Self-concordance for empirical likelihood

(and a little bit more)

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Overview

Statistical thinking can be very philosophical.

But practical implementation gets computational. The main tools are

1) Optimization
2) Sampling

Optimization

Convexity makes this much easier and gives gaurantees.

We often have that for parametric MLEs.

Also for empirical likelihood and estimating equations.

But profiling nuisance parameters is still hard.

Sampling

It turns original data (X_i, Y_i) into inferential data $\hat{\theta}_j$

Harder to know when it works.

I think prospects are good for Bayesian empirical likelihood Lazar (2003).

(E.g., Chaudhury's talk today.)

Motivation for today

Dylan Small and Dan Yang (2012) found a case where my old Levenberg-Marquardt iterations failed. Plain step reduction works better.

New optimization is

1) low dimensional
2) convex
3) unconstrained
4) **self-concordant**

The new ingredient is self-concordance (described below)

It gives mathematical guarantees of convergence.

Prior to convergence it lets us bound sub-optimality

Also

A quartic log likelihood Corcoran (1998) is also self-concordant.

Empirical Likelihood

Provides likelihood inferences without assuming a parametric family

For data $X_i \stackrel{\text{iid}}{\sim} F$

$$L(F) = \prod_{i=1}^{n} F(\{X_i\})$$ Likelihood

$$\hat{F} = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$$ Nonparametric MLE

$$R(F) = \prod_{i=1}^{n} nw_i, \quad w_i \equiv F(\{X_i\})$$ Empirical likelihood ratio

If $L(F) > 0$ then $w_i > 0$. Convenient to assume $\sum_{i=1}^{n} w_i = 1$ too.

Then we get a multinomial distribution on n items $X_1, \dots, X_n$.

EL properties

Empirical likelihood inherits many properties from parametric likelihoods.

- Wilks style χ^2 limit distribution
- automatic shape selection for confidence regions
- Bartlett correctability DiCiccio, Hall & Romano (1991) and Chen & Cui (2006)
- Very high power Kitamura and Lazar & Mykland
- Wide scope Hjort, McKeague & Van Keilegom (2009)

Statistical assumptions: independence and bounded moments.

Oddly

Having $n-1$ parameters for n observations does not lead to trouble.

Empirical likelihood for the mean

$$\mathcal{R}(\mu) = \max\Big\{\prod_{i=1}^{n} nw_i \mid w_i > 0 \sum_{i=1}^{n} w_i X_i = \mu, \sum_{i=1}^{n} w_i = 1\Big\}$$

Wilks-like: $-2\log(\mathcal{R}(\mu_0)) \xrightarrow{\mathrm{d}} \chi^2_{(d)}$ allows confidence regions and tests

Estimating equations $\mathbb{E}(m(X,\theta)) = 0$

$m(X,\theta) = X - \theta$	Mean
$m(X,\theta) = 1_{X<\theta} - 0.5$	Median
$m(X,Y,\theta) = (Y - X^{\mathsf{T}}\theta)X$	Regression
$m(X,\theta) = \frac{\partial}{\partial\theta}\log(f(X,\theta))$	MLE estimand

Computation

Maximize $\sum_{i=1}^{n} \log(nw_i)$ subject to $\sum_i w_i = 1$ and $\sum_i w_i Z_i = 0$

Here $Z_i = X_i - \mu_0$ or $Z_i = m(X_i, \theta)$.

The hull

If 0 is not in the convex hull of Z_i then $\log(\mathcal{R}(\cdot)) = -\infty$

Lagrangian

$$G = \sum_{i=1}^{n} \log(nw_i) - n\lambda^{\mathsf{T}} \sum_{i=1}^{n} w_i Z_i + \delta\Bigl(\sum_{i=1}^{n} w_i - 1\Bigr)$$

$$\frac{\partial G}{\partial w_i} = \frac{1}{w_i} - n\lambda^{\mathsf{T}} Z_i + \delta$$

$$0 = \sum_{i=1}^{n} w_i \frac{\partial G}{\partial w_i} = n - 0 + \delta$$

Therefore for some $\lambda \in \mathbb{R}^d$

$$w_i = \frac{1}{n} \frac{1}{1 + \lambda^{\mathsf{T}} Z_i}$$

Finding λ

$$w_i = \frac{1}{n}\frac{1}{1+\lambda^\mathsf{T} Z_i}, \quad \text{where} \quad \sum_{i=1}^{n} w_i(\lambda) Z_i = 0 \in \mathbb{R}^d.$$

We have to solve

$$\frac{1}{n}\sum_{i=1}^{n}\frac{Z_i}{1+\lambda^\mathsf{T} Z_i} = 0$$

The dual

$$\mathbb{L}(\lambda) = -\sum_{i=1}^{n}\log(1+\lambda^\mathsf{T} Z_i)$$

This function is convex in λ and,

$$\frac{\partial \mathbb{L}}{\partial \lambda} = \frac{1}{n}\sum_{i=1}^{n}\frac{Z_i}{1+\lambda^\mathsf{T} Z_i}.$$

Minimizing the dual maximizes the likelihood.

n constraints

Recall: $\mathbb{L}(\lambda) = -\sum_{i=1}^{n} \log(1 + \lambda^\mathsf{T} Z_i)$

Minimizer must have $1 + \lambda^\mathsf{T} Z_i > 0, \quad i = 1, \ldots, n$

This comes from $w_i > 0$.

Sharper

$$w_i < 1 \implies \frac{1}{n} \frac{1}{1 + \lambda^\mathsf{T} Z_i} < 1$$

Therefore

$$1 + \lambda^\mathsf{T} Z_i > \frac{1}{n}, \quad i = 1, \ldots, n$$

Removing the constraints

Replace $\log(x)$ by

$$\log_*(x) = \begin{cases} \log(x), & x \geqslant 1/n \\ Q(x), & x < 1/n \end{cases}$$

where Q is quadratic with

$$Q(1/n) = \log(1/n)$$
$$Q'(1/n) = \log'(1/n) \quad \text{and}$$
$$Q''(1/n) = \log''(1/n)$$

$$Q(x) = \log(1/n) - 3/2 + 2nx - (nx)^2/2$$

Now minimize

$$\mathbb{L}_* = -\sum_{i=1}^{n} \log_*(1 + \lambda^\mathsf{T} Z_i)$$

Same optimum as $\mathbb{L}$. No constraints. Always finite.

Newton steps

The gradient is $g(\lambda) \equiv \frac{\partial}{\partial\lambda}\mathbb{L}_*(\lambda)$.

The Hessian is $H(\lambda) \equiv \frac{\partial^2}{\partial\lambda\partial\lambda^\mathsf{T}}\mathbb{L}_*(\lambda)$

The Newton step is

$$\lambda \leftarrow \lambda + s \quad \text{where} \quad s = -H^{-1}g$$

Further analysis

Our H is of the form $J^\mathsf{T} J$ and $g = J^\mathsf{T}\eta$

So the Newton step can be solved by least squares (more numerically stable)

Step reductions

Newton steps still require some kind of step reduction methods. If there is not enough progress to the minimum, take a smaller multiple of s.

Levenberg-Marquardt: if the step gets too small start picking directions more near to $-g$.

Small and Yang's example

$$
\begin{aligned}
0 &= \mathbb{E}(Z_1(Y - \beta_1 W - \alpha_1)) \\
0 &= \mathbb{E}(Y - \beta_1 W - \alpha_1) \\
0 &= \mathbb{E}(Z_2(Y - (\beta_1 + \delta)W - \alpha_2)) \\
0 &= \mathbb{E}(Y - (\beta_1 + \delta)W - \alpha_2)
\end{aligned}
$$

Residuals $Y - \beta_1 W - \alpha_1$ and $Y - (\beta_1 + \delta)W - \alpha_2$.

