An Empirical Likelihood Method for Geo-referenced Spatial Data

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Overview

- Introduction
- Theoretical Framework
- Results on DFT for Irregularly Spaced Spatial Data
- Formulation of the Spatial Empirical Likelihood Method
- Examples
- Main results
- Some applications

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Introduction

- What is *Empirical Likelihood*?
- Consider a parametric model $\{f(\cdot;\theta) : \theta \in \Theta\}$ and let $X_1, \cdots, X_n$ be iid, $X_1 \sim f(\cdot;\theta)$. Then, the likelihood function for θ is

$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} f(x_i;\theta).$$

- Suppose we want to test

$$H_0 : \theta \in \Theta_0 \ vs \ H_1 : \theta \in \Theta_0^c, \ \Theta = \Theta_0 \cup \Theta_0^c.$$

- One way to test the above hypothesis is using the well-known *Likelihood Ratio Test*.

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Wilk's theorem

- Under some regularity conditions, Wilk's theorem asserts that

Wilk's Theorem

$$-2\log \mathcal{R}_n(\theta_0) \xrightarrow{d} \chi^2_p \text{ as } n \to \infty$$

- p is equal to the difference in dimensionality of Θ and Θ_0 and $\mathcal{R}_n(\theta_0)$ is the *LRT statistic* for testing the above null hypothesis. More specifically,

$$\mathcal{R}_n(\theta_0) = \frac{\sup_{\theta\in\Theta_0} \mathcal{L}_n(\theta)}{\sup_{\theta\in\Theta} \mathcal{L}_n(\theta)}$$

Empirical Likelihood

- Empirical Likelihood (EL) of Owen (1988) is a method that defines a likelihood for certain population parameters *without* requiring a parametric model.
- let $X_1, \cdots, X_n$ be iid with mean $\mu \in I\!R$. The EL for μ is

$$\mathcal{L}_n(\mu) = \sup_{\pi_i} \left\{ \prod_{i=1}^{n} \pi_i : \pi_i \geq 0, \sum \pi_i = 1, \sum \pi_i X_i = \mu \right\}$$

- The unconstrained maximum is at $\pi_i = n^{-1}$ for all i. Thus, the EL ratio statistic for testing $H_0 : \mu = \mu_0$ is $\mathcal{R}_n(\mu_0) = \frac{\mathcal{L}_n(\mu_0)}{n^{-n}}$.
- Owen (1988) proved a version of Wilk's Theorem:

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Literature Review

- Owen (1988, 1990) introduced the EL method for *independent* random variables.
- Extensions and refinements of the EL method to different problems under independence are given by
  - Chen and Hall (1993): Quantiles
  - Qin and Lawless (1994): Estimating equations
  - DiCiccio, Hall and Romano (1996): Bartlett corrections
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Independent EL (continued)

- Hjort, McKeague and Van Keilegom (2009): Functional nuisance parameters and increasing dimensions
- Chen, Variyath and Abraham (2006): Adjusted EL in the $p > n/2$ case
- Chen, Peng and Qin (2008): Increasing dimensions $p = o(n^{1/2})$
- Bertolucci (2007): Penalized EL.

EL under dependence: Time Series

- Under dependence, the standard EL fails in the sense that the limit involves population parameters.
- Kitamura (1997) introduced Block EL (in the time domain) and established Wilk's phenomenon.
  - Block EL requires scale adjustment involving known quantities, but the limit is still Chi-squared.
- Monti (1997) first considered EL for time series in the frequency domain, under weak dependence.
  - Major advantage: No blocking is necessary.
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Challenges

- The basic orthogonality property of the sine- and cosine- transforms of gridded data at Fourier frequencies (e.g., $\omega_j = 2\pi j/n$) no longer holds.
- One must deal with the unbounded frequency domain $\mathbb{R}^d$.
- The periodogram of irregularly spaced spatial data can be severely biased and must be preprocessed in order for it to be used for inference on the underlying spectral density function.
- More than one possible asymptotic structure can arise depending on the relative growth rates of the volume of the sampling region and the sample size.

