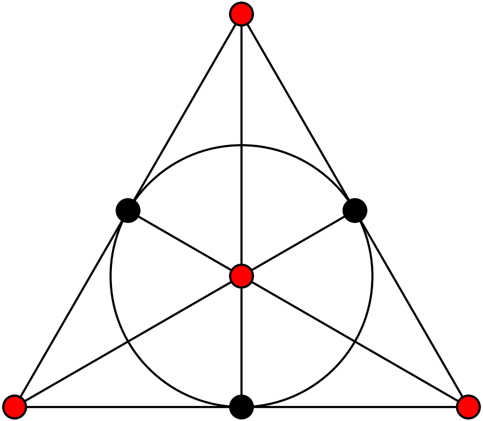

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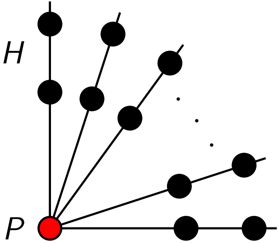

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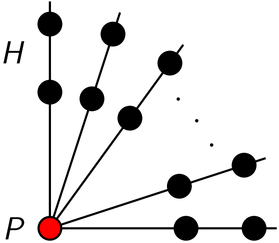

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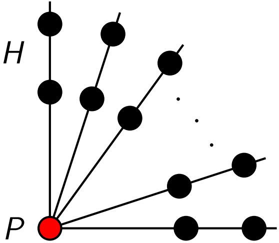

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An arc of size $q+1$ is called an oval.

Existence of ovals and hyperovals

All conics are ovals:

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Not in even characteristic.

Classification of hyperovals

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I know no-one of significance who shared his confidence in classifying hyperovals in the near future.

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Coordinisation

- Wlog: every hyperoval contains the quadrangle

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- The unique hyperoval in $\mathbb{P}^2(\mathbb{F}_4)$:

$$(1:0:0), (0:1:0),$$

$$(0:0:1), (1:1:1), (\alpha:\alpha^2:1), (\alpha^2:\alpha:1).$$

Hyperovals and o-polynomials

For $q > 2$, a set of $q+2$ points in $\mathbb{P}(\mathbb{F}_q)$ is a hyperoval if and only if it can be written as

$$\big\{(1:0:0),(0:1:0)\big\} \cup \big\{(f(c):c:1) : c \in \mathbb{F}_q\big\},$$

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- For each $a \in \mathbb{F}_q^*$, the mapping

  $$x \mapsto f(x) + ax$$

  is 2-to-1 on $\mathbb{F}_q$ (Carlet-Mesnager 2011).

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$$\sigma = 2^{(h+1)/2} \qquad \gamma = \begin{cases} 2^{(3h+1)/4} & \text{for } h \equiv 1 \pmod 4 \\ 2^{(h+1)/4} & \text{for } h \equiv 3 \pmod 4 \end{cases}$$

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Known classification results

Planes of small order:

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Polynomials of a certain form:

- Linearised o-polynomials (Payne 1971, Hirschfeld 1975)
- O-monomials of degree $2^i + 2^j$ (Cherowitzo-Storme 1998)
- O-monomials of degree $2^i + 2^j + 2^k$ (Vis 2010)

A new classification result

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Similar classification problems have been studied extensively for permutation polynomials, cyclic codes, and planar functions, but no complete solution is known in these cases.

The determinant condition

Every o-polynomial f of $\mathbb{F}_q$ satisfies

$$\det\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ f(x) & f(y) & f(z) \end{pmatrix} \neq 0 \quad \text{for all distinct } x, y, z \in \mathbb{F}_q.$$

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Associate with $f \in \mathbb{F}_q[x]$ the determinant polynomial:

$$\Phi_f = \frac{x(f(y)+f(z)) + y(f(x)+f(z)) + z(f(x)+f(y))}{(x+y)(x+z)(y+z)}.$$

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Goal

Show that, for most $f \in \mathbb{F}_q[x]$ of low degree, the determinant polynomial Φ_f has roots in $\mathbb{F}_q^3$ with x, y, z distinct.

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Lang-Weil Bound (Ghorpade-Lauchaud 2002)

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n]$ be an absolutely irreducible polynomial of degree d and let N be its number of roots in $\mathbb{F}_q^n$. Then

$$|N - q^{n-1}| \leq (d-1)(d-2)q^{n-3/2} + 12(d+3)^{n+1}q^{n-2}.$$

Strategy

The determinant polynomial:

$$\Phi_f = \frac{x(f(y) + f(z)) + y(f(x) + f(z)) + z(f(x) + f(y))}{(x + y)(x + z)(y + z)}.$$

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Why? Write $f(x) = \sum_{i=1}^{q-1} c_i x^i$ and expand:

$$\left.\frac{f(x+a)+f(a)}{x}\right|_{x=0} = c_1 + c_3 a^3 + \cdots + c_{q-1} a^{q-2}.$$

The monomial case

For monomials $f(x) = x^d$ we have

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Theorem (Hernando-McGuire 2012)

If d is an even positive integer, not equal to 6 or a power of two, then Φ_f has an absolutely irreducible factor over $\mathbb{F}_2$.