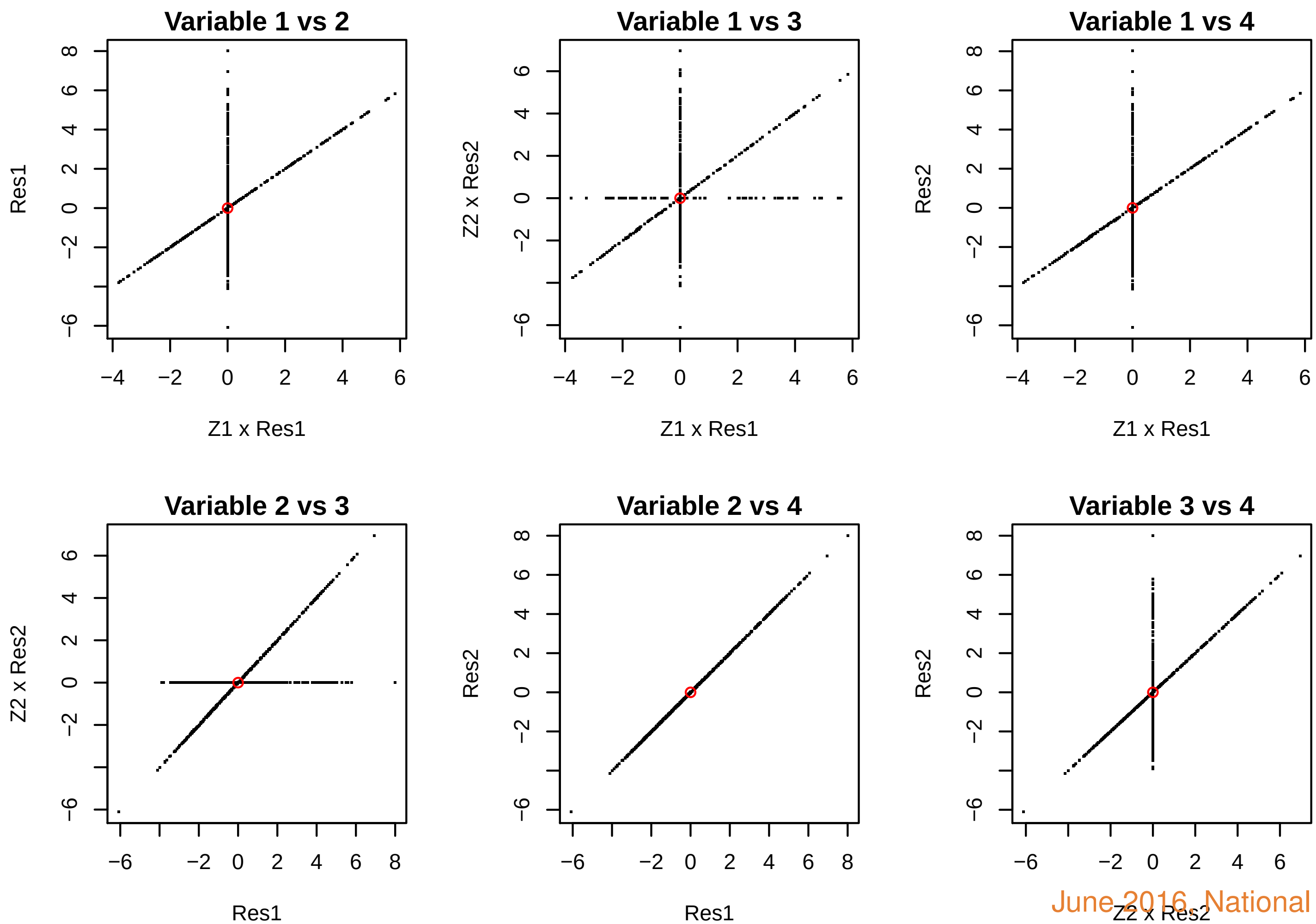
Instrumental variables $Z_1, Z_2 \in \{0, 1\}$

Problem arose in a bootstrap sample.

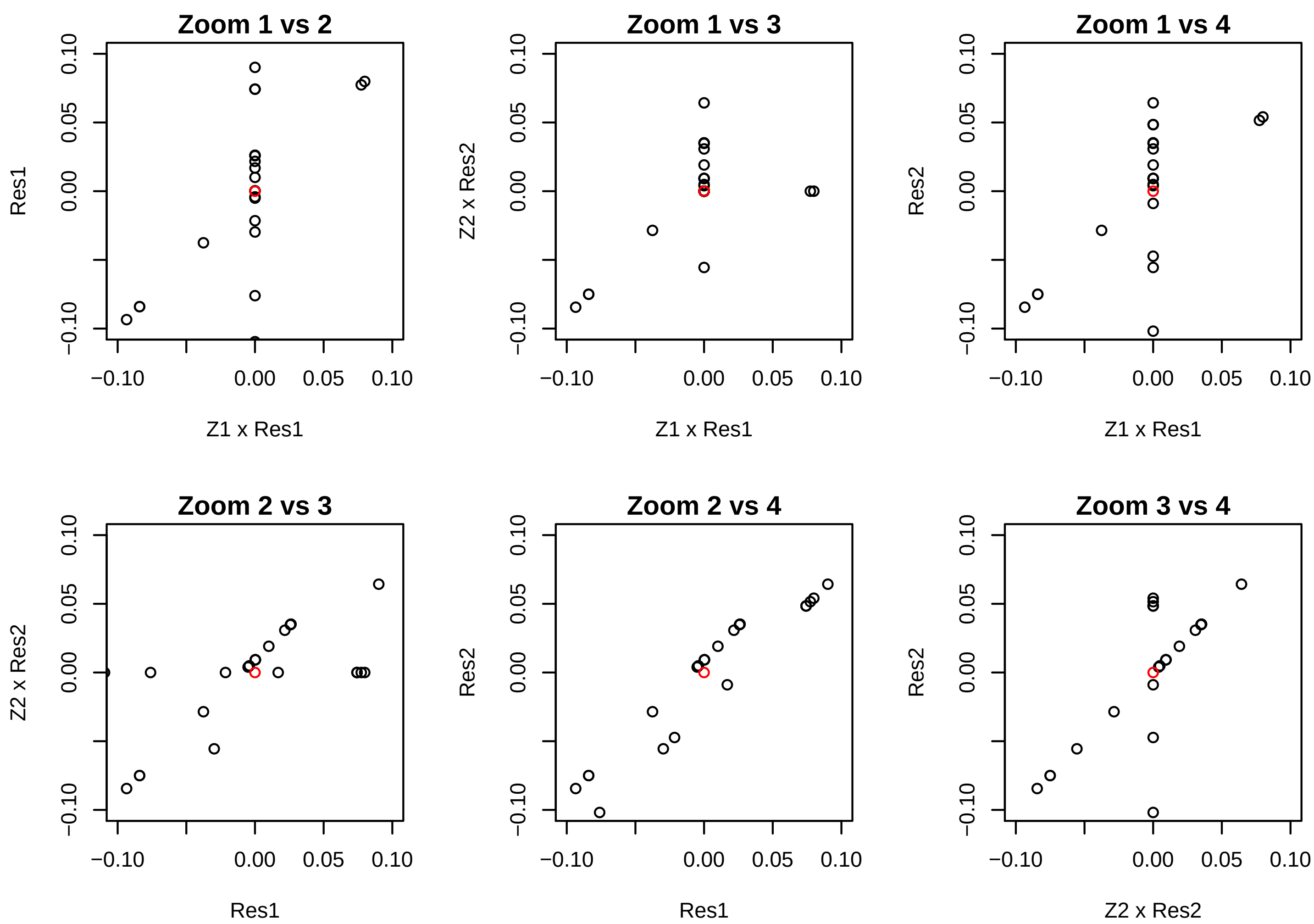
Small and Yang's example

They needed to test the mean of 1000 points in $\mathbb{R}^4$.