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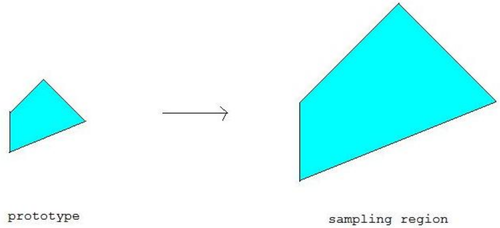
Advantages of SFDEL Method

- As with other EL methods, it does not require explicit variance estimation (which can be extremely difficult in this spatial setting)
- Applies in a unified manner to different asymptotic structures
- Valid without stringent distributional assumptions on the spatial process (i.e. assuming Gaussian processes)
- Does not involve block size selection issues associated with block-based EL method

We next introduce a theoretical framework and consider limiting behavior of the DFTs first.

Sampling Region

- Let $\mathcal{D}_0$ be an open connected subset of $(-1/2, 1/2]^d$, containing the origin.
- Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of positive real numbers such that $\lambda_n \to \infty$ as $n \to \infty$.
- The sampling region $\mathcal{D}_n$ is obtained by 'inflating' $\mathcal{D}_0$ by a multiplicative factor λ_n, i.e., $\mathcal{D}_n = \lambda_n \mathcal{D}_0$.



Sampling Design

- Let $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ be a zero mean stationary random field observed at irregularly spaced locations.
- $f(\mathbf{x})$: continuous, everywhere positive probability density function on $\mathcal{D}_0$.
- Let $\mathbf{X}_k \overset{iid}{\sim} f(\mathbf{x}),\ k \geq 1$.
- We assume that the sampling sites $\mathbf{s}_1, \cdots, \mathbf{s}_n$ are obtained by the relation: $\mathbf{s}_i \equiv \mathbf{s}_{in} = \lambda_n \mathbf{x}_i,\ \ 1 \leq i \leq n$.

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Asymptotic Structures

- Pure-increasing domain asymptotics (PID): (similar to gridded data) sampling region $\mathcal{D}_n$ expands as sample size $n \to \infty$, with $n \propto vol(\mathcal{D}_n)$ or $n/\lambda_n^d \to c^* \in (0, \infty)$ as $n \to \infty$.
- Mixed-increasing domain asymptotics (MID): expanding $\mathcal{D}_n$ but with a heavy infill of sampling sites: $vol(\mathcal{D}_n) \ll n$ or $n/\lambda_n^d \to \infty \equiv c^*$ as $n \to \infty$.

OBSERVATION: These differing structures impact even simple statistics

$$\lambda_n^{d/2} \bar{Z}_n = \frac{\lambda_n^{d/2}}{n} \sum_{i=1}^{n} Z(\mathbf{s}_i) \xrightarrow{d} N\left(0, \sigma(\mathbf{0})/c^* + \int_{\mathbb{R}^d} \sigma(\mathbf{s}) d\mathbf{s}\right)$$

with $\sigma(\mathbf{s}) = cov[Z(\mathbf{s}), Z(\mathbf{0})]$ (Lahiri, 2003).

Properties of Discrete Fourier Transforms (DFTs) for Irregularly Located Spatial Data

- Bandyopadhyay and Lahiri (2009) define the DFT $d_n(\boldsymbol{\omega})$, $\boldsymbol{\omega} \in \mathbb{R}^d$, of $\{Z(\mathbf{s}_1), ..., Z(\mathbf{s}_n)\}$ as

$$d_n(\boldsymbol{\omega}) = \lambda_n^{d/2} n^{-1} \sum_{j=1}^{n} Z(\mathbf{s}_j) \exp(\imath \boldsymbol{\omega}' \mathbf{s}_j).$$

- Two sequences of frequencies $\{\boldsymbol{\omega}_{1n}\}_{n \geq 1}, \{\boldsymbol{\omega}_{2n}\}_{n \geq 1} \subset \mathbb{R}^d$ are asymptotically independent if and only if the sequences $\{\boldsymbol{\omega}_{1n}\}_{n \geq 1}$ and $\{\boldsymbol{\omega}_{2n}\}_{n \geq 1} \subset \mathbb{R}^d$ are *asymptotically distant*:

$$\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\| \to \infty \qquad \text{as } n \to \infty$$

- Intuitively, to define a spatial EL method based on asymptotic independence of DFTs (similarly to time series versions), need to use this aspect.

