Reflection and the fine structure theorem

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joint work with

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Reflection

Let T be a theory.

Reflection is the statement

if ϕ is provable from T, then ϕ is true.

This statement should be understood internally.

Formalization of reflection

Formulas will coded using the standard Gödel numbers.

Definition (Provability predicate)

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Definition (Truth predicate)

- Truth predicate for Π_n-sentences, $\mathtt{True}_{\Pi_n}(x)$
- $\mathtt{True}_{\Pi_n}(\ulcorner\phi\urcorner) \leftrightarrow \phi$ for $\phi \in \Pi_n$

Formalization of reflection (cont.)

Theorem

$\texttt{True}_{\Pi_n}(x)$ *is* Π_n*-definable.*
(For $n = 0$*,* Δ_1*-definable.)*

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For $n = 1$ one can take for $\mathtt{True}_{\Pi_1}(x)$ the sentence:
If x codes $\forall n\, \phi_0(n)$,
the TM searching for a minimal n with $\neg\phi_0(n)$
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Definition (Reflection)

Reflection for a theory T and Π_n statements

$$\texttt{RFN}_T(\Pi_n) :\equiv \texttt{Prov}_T(x) \to \texttt{True}_{\Pi_n}(x).$$

Relation to induction

Let EA $:= I\Delta_0 + \mathsf{exp}$.

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$\leftarrow$: Cut-elimination.

Why reflection?

Theorem (partly K., Yokoyama '15)

The following are equivalent over $I\Sigma_1$:

- $\mathtt{RFN}_{I\Sigma_1}(\Pi_3)$,
- *well-foundedness of* ω^ω,
- *Hilbert Basis theorem (Simpson '88),*
- *Formanek/Lawrence Theorem (Hatzikiriakou, Simpson '15)*
- $P\Sigma_1$ *(introduced by Hájek, Paris '86/'87)*
- BME_1 *(introduced by Chong, Slaman, Yang, '14)*
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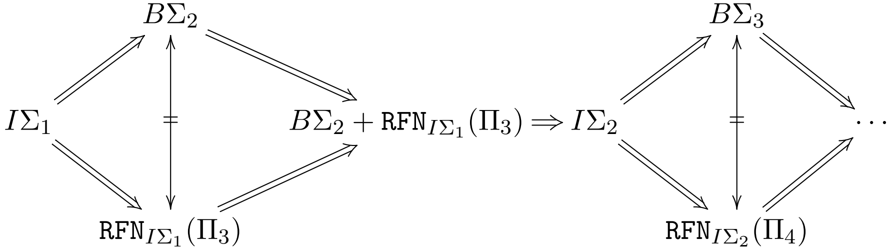
- In particular, $\mathtt{RFN}_{I\Sigma_1}(\Pi_3)$ lies strictly between $I\Sigma_1$ and $I\Sigma_2$.
- Observe $\mathtt{RFN}_{I\Sigma_1}(\Pi_3) \equiv \mathtt{RFN}_{\mathtt{RFN}_{EA}(\Pi_3)}(\Pi_3)$. (Iterated reflection!)

Extended Paris-Kirby hierarchy



Reflection and consistency

- Let $\texttt{Con}(T)$ be the consistency of T.
- This can be formulated as $\neg\texttt{Prov}_T(\ulcorner\bot\urcorner)$

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- By $\texttt{RFN}_T(\Pi_1)$ one gets $\texttt{True}_{\Pi_1}(\ulcorner\bot\urcorner)$, i.e., $\bot$. $\lightning$

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- Let $\texttt{Con}(\Pi_n + T)$ be the consistency of T plus all Π_n-sentences.
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Theorem

$\texttt{RFN}_T(\Pi_{n+1}) \leftrightarrow \texttt{Con}(\Pi_n + T)$.

Existence of models

Theorem (Simpson)

WKL_0^* *proves the completeness theorem, i.e., every consistent theory has a model.*

Model $\mathcal{M}$ is here coded a the set of (Gödel numbers of) sentences true in $\mathcal{M}$.

Existence of models (cont.)

Theorem

Let $n \geq 1$ *and* T *be a theory.*
A model $(\mathcal{M}, \mathcal{S}) \models \mathsf{RCA}_0 + B\Sigma_{n+1} + \mathtt{Con}(\Pi_n + T)$ *has an* n*-elementary end extension* $\mathcal{I}$ *satisfying* T.