The specific problem arose in an instrumental variables context.



Zooming in


Zoom 1 vs 2
Zoom 1 vs 3
Zoom 1 vs 4
Zoom 2 vs 3
Zoom 2 vs 4
Zoom 3 vs 4
Res1
Z2 x Res2
Res2
Z1 x Res1
Res1
Z2 x Res2
−0.10
0.00
0.05
0.10

True empirical log likelihood

$$\mathcal{R}(0) = -399.6937$$

Old algorithm got stuck; stepsize got small ad hoc Levenberg-Marquardt reductions did not help.

They used step reducing line search instead.

Self-concordance

A convex function g from $\mathbb{R}$ to $\mathbb{R}$ is **self-concordant** if

$$|g'''(x)| \leqslant 2g''(x)^{3/2} \qquad \text{N.B. } g'' \geqslant 0$$

Nesterov & Nemirovskii (1994) Boyd & Vandeberghe (2004)

A convex function from g from $\mathbb{R}^d$ to $\mathbb{R}$ is self-concordant if

$$g(\boldsymbol{x}_0 + t\boldsymbol{x}_1)$$

is a self-concordant function of $t \in \mathbb{R}$.

Implications

The Hessian of self-concordant $g(\boldsymbol{x})$ cannot change too rapidly with $\boldsymbol{x}$.

Newton updates with line search step-reduction are guaranteed to converge.

Also the Newton decrement (below) bounds the suboptimality.

The 2 is not essential

If $|g'''(x)| \leqslant Cg''(x)^{3/2}$ then $\frac{C^2}{4}g$ is self-concordant.

Backtracking Newton

1) Select starting point $\boldsymbol{x}$
2) Repeat until Newton decrement $\nu(\boldsymbol{x})$ below tolerance
   a) $\boldsymbol{s} \leftarrow -H(\boldsymbol{x})^{-1} g(\boldsymbol{x}), \quad t \leftarrow 1$
   b) While $f(\boldsymbol{x} + t\boldsymbol{s}) > f(\boldsymbol{x}) + \alpha t \boldsymbol{s}^{\mathsf{T}} g$
      i) $t \leftarrow t \times \beta$
3) $\boldsymbol{x} \leftarrow \boldsymbol{x} + t\boldsymbol{s}$

Guaranteed convergence if

$\alpha \in (0, 1/2)$, $\beta \in (0, 1)$, f bounded below, sublevel set of $\boldsymbol{x}$ is closed

Newton decrement

$$\nu(\boldsymbol{x}) = (g(\boldsymbol{x})^{\mathsf{T}} H(\boldsymbol{x})^{-1} g(\boldsymbol{x}))^{1/2}$$

If f is strictly convex self-concordant and $\nu(\widetilde{\boldsymbol{x}}) \leqslant 0.68$ then

$$\inf_{\boldsymbol{x}} f(\boldsymbol{x}) \geqslant f(\widetilde{\boldsymbol{x}}) - \nu(\widetilde{\boldsymbol{x}})^2$$

Chen, Sitter, Wu

- Biometrika (2002)
- Use backtracking line search with step halving when objective not improved (i.e., improvement factor $\alpha = 0$ and step factor $\beta = 1/2$)
- Show convergence via results in Polyak (1987)
- Starts k'th search at size $t = (k+1)^{-1/2}$.
- Starting with $t < 1$ will slow Newton from quadratic convergence. They observe that starting at $t = 1$ works.

Back to $\mathbb{L}_*$

$$\mathbb{L}_*(\lambda) = -\sum_{i=1}^{n} \log_*(1+\lambda^\mathsf{T} Z_i) \qquad \text{where} \quad \log_*(x) = \begin{cases} \log(x), & x \geqslant 1/n \\ Q(x), & x < 1/n \end{cases}$$

$\log_*$ is self-concordant on $(-\infty, 1/n)$ and on $(1/n, \infty)$.

But it lacks a third derivative at $1/n$

Hence not self-concordant.

Higher order approximations

$$-\log_{(k)}(x) = \begin{cases} -\log(x), & x \geqslant \epsilon > 0 \\ h_k(x-\epsilon) & x < \epsilon \end{cases}$$

Taylor approx to $-\log$ at ϵ

$$h_k(y) = h_k(y;\epsilon) = -\sum_{t=0}^{k} \log^{(t)}(\epsilon)\frac{y^t}{t!}$$

$k=2$	Convex but not self-concordant (fails at ϵ)	$-\log_{(2)} = -\log_*$
$k=3$	Not even convex	
$k=4$	Convex and self-concordant	$\overset{0\,0}{\smile}$

Back to the example

Self-concordant version also gets $\log \mathcal{R}() = -399.6937$

Newton decrement

$$\eta \equiv (g^{\mathsf{T}} H^{-1} g)^{-1/2} = 6.74277 \times 10^{-16}$$

Estimate has $\log(\mathcal{R})$ within η^2 of true optimum.

I.e. good to within given precision.

Sketch of proof

We need to show that $h_4(y)$ is self-concordant on $(-\infty, 0]$.

- i.e., $|h_4'''| \leqslant 2(h_4')'^{3/2}$
- Suffices to show $h_4(\epsilon \times \cdot)$ self-concordant
- $h_4'''(t\epsilon) = \epsilon^{-3}(-2 + 6t)$
- $h_4''(t\epsilon) = \epsilon^{-2}((1-t)^2 + t^2)$
- $\rho(t) \equiv \dfrac{|h_4'''(t\epsilon)|}{h_4''(t\epsilon)^{3/2}} = \dfrac{2-6t}{(t-1)^2 + t^2}$ on $t \leqslant 0$.
- $\rho(0) = 2$
- $\rho'(t) \geqslant 0$ for $t \leqslant 0$

So the ratio ρ increases to 2 as $t \uparrow 0$

Quartic log likelihood

$$\text{use} \quad \mathcal{R}_Q = -\sum_{i=1}^{n} \widetilde{\log}(nw_i)$$

$$\widetilde{\log}(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4$$

Properties

Bartlett correctable Corcoran (1998)