Spatial Periodogram

- The periodogram is defined as

$$I_n(\boldsymbol{\omega}) = |d_n(\boldsymbol{\omega})|^2, \qquad \boldsymbol{\omega} \in \mathbb{R}^d$$

- Let $\sigma(\cdot)$ be the covariance function of $Z(\cdot)$ and $K = (2\pi)^d \int_{\mathbb{R}^d} f^2$. BL(2009) and Matsuda and Yajima (2009) showed the periodogram is *biased* under PID for the process spectral density $\psi(\cdot)$:

$$EI_n(\boldsymbol{\omega}) = \left[n^{-1}\lambda_n^d \sigma(\mathbf{0}) + K\psi(\boldsymbol{\omega})\right](1 + o(1)) \qquad \text{for all } \boldsymbol{\omega} \in \mathbb{R}^d$$

where $n^{-1}\lambda_n^d \to c_*^{-1} \in (0, \infty)$ under PID ($n^{-1}\lambda_n^d \to 0$ under MID)

- Let $\bar{Z}_n = n^{-1}\sum_{j=1}^n Z(\mathbf{s}_i)$ and $\hat{\sigma}_n = n^{-1}\sum_{j=1}^n \left(Z(\mathbf{s}_j) - \bar{Z}_n\right)^2$. Define the bias-corrected periodogram as

$$\tilde{I}_n(\boldsymbol{\omega}) = I_n(\boldsymbol{\omega}) - n^{-1}\lambda_n^d \hat{\sigma}_n(\mathbf{0}), \qquad \boldsymbol{\omega} \in \mathbb{R}^d$$

Formulation of the Spatial FDEL

- Suppose that the information about $\theta \in \Theta \subset I\!R^p$ exists through a system of spectral estimating equations.
- Let $G_\theta(\boldsymbol{\omega}) : I\!R^d \times \Theta \to I\!R^p$ denote a vector of estimating functions, satisfying the spectral moment condition

$$\int_{I\!R^d} G_\theta(\boldsymbol{\omega})\psi(\boldsymbol{\omega})d\boldsymbol{\omega} = \mathbf{0}$$

- Some Examples next

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Formulation of the Spatial FDEL:examples

Example 1 (Autocorrelation). Let $\sigma(\cdot)$ and $\rho(\cdot)$ be the auto-covariance and auto-correlation functions of $Z(\cdot)$, respectively. Let

$$\theta = (\rho(\mathbf{h}_1), \cdots, \rho(\mathbf{h}_p))^{'}$$

for a given set of lags $\mathbf{h}_1, \cdots, \mathbf{h}_p \in I\!R^d$. Define,

$$G_\theta(\boldsymbol{\omega}) = \left(\cos(\boldsymbol{\omega}^{'}\mathbf{h}_1), \cdots, \cos(\boldsymbol{\omega}^{'}\mathbf{h}_p)\right)^{'} - \theta,\ \boldsymbol{\omega} \in I\!R^d$$

Then, it is easy to check that

$$\int G_\theta(\boldsymbol{\omega})\psi(\boldsymbol{\omega})d\boldsymbol{\omega} = (\sigma(\mathbf{h}_1), \cdots, \sigma(\mathbf{h}_p))^{'} - \sigma(\mathbf{0})\theta = \mathbf{0}_p.$$

Formulation of the Spatial FDEL:examples

Example 2 (Spectral distribution function). For $\mathbf{x} \in I\!R^d$, denote the spectral distribution function by

$$\Psi(\mathbf{x}) = \int \mathbf{1}_{(-\infty,\mathbf{x}]}(\boldsymbol{\omega})\psi(\boldsymbol{\omega})d\boldsymbol{\omega},$$

where $(-\infty, \mathbf{x}] = (-\infty, x_1] \times \cdots \times (-\infty, x_d]$. Suppose that

$$\theta = (\Psi(\mathbf{x}_1), \cdots, \Psi(\mathbf{x}_p))' \Big/ \int \psi(\boldsymbol{\omega})d\boldsymbol{\omega}$$

for some fixed $x_1, \cdots, x_p \in I\!R^d$. In this case, it is easy to check that the relevant estimating function is given by

$$G_\theta(\boldsymbol{\omega}) = \left(\mathbf{1}_{(-\infty,x_1]}(\boldsymbol{\omega}), \cdots, \mathbf{1}_{(-\infty,x_p]}(\boldsymbol{\omega})\right)' - \theta,\ \boldsymbol{\omega} \in I\!R^d.$$

