Distance sets on circles and Kneser's addition theorem

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joint work with
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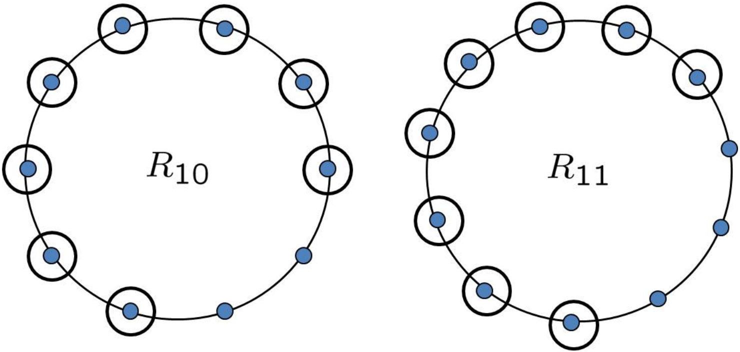
8 points on the unit circle with 5 distances

Such 8 points lie on a decagon or a hendecagon.



$\boldsymbol{R_n}$: a regular $\boldsymbol{n}$-sided polygon

100 points on the unit circle with 70 distances

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Yes! Such 100 points lie on $\boldsymbol{R}_{140}$ or $\boldsymbol{R}_{141}$!

How about the general case?

Background

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  - If $n = 2k+1$, then $X = R_{2k+1}$.
- Fishburn (1995)
  - If $n = 2k$, then $X = R_{2k}$ or $X \subset R_{2k+1}$.
  - For $(n, k) = (7, 4)$, $X \subset R_{2k}$ or $X \subset R_{2k+1}$.

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  - For $(n, k) = (7, 4)$, $X \subset R_{2k}$ or $X \subset R_{2k+1}$.

Conjecture (Fishburn, 1995)

If $n = 2k-1$, then $X \subset R_{2k}$ or $X \subset R_{2k+1}$.

This conjecture is correct for $(n, k) = (9, 5)$ and $(11, 6)$ (Erdős-Fishburn, 1996).

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Problem

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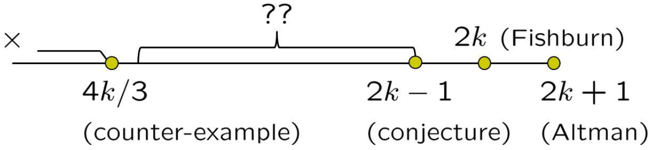
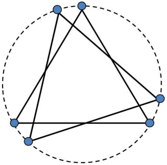
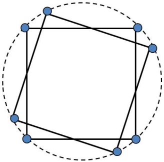
We can construct infinite many examples of sets of n points not lying on regular polygons if $s \geq 2k/3$ (or $n \leq 4k/3$).



Counterexamples



6-points 4-distances



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Remark

For a set X of points on S^1, the number of Euclidean distances between distinct points in X is equal to that of (shorter) arc lengths.

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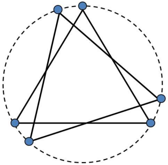
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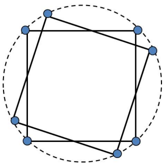
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Remark

For a set X of points on S^1, the number of Euclidean distances between distinct points in X is equal to that of (shorter) arc lengths.

Example

Let

$$M_n = \begin{cases} 3t, & \text{if } n = 4t \text{ or } 4t - 1, \\ 3t - 2, & \text{if } n = 4t - 2 \text{ or } 4t - 3. \end{cases}$$

Then, there exist (infinitely many) n-point sets with $k = M_n$ distances on S^1 not lying on regular polygons.

Can you construct counter-examples for $k < M_n$?

Theorem (M.-Shinohara, 2016)

Let

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If $k < M_n$, then any n-point set on S^1 with k distances lies on R_{2k} or R_{2k+1}.

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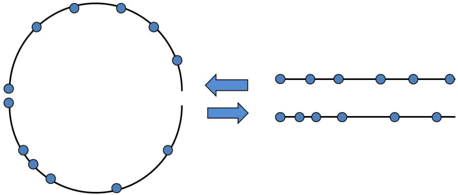
Proposition 1

If $k < M_n$, then any n-point set on S^1 with k distances lies on R_m for some integer m.

Proposition 2

Assume that $k < M_n$ and an n-point set on S^1 with k distances lies on R_m. Then, $m \in \{2k, 2k+1\}$.

Proposition 1: cut & join method



Assume the existence of a $4t+1$-point set X with $k < 3t+1$ distances.

- We can cut the circle into two half circles so that each of them contains exactly $2t+1$ of the points in X.
- We can classify $2t+1$-point sets on $\mathbb{R}$ with $k < 3t+1$ distances having both rational and irrational intervals.
- We can show that the circle as a join of such two distance sets on $\mathbb{R}$ satisfies $k \geq 3t+1$.

Proposition 2

Proposition 2 is due to *Kneser's addition theorem*.

Remark

A subset X of points of R_m can be viewed as a subset $\overline{X}$ of $\mathbb{Z}_m$.

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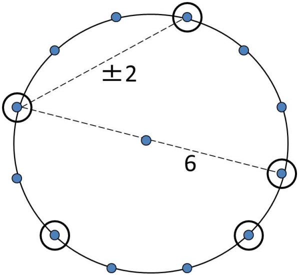
A subset $\boldsymbol{X}$ of points of $\boldsymbol{R_m}$ can be viewed as a subset $\overline{\boldsymbol{X}}$ of $\mathbb{Z}_{\boldsymbol{m}}$.
Then, # of distances between points in $\boldsymbol{X} \subset \boldsymbol{R_m}$ is equal to
(# of differences between elements in $\overline{\boldsymbol{X}} \subset \mathbb{Z}_{\boldsymbol{m}} + \boldsymbol{\epsilon})/\mathbf{2}$, where $\boldsymbol{\epsilon} = \mathbf{1}$ or $\mathbf{0}$ depending on whether $\boldsymbol{X}$ contains a point having its antipodal in $\boldsymbol{X}$ or not.



Kneser's addition theorem

Theorem (Kneser, 1953)

G: a finite abelian group
$A, B \subseteq G$
$\Longrightarrow \exists H \leq G$ s.t. $|A + B| \geq \min\{|G|, |A| + |B| - |H|\}$.

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Corollary

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Application of Kneser's addition theorem

Proposition 3

A: an n-subset of $G = \mathbb{Z}_m$ s.t. $\langle A \rangle = \mathbb{Z}_m$
$\Longrightarrow |A - A| \geq \min\{m, s_n\}$, where

$$s_n = \begin{cases} 3n/2, & \text{if } n \equiv 0 \pmod{2}, \\ 3(n+1)/2, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

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The even case was already proved by Hamidoune-Plagne, 2002.
The odd case needs a bit complicated modification.

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Proposition 4

X: an n-point subset of R_m with k distances satisfying $\langle \overline{X} \rangle = \mathbb{Z}_m$
$\Longrightarrow$ If $k < M_n$, then $m \in \{2k, 2k+1\}$, where

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- $\lceil (m-1)/2 \rceil < M_n \Longleftrightarrow n > 4\lceil (m-1)/2 \rceil/3 - 1 > \lfloor m/2 \rfloor$
- X have $k = \lfloor m/2 \rfloor$ distances.

Open problem

Problem

- How about the case where $k \geq M_n$?
  For example, can you say anything about the structure of 100-points with 75 distances on S^1?
  (Can you show that if $k = M_n$, X lies on a regular polygon or $R_m \cup \sigma_c(R_m)$?)

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Thank you very much for your attention!