Match 4 derivatives & match 4 moments

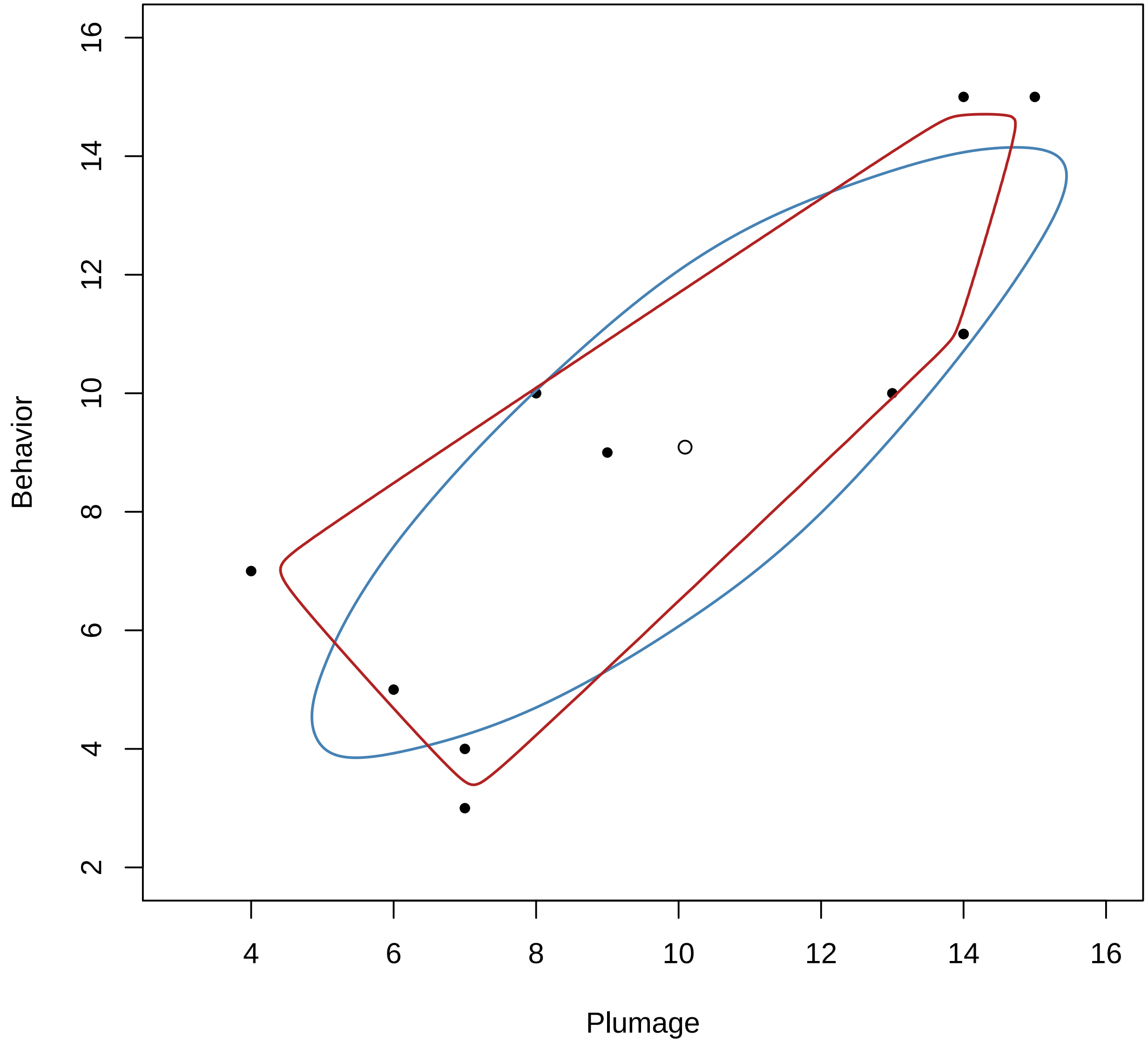
Self-concordant O (2013) [$C = 3.92$ instead of $C = 2$]

Convex confidence regions for the mean O (2013)

Lagrange multiplier for $\sum w_i = 1$ cannot be eliminated.

Primal-dual algorithm in Boyd & Vandeberghe available

Duck data



Extreme confidence region. Red $\mathcal{R}$; Blue $\mathcal{R}_Q$

Larsen & Marx (1986)

Next thoughts

Maybe it is not necessary to enforce $1 + \lambda^{\mathsf{T}} Z_i > 1/n$

Avoid piece-wise pseudo-logarithm altogether

Step reduction keeps $1 + \lambda^{\mathsf{T}} Z_i > 0$

$-\sum_{i=1}^{n} \log(1 + \lambda^{\mathsf{T}} Z_i)$ also self-concordant

Simpler, but

$\log(z)$ may be slightly worse conditioned than z^4

Maximizing over nuisance parameters might be easier without linearly constraining λ

Time permitting . . .

Some computational challenges.

Profiling for regression

Maximize $\sum_{i=1}^{n} \log(nw_i)$ subject to $w_i \geqslant 0$ $\sum_i w_i = 1$

$$\sum_i w_i(Y_i - \boldsymbol{x}_i^\mathsf{T}\beta)\boldsymbol{x}_i = 0$$

and $\beta_j = \beta_{j0}$.

Not quite convex optimization

The free variables are β_k for $k \neq j$ as well as $w_1, \dots, w_n$.

The computational challenge comes from **bilinearity** of the constraint.

If β is held fixed the normal equation constraint is linear in w and vice versa.

Multisample EL

Chapter 11.4 of the text “Empirical likelihood” looks at a multi-sample setting. Observations $\boldsymbol{X}_i \overset{\text{iid}}{\sim} F$ for $i = 1, \dots, n$ independent of $\boldsymbol{Y}_j \overset{\text{iid}}{\sim} G$ for $j = 1, \dots, m$. The likelihood ratio is

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (nu_i)(mv_j)$$

with $u_i \geqslant 0$, $v_j \geqslant 0$, $\sum_i u_i = 1$, $\sum_j v_j = 1$ and

$$\sum_i \sum_j u_i v_j h(\boldsymbol{x}_i, \boldsymbol{y}_j, \theta) = 0 \tag{1}$$

For example: $h(X, Y, \theta) = 1_{X-Y>\theta} - 1/2$. The computational problem is a challenge. The log likelihood is convex but constraint (1) is bilinear. So computation is awkward.

Regression again

$$Y \approx \boldsymbol{x}^{\mathsf{T}}\beta, \quad \boldsymbol{x} \in \mathbb{R}^d \quad y \in \mathbb{R}$$

Estimating equations*

$$\mathbb{E}\big((Y - \boldsymbol{x}^{\mathsf{T}}\beta)\boldsymbol{x}\big) = 0$$

Normal equations

$$\sum_{i=1}^{n}(y_i - \boldsymbol{x}_i^{\mathsf{T}}\beta)\boldsymbol{x}_i = 0 \in \mathbb{R}^d$$

In principle we let $\boldsymbol{z}_i = \boldsymbol{z}_i(\beta) \equiv (y_i - \boldsymbol{x}_i^{\mathsf{T}}\beta)\boldsymbol{x}_i \in \mathbb{R}^d$, adjoin $\boldsymbol{z}_{n+1}$ and $\boldsymbol{z}_{n+2}$, and carry on.

*residuals $\varepsilon = y - \boldsymbol{x}^{\mathsf{T}}\beta$ are uncorrelated with $\boldsymbol{x}$.

They have mean zero too, when as usual, $\boldsymbol{x}$ contains a constant.

Regression hull condition

$$\mathcal{R}(\beta) = \sup\left\{ \prod_{i=1}^{n} nw_i \,\middle|\, w_i \geqslant 0,\ \sum_{i=1}^{n} w_i = 1,\ \sum_{i=1}^{n} w_i(y_i - \boldsymbol{x}_i^{\mathsf{T}}\beta)\boldsymbol{x}_i = 0 \right\}$$

$$\mathcal{P} = \mathcal{P}(\beta) = \{\boldsymbol{x}_i \mid y_i - \boldsymbol{x}_i^{\mathsf{T}}\beta > 0\} \qquad \boldsymbol{x} \text{ with pos resid}$$

$$\mathcal{N} = \mathcal{N}(\beta) = \{\boldsymbol{x}_i \mid y_i - \boldsymbol{x}_i^{\mathsf{T}}\beta < 0\} \qquad \boldsymbol{x} \text{ with neg resid}$$

Convex hull condition O (2000)

$$\mathrm{chull}(\mathcal{P}) \bigcap \mathrm{chull}(\mathcal{N}) \neq \varnothing \implies \beta \in C(0)$$

For $\boldsymbol{x}_i = (1, t_i)^{\mathsf{T}} \in \mathbb{R}^2$ $\quad\mathcal{P}$ and $\mathcal{N}$ are intervals in $\{1\} \times \mathbb{R}$.