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Their proof uses Bezout's Theorem: If g and h are projective curves over an algebraically closed field $\mathbb{K}$ with no common component, then

$$(\deg g)(\deg h) = \sum_{P \in \mathbb{P}^2(\mathbb{K})} I_P(g, h),$$

where $I_P(g, h)$ is the intersection number of g and h at P.

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Corollary: The determinant polynomial of $f(x) = x^d$ has no absolutely irreducible factor over $\mathbb{F}_2$ if and only if

$$F(x) = \frac{(x+1)^d + 1}{x}$$

is an exceptional polynomial.

Zieve's approach

A polynomial $F \in \mathbb{F}_q[x]$ is indecomposable if there do not exist $G, H \in \mathbb{F}_q[x]$ of degree at least two such that $F(x) = G(H(x))$.

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If $F \in \mathbb{F}_p[x]$ is an indecomposable exceptional polynomial of degree coprime to p, then there are polynomials $\mu, \nu \in \mathbb{F}_p[x]$ of degree one such that $\mu \circ F \circ \nu$ is either a monomial or a Dickson polynomial of degree coprime to $p^2 - 1$.

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Let d be even and suppose that

$$F(x) = \frac{(x+1)^d + 1}{x} = \sum_{i=1}^{d} \binom{d}{i} x^{i-1}$$

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is an exceptional polynomial. Write $F = G \circ H$ for an indecomposable polynomial H and apply the theorem to H.

Finishing the proof

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where H is (up to compositions with linear polynomials) either

- a monomial, which forces d to be a power of 2; or
- a Dickson polynomial of degree s coprime to 6, thus

  $$H(x + x^{-1}) = x^s + x^{-s}$$

  and

  $$\sum_{i=1}^{d} \binom{d}{i} (x + x^{-1})^{i-1} = G(x^s + x^{-s}),$$

  which forces $d = 6$.

Recap: The monomial case

For monomials $f(x) = x^d$ we have

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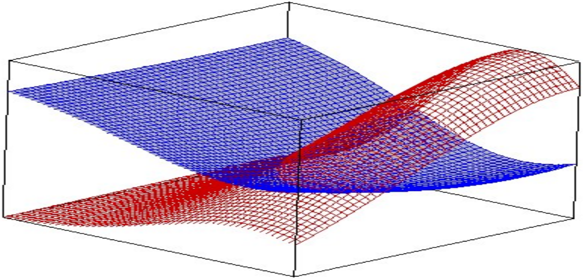
Corollary

If $f(x) = x^d$ is an o-polynomial of $\mathbb{F}_q$ with d less than $\frac{1}{2}q^{1/4}$, then d is either 6 or a power of 2.

Intersecting surfaces

Lemma (Aubry-McGuire-Rodier 2010)

Let S and P be projective surfaces in $\mathbb{P}^3(\overline{\mathbb{F}}_q)$, where P is defined over $\mathbb{F}_q$. If $S \cap P$ has a reduced absolutely irreducible component defined over $\mathbb{F}_q$, then S has an absolutely irreducible component defined over $\mathbb{F}_q$.



The reduction step

- Let $f(x) = \sum_i c_i x^i$ be of degree d. Then

$$\Phi_f(x, y, z) = \sum_{i=2}^{d} c_i \phi_i(x, y, z),$$

  where

$$\phi_i(x, y, z) = \frac{x(y^i + z^i) + y(x^i + z^i) + z(x^i + y^i)}{(x+y)(x+z)(y+z)}.$$

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- We are left with o-polynomials of degree 6 or a power of 2.

The remaining degrees

Degree 6:

Lemma (Hirschfeld 1971)

If f is an o-polynomial of degree 6, then f is equivalent to x^6.

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Proposition (Caullery-S. 2015)

If f is an even polynomial of degree 2^k, then Φ_f is absolutely irreducible or f is a linearised polynomial.

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Lemma (Payne 1971, Hirschfeld 1975)

If f is a linearised o-polynomial, then it is of the form x^{2^k}.

Hyperovals in $\mathbb{P}^2(\mathbb{F}_q)$

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