Erdős-Ko-Rado theorems in polar spaces

Klaus Metsch, Justus Liebig Universität, Gießen

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The theorem of Erdős-Ko-Rado

Theorem, 1961:

If S is an intersecting family of k-subsets of an n-set Ω, $n \geq 2k$, then $|S| \leq \binom{n-1}{k-1}$. For $n \geq 2k+1$ equality holds iff S consists of all k-subsets of Ω containing a fixed element.

Proof. Different techniques available

Generalisations

- t-intersection
- r-wise intersecting
- stability results
- other structures
- flags
- combinations

Vector spaces

Theorem (Hsieh, Frankl, Wilson, Godsil, Newman, Tanaka)

The largest sets of k-subspaces of F_q^n, $n \geq 2k$, which pairwise intersect non-trivially consists of all k-spaces on a 1-dimensional subspace, or if $n = 2k$, all k-subspaces in a hyperplane.

Proof: Geometric techniques for $n > 2k$, algebraic for $n \geq 2k$.

Vector spaces

Theorem (Blokhuis, Brouwer, Chowdhury, Frankl, Mussche, Patkos, Sőnyi, 2010)
Largest example of an intersecting set of k-subspaces of F_q^n not centered at a point when $q \geq 2$, $n \geq 2k+1$, $(q,n) \neq (2,2k+1)$.

Theorem (Blokhuis, Brouwer, Güven, 2014)
The largest EKR-sets of point-hyperplane flags in $\mathrm{PG}(n,q)$ are known.

EKR-sets of generators in polar spaces

Theorem (Stanton 1980)
All examples of the following list are largest sets of mutually intersecting generators in a polar space

- For polar spaces other than $Q^+(2d-1,q)$ and $H(2d-1,q^2)$ for odd $d \geq 3$: all generators on a point.
- For $Q^+(2d-1,q)$, $d \geq 3$ odd: All Latin (Greek) generators.

The correct bound for $H(2d-1,q^2)$, $d \geq 3$ odd remained open.

Classification of largest examples

Theorem, (Pepe, Storme, Vanhove, 2011)
Except when the polar space is $H(2d-1, q^2)$, $d \geq 5$ odd, the following is a complete list of largest EKR-sets of generators

- the above ones
- polar spaces $Q(6, q)$, $W(5, q)$ and $H(5, q^2)$:
  all planes centered at a plane.
- for polar spaces $Q(2d, q)$, $d \geq 3$ odd:
  all Latin generators contained in a $Q^+(2d-1, q)$

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Remarks

1. The example for $H(5,q^2)$ is larger than the point-example!
2. The correct bound for $H(2d-1,q^2)$, $d \geq 5$ odd, is still open.

EKR-sets of generators in $H(2d-1,q^2)$, $d \geq 5$ odd

- Point-example has size $\approx q^{(d-1)^2}$.
- Hoffman bound: $\approx q^{d(d-1)}$

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- **Theorem** (M, 2016)

$$|S| \leq \left(\frac{q^{2d}-q^{2d-3}}{q-1}+1\right) \prod_{\substack{i=1 \\ 2i \neq d\pm 1}}^{d-1} (q^{2i-1}+1)$$

$$= q^{(d-1)^2+1} + const \cdot q^{(d-1)^2}.$$

Proof: Hoffman-bound for matrix $A_d - fA_{d-2}$.

EKR-sets of n-spaces in polar spaces of rank d

Different question are possible.

- Sets of n-spaces no two of which are disjoint.
- Sets of n-spaces, no two of which are opposite (general position)

For generators both questions are the same.

EKR of n-spaces in polar spaces of rank d

Theorem (M, 2016):
A largest intersecting set of n-subspaces of a polar space of rank d with $1 < n < d$, consists of all n-spaces on a point.

Proof:

- Geometric arguments for $n \neq d - 1$.
- Algebraic arguments for all n, calculation of eigenvalues difficult.
- For $n = d - 1$ this is possible and Hoffman's bound for the matrix $f \cdot A_{0,d-1} + A_{0,d}$ proves the upper bound. The characterisation of the largest sets C is also done algebraically: Characteristic vector $c = \alpha j + v_0 + v_1$. Calculate $A_{0,d}c$.
  $\Rightarrow$ C has same distribution as the point-example.
  Now do geometry.

EKR-sets of non opposite lines

Problem Determine the largest sets C of lines in a polar space of rank d such that

$$\ell_1^\perp \cap \ell_2 = \emptyset \quad \forall \ell_1, \ell_2 \in C.$$

Example: Take a flag $(U_1, \ldots, U_{d-1})$ and all lines ℓ such that meet some U_i and are perpendicular to this U_i.

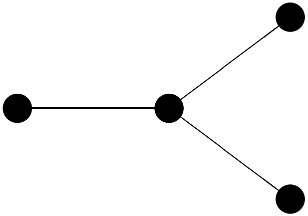
Theorem (Ihringer, M, Mühlherr)
The above example is best possible for rank three polar spaces.

Theorem (Ihringer, M, Mühlherr)
A set $\mathcal{S}$ of mutually non-opposite comaximal subspaces in $Q^+(2d-1,q)$, $d \geq 2$, even has at most ... elements. For $q > 2$ equality holds only in the following two cases:

- For some point P, the set $\mathcal{S}$ consists of all comaximal subspaces that lie in a Latin generator on P.
- $d = 4$ and $\mathcal{L}$ consists of all planes that meet a given generator at least in a line.

Technique: algebraic and geometric arguments

D_4



Polar spaces, higher intersection

Theorem (Ihringer, M, 2016)

Consider a finite classical polar space of rank d and an integer t such that $0 \leq t \leq \sqrt{\frac{8d}{9}} - 2$. Then the largest sets of generators that mutually meet in a subspace of codimension at most t consists of

- all generators that meet a fixed d-space in a subspace of dimension at least $d - \frac{t}{2}$, if t is even.
- all generators that meet a fixed $(d-1)$-space in a subspace of dimension at least $d - \frac{t}{2} - \frac{1}{2}$, if t is odd.

END

Thank you very much for your attention!