Converse

Suppose that $\tau \notin \{t_1, \ldots, t_n\}$ and

$$\text{Sign}(y_i - \beta_0 - \beta_1 t_i) = \begin{cases} 1, & t_i > \tau \\ -1, & t_i < \tau \end{cases}$$

Suppose also that

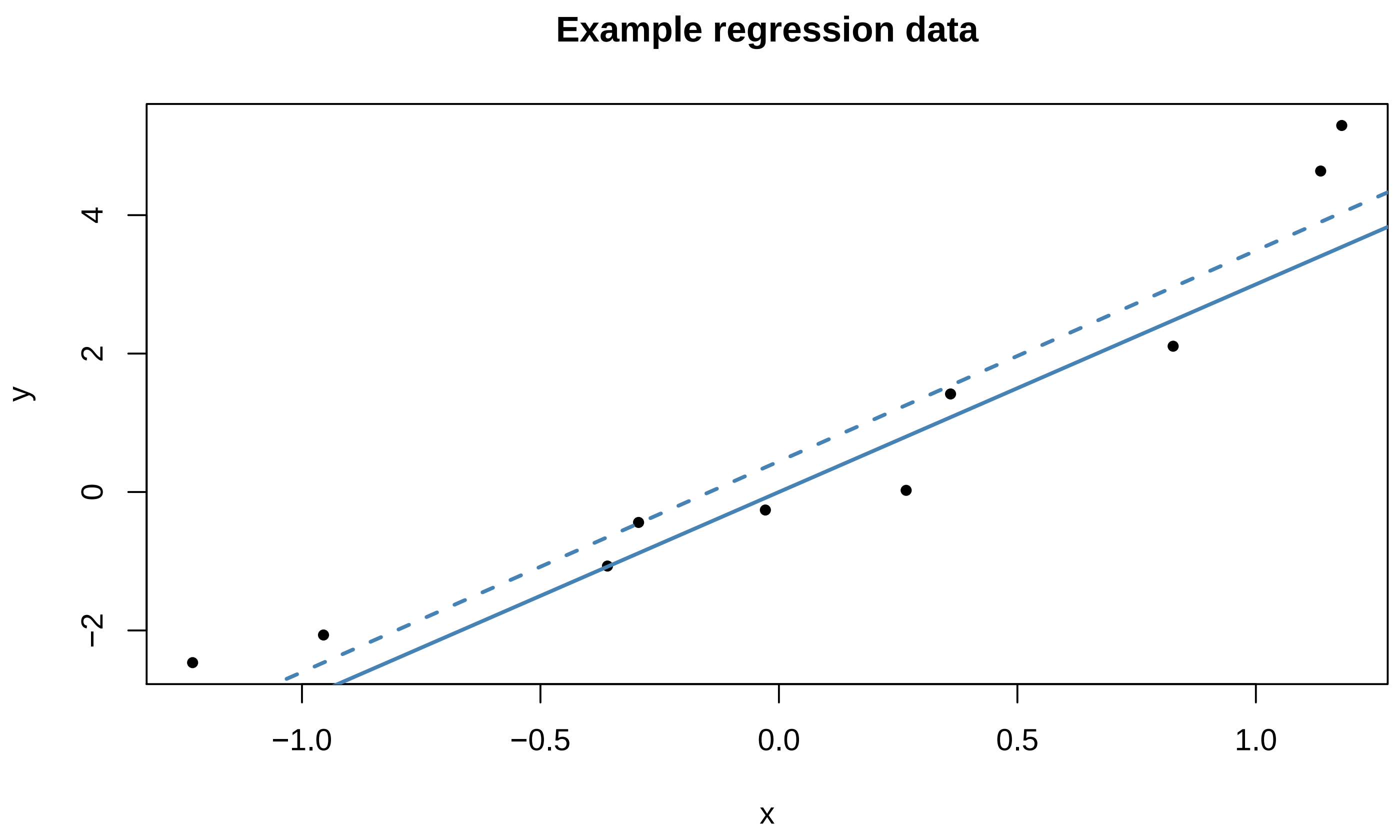
$$\sum_i w_i \begin{pmatrix} 1 \\ t_i \end{pmatrix} (y_i - \beta_0 - \beta_1 t_i) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then

$$\sum_i w_i (y_i - \beta_0 - \beta_1 t_i)(t_i - \tau) = 0$$

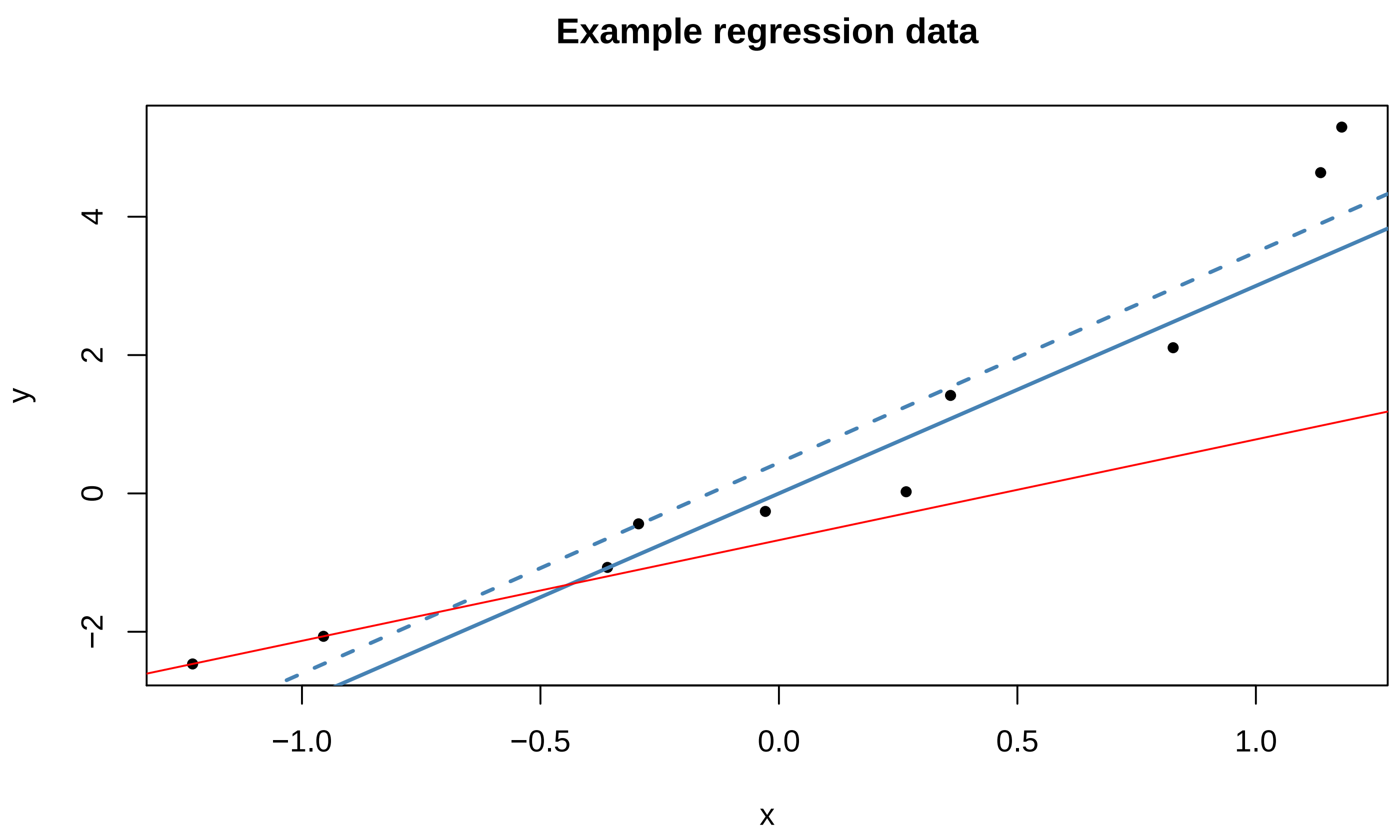
But $(y_i - \beta_0 - \beta_1 t_i)(t_i - \tau) > 0 \; \forall i$

Therefore the hull condition is necessary.