Formulation of the Spatial FDEL:examples

Example 3 (Variogram model fitting). Let $\{2\gamma(\cdot;\theta) : \theta \in \Theta\}, \Theta \subset \mathbb{R}^p$ be a class of variogram models for the (scale-invariant) variogram

$$2\gamma(\boldsymbol{h}) \equiv \mathrm{Var}(Z(\boldsymbol{h}) - Z(\mathbf{0}))/\mathrm{Var}(Z(\mathbf{0})), \boldsymbol{h} \in \mathbb{R}^d.$$

- Least squares variogram fitting (Cressie, 1993) corresponds to minimizing a population criterion $\sum_{i=1}^m \left[2\gamma(\boldsymbol{h}_i) - 2\gamma(\boldsymbol{h}_i;\theta))\right]^2$.
- Taking partial derivatives ∇, the true value θ_0 solves (cf. Lahiri, 2002)

$$\sum_{i=1}^m \left[2\gamma(\boldsymbol{h}_i) - 2\gamma(\boldsymbol{h}_i;\theta)\right] \nabla \left[2\gamma(\boldsymbol{h}_i;\theta)\right] = \mathbf{0}_p,$$

- or an equivalent spectral estimating equation:

$$\int \left[\sum_{i=1}^m \{1 - \cos(\boldsymbol{h}_i'\boldsymbol{\omega}) - \gamma(\boldsymbol{h}_i;\theta)\} \nabla \left[2\gamma(\boldsymbol{h}_i;\theta)\right]\right] \psi(\boldsymbol{\omega}) d\boldsymbol{\omega} = \mathbf{0}_p.$$

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To re-iterate,

- Suppose that the parameter of interest is $\theta \in \Theta \subset \mathbb{R}^p$ which satisfies a system of spectral estimating equations:

$$\int_{\mathbb{R}^d} G_\theta \psi = \mathbf{0},$$

where $G_\theta(\boldsymbol{\omega}) : \mathbb{R}^d \times \Theta \to \mathbb{R}^p$ and where ψ is the spectral density of the $Z(\cdot)$ process.

- Our goal is to define an EL for θ.
- FDEL involves $I_n(\cdot)$ over a grid of asymptotically distant frequencies in $\mathbb{R}^d$.

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- FDEL involves $I_n(\cdot)$ over a grid of asymptotically distant frequencies in $I\!R^d$.

Formulation of the Spatial FDEL

To re-iterate,

- Suppose that the parameter of interest is $\theta \in \Theta \subset I\!R^p$ which satisfies a system of spectral estimating equations:

$$\int_{I\!R^d} G_\theta \psi = \mathbf{0},$$

  where $G_\theta(\boldsymbol{\omega}) : I\!R^d \times \Theta \to I\!R^p$ and where ψ is the spectral density of the $Z(\cdot)$ process.

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Formulation of the Spatial FDEL

- Define a grid of N frequencies as

$$\{\boldsymbol{\omega}_{kn}\}_{k=1}^{N} = \left\{\mathbf{j}\lambda_n^{-\kappa} : \mathbf{j} \in \mathbb{Z}^d \cap [-C\lambda_n^{\eta}, C\lambda_n^{\eta}]^d\right\}$$

where, $0 < \kappa < 1$, $\kappa < \eta < \infty$ and $C \in (0, \infty)$.

- Frequencies $\{\boldsymbol{\omega}_{kn}\}_{k=1}^{N}$ form a regular lattice over the hyper-cube $[-C\lambda_n^{\eta-\kappa}, C\lambda_n^{\eta-\kappa}]^d \uparrow \mathbb{R}^d$ as $n \to \infty$ and covering the frequency domain.
- Any frequency-pairs are asymptotically distant

$$\lambda_n^d \|\boldsymbol{\omega}_{jn} - \boldsymbol{\omega}_{kn}\| \geq \lambda_n^{d(1-\kappa)} \to \infty.$$

and the corresponding DFTs are asymptotically independent.