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Proof.

- Let $(\mathcal{M}, \mathcal{S})$ be a second-order model with $(\mathcal{M}, \mathcal{S}) \models \mathsf{RCA}_0 + B\Sigma_{n+1} + \mathtt{Con}(\Pi_n + T)$.

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- Let $T' := T + X +$ "constants for each element of $\mathcal{M}$".
- By $\mathtt{Con}(\Pi_n + T)$, the theory T' is consistent. By WKL there exists a model of T'. By definition T' is an n-elementary end-extension. □

Existence of models (cont.)

Remark

To make sure that is a true end-extension one can replace T' by $T' + \neg\mathtt{Con}(T')$. By Gödel's incompleteness theorem, $T' + \neg\mathtt{Con}(T')$ is also consistent.

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Remark

In the previous proof we used $B\Sigma_n$ only to get the set of all true Π_n-sentences. If the end-extension $\mathcal{I}$ should satisfy one sentence Π_n-sentence then RCA_0^* is sufficient.

Existence of models (cont.)

Example

Over RCA_0^* the statement $\mathtt{Con}(\Pi_1 + I\Delta_0 + \mathsf{exp})$ proves the totality of superexp, i.e., $n \mapsto \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{n}}}}}}_{n \text{ times}}$.

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  Then there is an $c \in \mathcal{M}$, such that superexp(c) is does not exists.
  In detail, let $\phi(x, y)$ be the Σ_1-formula defining superexp.
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$$\mathcal{M} \models \exists y \, \phi(0, y), \forall x \; (\exists y \, \phi(c, y) \rightarrow \exists y \, \phi(c + 1, y)) \, .$$

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- Working in $\mathcal{M}$, using $I\Delta_0(\mathcal{I})$, we can apply the implication c times and obtain that $\mathcal{I} \models \exists y\, \phi(c, y)$. ↯ □

Iterated reflection

Notation

Let T be a theory.

- $(T)^n_0 := T$,
- $(T)^n_{\alpha+1} := (T)^n_\alpha + \mathtt{RFN}_{(T)^n_\alpha}(\Pi_n)$,
- $(T)^n_\lambda := \bigcup_{\alpha<\lambda}(T)^n_\alpha$.

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Example

- $(\mathsf{EA})^3_1 = I\Sigma_1$,
- $(\mathsf{EA})^3_2 = (I\Sigma_1)^3_1 =$ "well-foundedness of ω^ω",
- $(\mathsf{EA})^4_1 = I\Sigma_2$,
- $(\mathsf{EA})^2_1 = I\Delta_0 + \mathsf{exp} + \mathsf{superexp}$.
- $(\mathsf{EA})^2_\omega = \mathsf{PRA}$.

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- $(\mathsf{EA})^4_1 = I\Sigma_2$,
- $(\mathsf{EA})^2_1 = I\Delta_0 + \exp + \text{superexp}$.
- $(\mathsf{EA})^2_\omega = \mathsf{PRA}$.

Theorem (Beklemishev '97)

$(EA)^2_\alpha$ is the same as Grzegorczyk arithmetic of level $\alpha + 3$.

Fine structure theorem

Theorem (Schemerl's formula, '79,)

Let $n \geq 1$ *and* T *be a* Π_{n+1}*-axiomatic extension of* EA*.*
$(T)_1^{n+1}$ *is* Π_n*-conservative over* $(T)_\omega^n$*.* *(*$n \geq 1$*)*

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- We prove the case $n = 2$, $T = I\Sigma_1$.
- Proof we proceed by contraposition:
  For $\phi \in \Pi_2$:

  $$\text{If } (T)_\omega^2 \nvdash \phi \quad \text{then} \quad (T)_1^3 \nvdash \phi.$$

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- This will be shown by a model construction.
- The construction is a refinement of McAllon '78.

Proof of Schmerl's formula

Given is a non-standard model $\mathcal{I}_0 \models (T)^2_\omega + \neg\phi$.

Goal: Construct a model $\mathcal{M} \models (T)^3_1 + \neg\phi$.