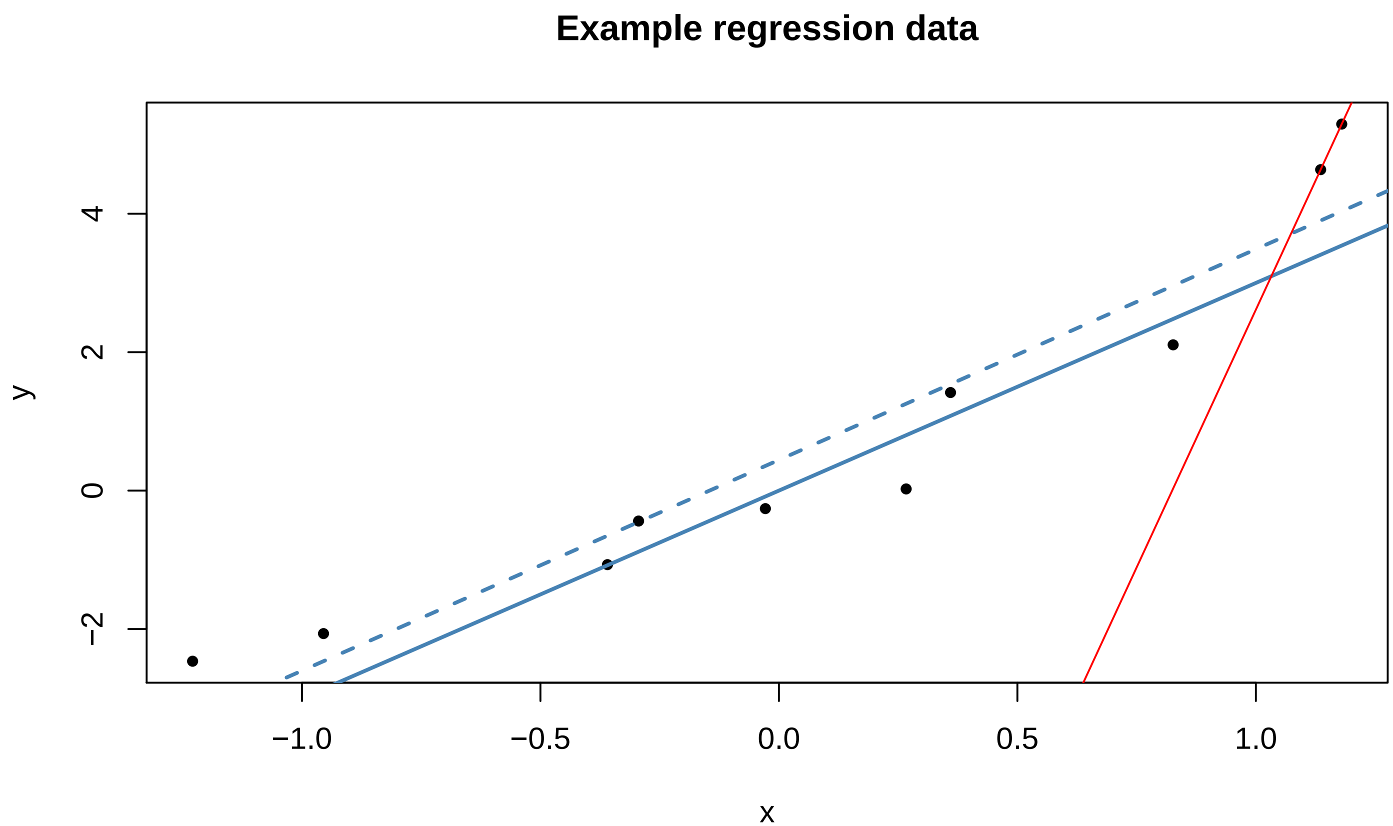


$$Y = \beta_0 + \beta_1 X + \sigma\varepsilon \quad \beta = (0,3)^{\mathsf{T}}, \sigma = 1$$

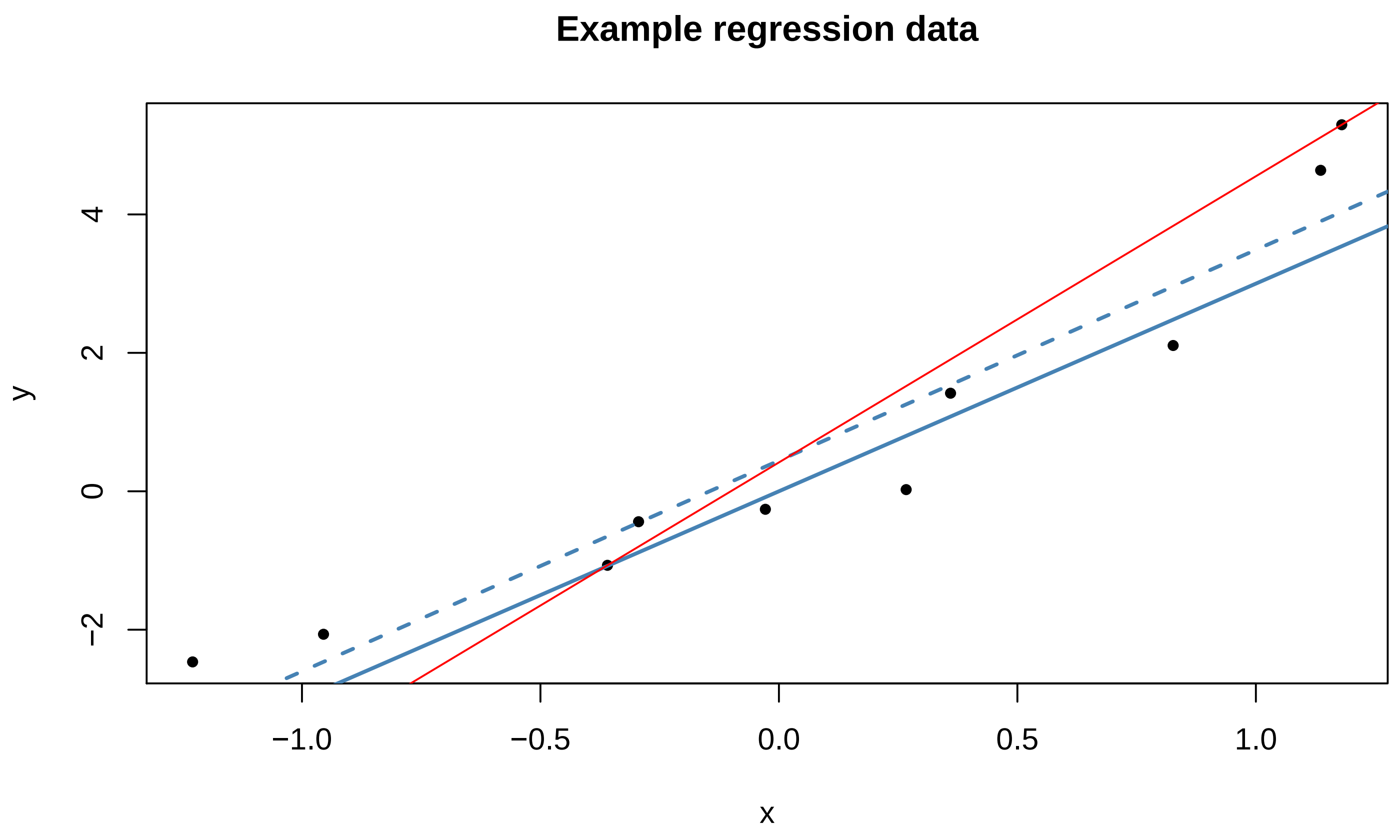
β solid $\quad \hat{\beta}$ dashed



Red line is on boundary of set of (β_0, β_1) with positive empirical likelihood



Another boundary line.