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Formulation of the Spatial FDEL

- RECALL: The ordinary periodogram is NOT asymptotically unbiased under PID:

$$E|d_n(\boldsymbol{\omega})|^2 \to c_*^{-1} I_\psi + K.(2\pi)^d \psi(\boldsymbol{\omega}),\ \boldsymbol{\omega} \in \mathbb{R}^d.$$

- To define EL here, we use the bias corrected periodogram

$$\tilde{I}_n(\boldsymbol{\omega}) = |d_n(\boldsymbol{\omega})|^2 - n^{-1}\lambda_n^d \widehat{\sigma}_n,\ \boldsymbol{\omega} \in \mathbb{R}^d.$$

- With this, we define the **Spatial Frequency Domain Empirical Likelihood (SFDEL) function** for $\theta \in \Theta$ as

$$\mathcal{L}_n(\theta) = \sup\left\{\prod_{k=1}^N p_k : \sum_{k=1}^N p_k = 1, p_k \geq 0, \sum_{k=1}^N p_k G_\theta(\boldsymbol{\omega}_{kn})\tilde{I}_n(\boldsymbol{\omega}_{kn}) = \mathbf{0}\right\}.$$

- $\mathcal{R}_n(\theta) = \mathcal{L}_n(\theta)/N^{-N}$.

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PID asymptotic structure

Theorem 3 (cf. Bandyopadhyay, Lahiri and Nordman (2015))

Let $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ be a zero-mean second order stationary process satisfying some standard moment and mixing conditions and that $n/\lambda_n^d \to c_* \in (0, \infty)$. Then,

$$-\log \mathcal{R}_n(\theta_0) \xrightarrow{d} \chi_p^2 \; as \; n \to \infty, \; a.s. \; (P_{\mathbf{X}}).$$

MID with a slow rate of infilling

Theorem 4 (cf. BLN (2015))

Let $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ be a zero-mean second order stationary process satisfying some standard moment and mixing conditions and that $1 \ll c_n^2 \ll N\lambda_n^{-\kappa d}$. Then,

$$-\log \mathcal{R}_n(\theta_0) \xrightarrow{d} \chi_p^2 \; as \; n \to \infty, \; a.s. \; (P_{\mathbf{X}}).$$

MID with a fast rate of infilling

Theorem 5 (cf. BLN (2015))

Let $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ be a zero-mean second order stationary process satisfying some standard moment and mixing conditions and that $c_n^2 \gg N\lambda_n^{-\kappa d}$. Then,

$$-2\log \mathcal{R}_n(\theta_0) \xrightarrow{d} \chi_p^2 \; as \; n \to \infty, \; a.s. \; (P_{\mathbf{X}}).$$

A unified scaled spatial FDEL method

- Previous results show the standard calibration $-2\log(\cdot)$ of the EL ratio statistic may be incorrect depending on the relative rate of infilling.
- While this gives rise to a clear dichotomy in the limit, the choice of the correct scaling constant and, hence, the correct calibration may not be obvious in a finite sample application.
- Remedy! We develop a data based scaling factor that adjusts itself to the relative rates of infilling and delivers a unified χ^2_p limit law.

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Modified SFDEL

- Define modified FDEL statistic

$$-2a_n(\theta_0)\log\mathcal{R}_n(\theta_0)$$

where $a_n(\theta) = \frac{\sum_{k=1}^{N} \|G_\theta(\mathbf{w}_{kn})\|^2 \tilde{I}_n(\mathbf{w}_{kn})}{\sum_{k=1}^{N} \|G_\theta(\mathbf{w}_{kn})\|^2 I_n(\mathbf{w}_{kn})}$, a ratio of biased corrected and uncorrected periodograms

Theorem 6 (cf. BLN (2015))

Suppose that the conditions of one of Theorems 3-5 hold. Then, under $\theta = \theta_0$,

$$-2a_n(\theta_0)\log\mathcal{R}_n(\theta_0) \xrightarrow{d} \chi^2_p, as\ n \to \infty,\ a.s.\ (P_{\mathbf{X}}).$$

Application Examples: Variogram model fitting

Variogram model: $2\gamma(\boldsymbol{h};\theta_1,\theta_2) = 1 - \exp\left[-\theta_1|h_1| - \theta_2|h_2|\right]$.