Take a non-standard $b \in \mathcal{I}_0$ such that $\mathcal{I}_0 \models (T)^2_b$.
Let $\mathcal{I}_1$ be a true Π_1-elementary end extension satisfying $(T)^2_{b-1}$, as constructed before.
By construction $\mathcal{I}_0 \models \mathtt{Prov}(\ulcorner\psi\urcorner)$ then $\mathcal{I}_1 \models \psi$.
Iterate this construction to get $\mathcal{I}_n \models (T)^2_{b-n}$.
Let $\mathcal{M} := \bigcup_{n\in\mathbb{N}} \mathcal{I}_n$.

Lemma

$\mathcal{M} \models \mathtt{RFN}_{I\Sigma_1}(\Pi_3)$

Proof of Schmerl's formula (cont.)

Lemma

$\mathcal{M} \models \mathtt{RFN}_{I\Sigma_1}(\Pi_3)$

Proof.

- Let $\psi = \forall x \, \exists y \, \forall z \, \psi_0(x, y, z)$.
- Suppose $\mathcal{M} \models \mathtt{Prov}(\ulcorner \psi \urcorner)$. Then there is a derivation of ψ in $\mathcal{I}_{k_1}$ for some $k_1 \in \mathbb{N}$.
- Given $c_x \in \mathcal{M}$. Then $c_x \in \mathcal{I}_{k_2}$ for a $k_2 \in \mathbb{N}$.
- $\mathcal{I}_{\max(k_1,k_2)+1} \models \exists y \, \forall z \, \psi_0(c_x, y, z)$.
- In other words, there exists $c_y \in \mathcal{I}_{\max(k_1,k_2)+1}$, s.t. $\mathcal{I}_{\max(k_1,k_2)+1} \models \forall z \, \psi_0(c_x, c_y, z)$.
- By Π_1-elementarity

$$\mathcal{I}_n, \mathcal{M} \models \psi_0(c_x, c_y, z)$$

for $n \geq \max(k_1, k_2) + 1$. □

Proof of Schmerl's formula

- Original proof of Schmerl proceeds by comparing well-orders.
- This model-theoretic proof is new.
  - Note that it only works for reasonably strong theories T.
    $I\Sigma_1$ is certainly enough.
    That means $n \geq 2$ or T contains $I\Sigma_1$.
    This is need to extend the model to a model of WKL. Here we use Baire Category theorem for the forcing extension.
    By Simpson '14 the Baire Category theorem is equivalent to $I\Sigma_1$.

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  - Note that it only works for reasonably strong theories T.
    $I\Sigma_1$ is certainly enough.
    That means $n \geq 2$ or T contains $I\Sigma_1$.
    This is need to extend the model to a model of WKL. Here we use Baire Category theorem for the forcing extension.
    By Simpson '14 the Baire Category theorem is equivalent to $I\Sigma_1$.
  - Natural reflection model.

Question

What is the strength of extending a model $\mathcal{M} \models \mathsf{RCA}_0^*$ to a model of $\mathcal{M} \models \mathsf{WKL}_0^*$?

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Theorem (Chong, Slaman, Yang, '14)

$\mathsf{RCA}_0 + \mathsf{SRT}^2_2$ *does not prove* $I\Sigma_2$.

Proof proceeds in two steps:

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This theorem follows also from K. Yokoama and L. Patey.

Full fine structure theorem

Theorem (Fine structure theorem, Schmerl '79)

For each $n, k \geq 1$, and all ordinals $\alpha \geq 1$, β, the theory $((\mathsf{EA})^{n+k}_{\alpha})^{n}_{\beta}$ proves the same Π_n-sentences as $(\mathsf{EA})^{n}_{\omega_k(\alpha)\cdot(1+\beta)}$.

Follows from iterations of Schmerl's formula.
For this to work it is sufficient if $n \geq 3$.

Note on the Existence of Models theorem

Theorem

Let $n \geq 1$ *and* T *be a theory.*
$\mathsf{RCA}_0 + B\Sigma_{n+1} + \mathtt{Con}(\Pi_n + T)$ *proves that there exists an* n*-elementary* ~~*end extension*~~ $\mathcal{I}$ *satisfying* T*.*

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sometimes also called *reflection*. This theorem say that these two forms of reflection coincide.
For stronger Σ^1_k sets this has been analyzed. This is on the level Π^1_∞-TI. (Friedman, see Simpson's Subsystems of Second Order Arithmetic.)

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Thank you for your attention!