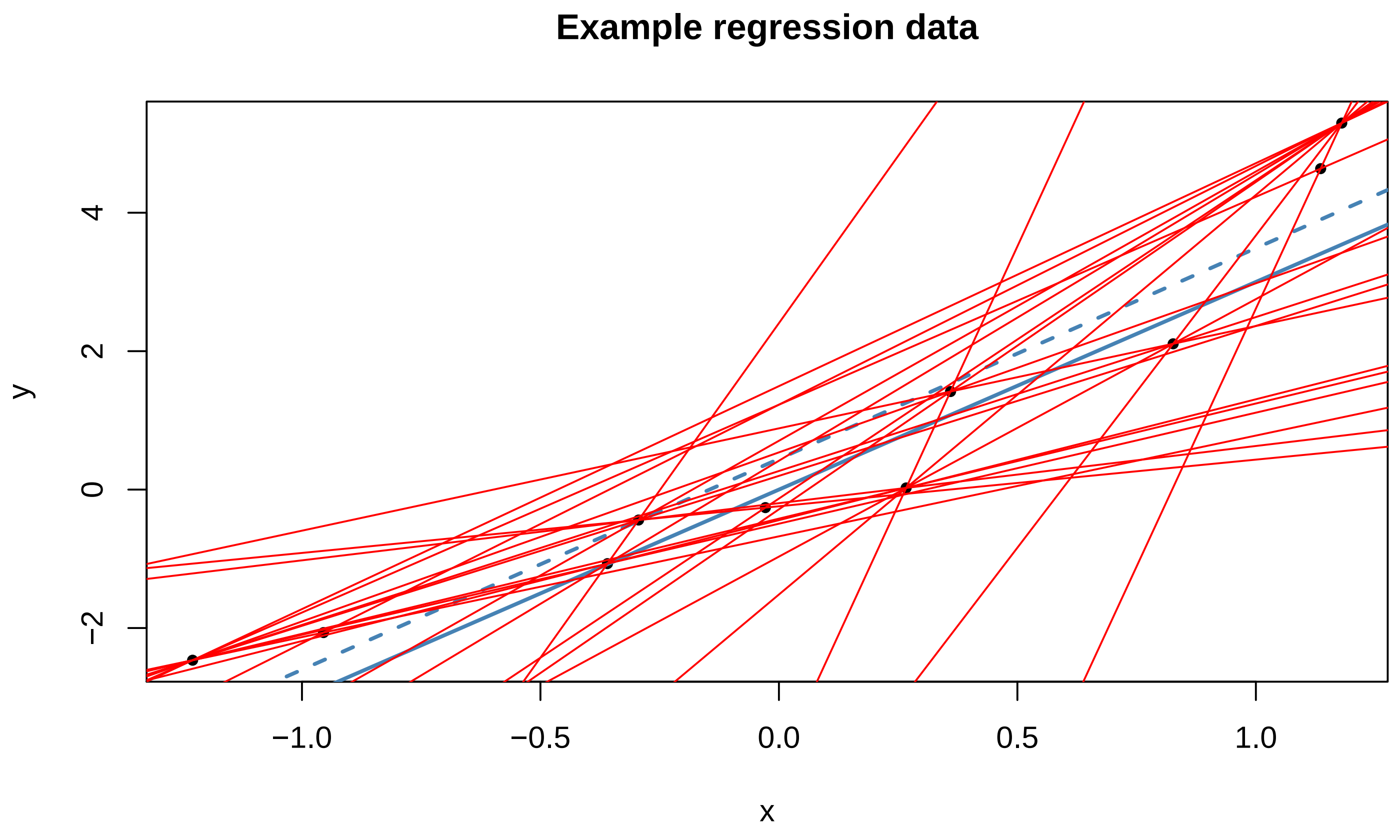


Yet another boundary line.

Left side has positive residuals; right side negative.

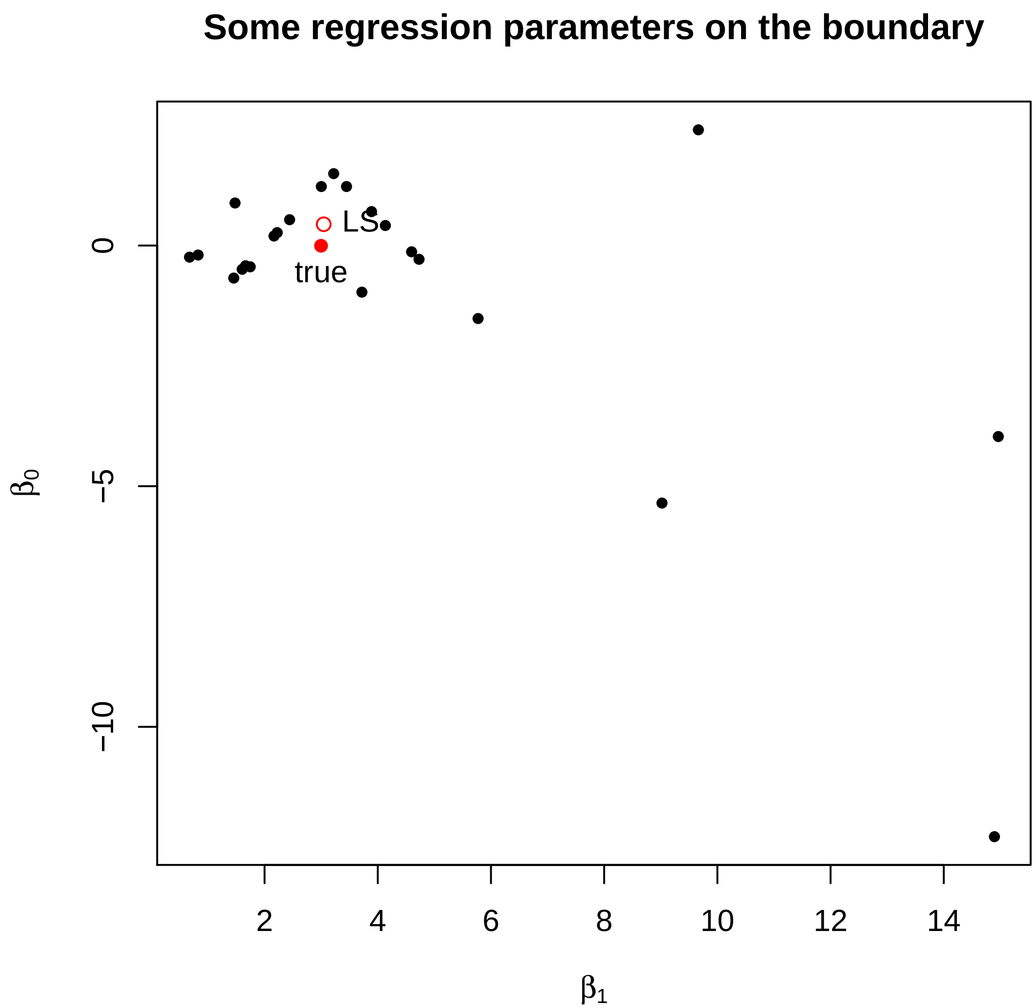
Wiggle it up and point 3 gets a negative residual $\Longrightarrow$ ok.

Wiggle down $\Longrightarrow$ NOT ok.



All the boundary lines that interpolate two data points.

They are a subset of the boundary.