- Gaussian r.f. $\{Z(\mathbf{s} : \mathbf{s} \in I\!R^2)\}, \theta_1 = \theta_2 = 1$.
- iid uniform sites $n = 100, 400, 900, 1400$; region $\mathcal{D}_n = \lambda_n[-1/2, 1/2)^2$.
- 90% modified FDEL regions for (θ_1, θ_2) using variogram estimating functions with $\boldsymbol{h}_1 = (1,1)', \boldsymbol{h}_2 = (1,-1)'$.
- Frequency grid: $\left\{\mathbf{j}\lambda_n^{-\kappa} : \mathbf{j} \in \mathbb{Z}^2 \cap \left[-C\lambda_n, C\lambda_n\right]^d\right\}$, varying C, κ

Application Examples: Variogram model fitting

		$\lambda_n = 24$				$\lambda_n = 48$			
C	κ	100	400	900	1400	100	400	900	1400
1	0.05	88.9	87.8	87.8	89.9	89.3	89.4	89.7	87.9
1	0.1	89.0	90.2	89.6	90.4	89.0	91.4	91.5	90.0
1	0.2	90.0	88.7	90.1	89.7	87.6	87.9	87.9	88.9
2	0.05	89.0	88.6	89.7	87.9	89.2	88.9	90.5	89.7
2	0.1	89.2	88.4	91.1	89.9	90.6	90.0	90.0	91.4
2	0.2	88.9	89.9	89.9	89.2	89.9	89.3	88.1	89.4
4	0.05	89.3	89.0	90.1	90.2	92.9	88.2	90.6	89.9
4	0.1	90.3	89.4	90.3	89.2	92.0	87.8	90.8	89.1
4	0.2	88.7	88.9	90.0	89.6	92.8	88.6	88.5	88.8

Applications: Massive Spatial Data

- FDEL computations require the periodogram on the *frequency grid*
  - The grid itself can be *orders smaller* than the spatial sample size n
  - Only have to compute the periodogram *once* & only on half the frequency grid (by symmetry)
- **Whittle Estimation** (fitting spectral densities $\{\psi_\theta : \theta \in \Theta \subset \mathbb{R}^p\}$)
  - Use Whittle estimation to avoid computational issues with pure likelihood estimation (probability distributions) for large data sets
  - Matsuda and Yajima (2009) introduced a version of the Whittle likelihood for Gaussian processes under MID spatial structure
  - FDEL applies to Whittle estimation for PID/MID structures and for potentially non-Gaussian process.

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Applications: Dependence structure assessments

- We discussed FDEL with p parameters and p estimating functions.
- But, using $r > p$ estimating functions and p parameters, one can maximize the FDEL ratio $\mathcal{R}_n(\theta)$ to obtain $\widehat{\theta}_n$ and use

$$-2a_n(\widehat{\theta}_n) \log \mathcal{R}_n(\widehat{\theta}_n) \xrightarrow{d} \chi^2_{r-p}, \text{ as } n \to \infty, \ a.s. \ (P_{\mathbf{X}}).$$

  to test H_0 : moment $\int_{\mathbb{R}^d} G_{\theta_0}(\boldsymbol{\omega})\psi(\boldsymbol{\omega})d\boldsymbol{\omega} = \mathbf{0}_r$ holds for some θ_0.
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Based on:

1. **Bandyopadhyay, S.** and Lahiri, S., N. (2009). Asymptotic Properties of Discrete Fourier Transforms for Spatial Data, *Sankhya, Series A*, **71(2)** 221-259.
2. **Bandyopadhyay, S.**, Lahiri, S. N., & Nordman, D. (2015). Frequency Domain Empirical Likelihood Method for Irregularly Spaced Spatial Data. *Annals of Statistics*, **43 (2)**, 519-545.
3. **Bandyopadhyay, S.** and Subba Rao, S. (2015). A Test for Stationarity for Irregularly Spaced Spatial Data. *JRSS, Series B*, in press.
4. Van Hala, M., **Bandyopadhyay, S.**, Lahiri, S. N., & Nordman, D. A Frequency Domain Empirical Likelihood for Estimation and Testing of Spatial Covariance Structure. *submitted*
5. **Bandyopadhyay, S.**, Lahiri, S. N. & Nordman, D. A Central Limit Theorem for Periodogram Based Statistics for Irregularly Spaced Spatial Data and Whittle Estimation. *in preparation*

Thank you!