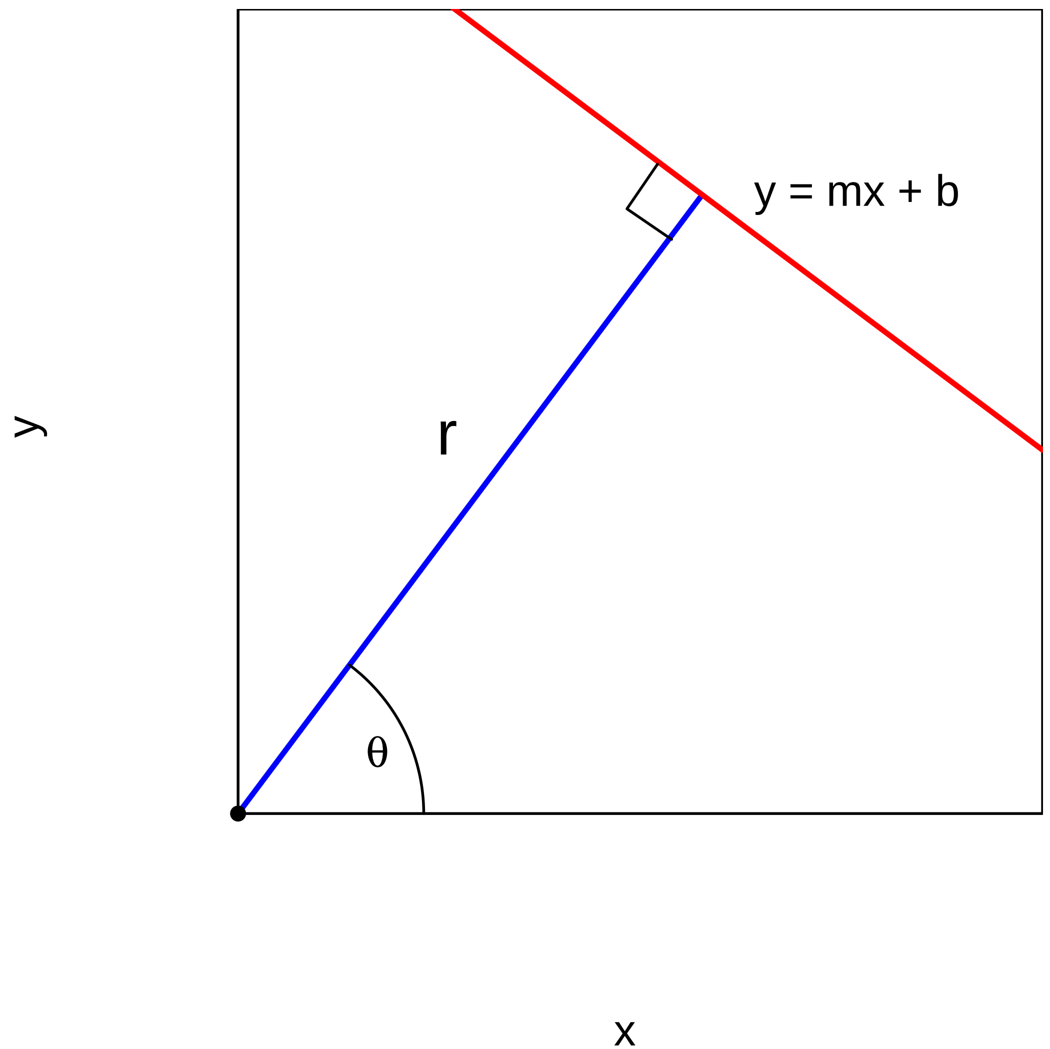
Boundary points (β_0, β_1). Region is not convex.

It **is** convex in β_0 (vertical) for fixed β_1 (horizontal).

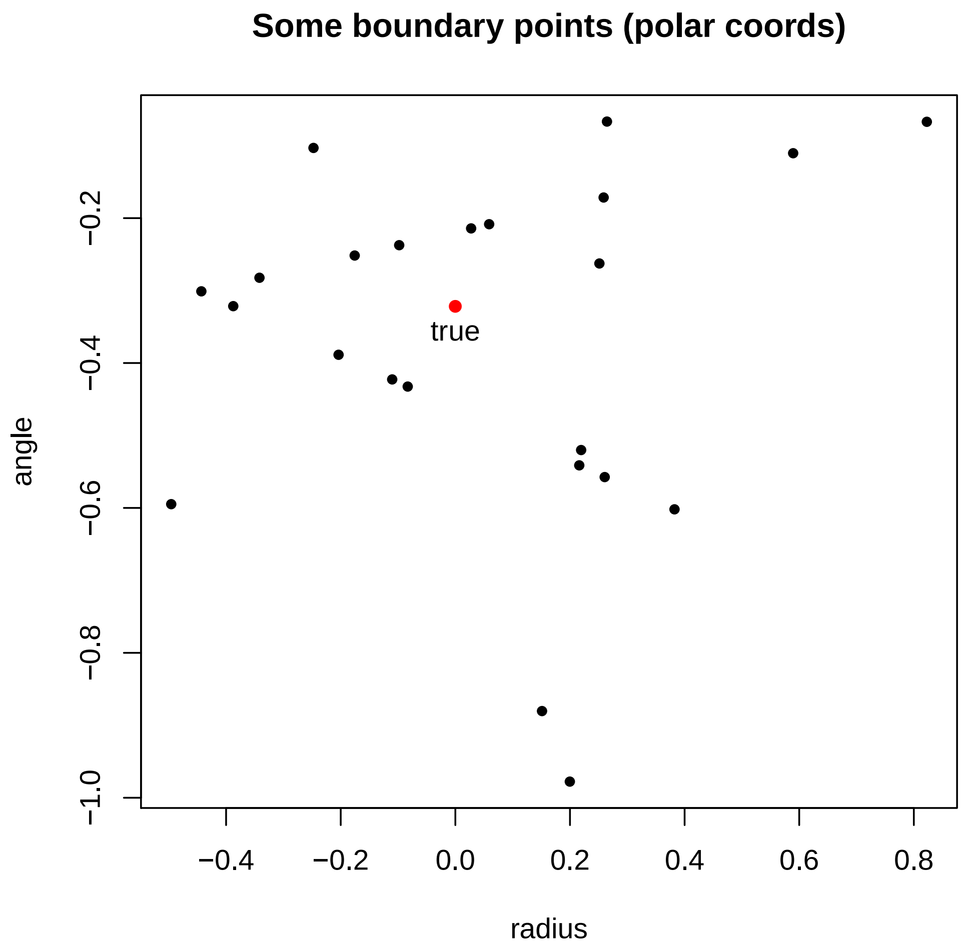
What is a convex set of lines?

- convex set of (β_0, β_1)?
- convex set of (ρ, θ)? (polar coordinates)
- convex set of (a, b) ($ax + by = 1$)?

Polar coordinates of a line



Boundary pts in polar coords



Not convex here either.

Intrinsic convexity

There is a geometrically intrinsic notion for a convex set of linear flats.

J. E. Goodman (1998) “When is a set of lines in space convex?”

Maybe $\cdots$ that can support some computation.

Dual definition

The set of flats that intersects a convex set $C \subset \mathbb{R}^d$ is a convex set of flats.

So is the set of flats that intersect **all of** $C_1, \dots, C_k \subset \mathbb{R}^d$ for convex C_j.

Convex functions

This notion of convex set does not yet seem to have a corresponding notion of convex function. There could be quasi-convex functions, those where the level sets are convex. But quasi-convexity is much less powerful computationally than convexity.

Bayesian empirical likelihood

Basic idea:

use $\pi(\theta) \times \mathcal{R}(\theta)$, prior times empirical likelihood.

Philosophy

We might have a good idea about the prior but prefer not to specify a likelihood.

Lazar (2003) shows some good frequentist calibrations.

The EL is asymptotically a likelihood on a least favorable family.

Placing the prior on that same family unites the two.

Computation

There have been recent strides in Hamiltonian MCMC.

Faster convergence.

Better user interface via STAN.

Thanks

1) Dylan Small and Dan Yang

2) Jiahua Chen, sharing an early paper

3) NSF DMS-0906056

4) Sanjay Chaudhuri

5) Eileen Tan