Recursion theory over a model

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General motivation

A theory T_1 is Π^1_1 *conservative* over a second theory T_0 if every Π^1_1 sentence provable from $T_0 \cup T_1$ is already provable from T_0.

Examples:

1. (Harrington) WKL_0 is Π^1_1 conservative over RCA_0.
2. (Chong, Slaman, Yang) COH is Π^1_1 conservative over $\mathrm{RCA}_0 + \mathbf{B\Sigma_2}$.
3. (H. Friedman) ACA_0 is Π^1_1 conservative over RCA_0+PA.

Proof recipe:

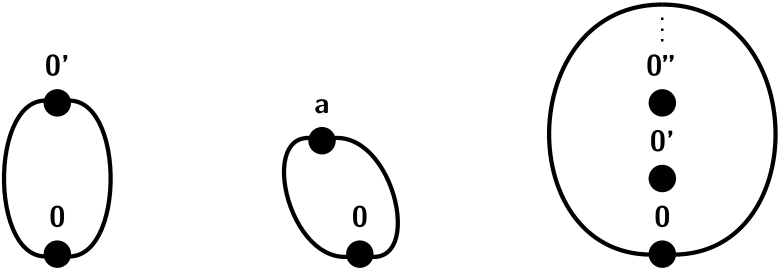
- Suppose $T_0 \not\vdash (\forall X)\Theta$ with Θ arithmetical.
- Then there is a countable model $\mathcal{M} \models T_0 + (\exists X)\neg\Theta$.
- By adding sets to $\mathcal{M}$, expand to a model $\mathcal{N}$ of $T_0 \cup T_1$.
- Then $\mathcal{N} \models (\exists X)\neg\Theta$, so $\mathcal{N} \models (\exists X)\neg\Theta$, so $\mathcal{N} \models (\exists X)\neg\Theta$, so $\mathcal{N} \models (\exists X)\neg\Theta$, so $T_0 \cup T_1 \not\vdash (\forall X)\Theta$.

The second-order part of a model

If $\mathcal{M} = (M, \mathcal{S})$ is a model of RCA_0, then $\mathcal{S}$ is

- closed under $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$
- downward-closed in the 'Δ_1 in' relation.

If $\mathcal{M}$ is an ω-model, $\mathcal{S}$ is a Turing ideal.



Some definitions

Let $\mathcal{M} = (M, \mathcal{S})$ be a model of RCA_0. Let $X \subseteq M$ be any set.

X is *bounded* if some $a \in M$ is greater than every $x \in X$.

X is *finite* if it is bounded and is encoded as an element of M (perhaps in binary).

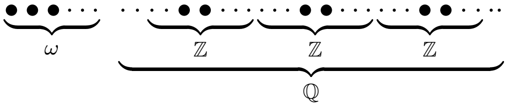
X is *regular* if each initial segment $X \cap \{0, \ldots, a\}$ is finite.

X is $\boldsymbol{\Sigma_n}$ if it is Σ_n with parameters from $\mathcal{M}$.
In particular, X is $\boldsymbol{\Delta_1}$ iff $X \in \mathcal{S}$.

$\mathbf{I\Sigma_n}$: Every $\boldsymbol{\Sigma_n}$ set is regular.
$\mathbf{B\Sigma_n}$: Every $\boldsymbol{\Delta_n}$ set is regular.

Some bad sets

Say we want to add sets to a model $\mathcal{M}$ while preserving $\mathbf{I\Sigma_1}$.



We want to expand while preserving $\mathbf{I\Sigma_1}$

Cannot adjoin: A proper cut

because then it would be bounded and $\mathbf{\Delta_1}$ with no maximum.

Cannot adjoin: A cofinal sequence of order type ω

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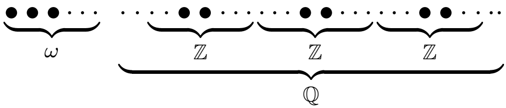

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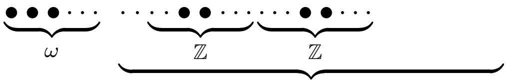

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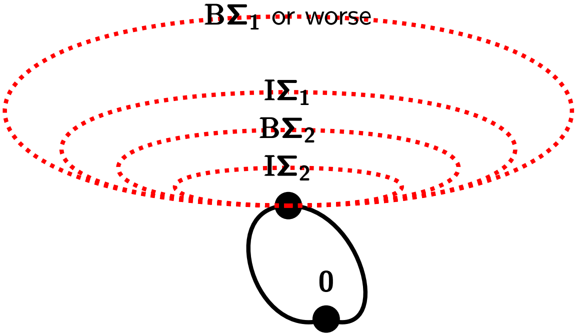
Many possible extensions

$\mathcal{M} \models \mathrm{RCA}_0 + \mathbf{I\Sigma_2}$ is topped.



Some good sets

Suppose $\mathcal{M} = (M, \mathcal{S})$ is *topped*, i.e., there is a single $A \in \mathcal{S}$ which can be used as the parameter for defining any $\mathbf{\Sigma_n}$ set.

A set $X \subseteq M$ is *low* if $\mathcal{M}[X]$ has the same $\mathbf{\Delta_2}$ sets as $\mathcal{M}$.

Recall: $\mathrm{B}\mathbf{\Sigma_n} \iff$ Every $\mathbf{\Delta_n}$ set is regular.

Thus if $\mathcal{M} \models \mathrm{RCA}_0 + \mathrm{B}\mathbf{\Sigma_2}$ and X is low, $\mathcal{M}[X] \models \mathrm{RCA}_0 + \mathrm{B}\mathbf{\Sigma_2}$.

Lemma (Formalized Low Basis Theorem)

If $\mathcal{M} \models \mathrm{RCA}_0 + \mathrm{B}\mathbf{\Sigma_2}$ then every infinite $\mathbf{\Delta_1}$ binary tree has a low infinite path.

Corollary (Hajek)

WKL_0 *is* Π^1_1 *conservative over* $\mathrm{RCA}_0 + \mathrm{B}\mathbf{\Sigma_2}$.

Similarly for all $\mathrm{I}\mathbf{\Sigma_n}$, $\mathrm{B}\mathbf{\Sigma_n}$, $n \geq 2$.

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If $\mathcal{M} \models \mathrm{RCA}_0 + \mathbf{B\Sigma_2}$ *then every infinite* $\mathbf{\Delta_1}$ *binary tree has a low infinite path.*

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Some better sets

A set $X \subseteq M$ is *ω-r.e.* if there are a uniformly $\mathbf{\Delta_1}$ sequence $\langle X_0, X_1, \ldots \rangle$ and a $\mathbf{\Delta_1}$ function f such that

- $\langle X_s \rangle_s$ converges pointwise to X, and
- $\left|\{s : k \text{ enters or leaves } X_s\}\right| < f(k)$ for each k.

Recall: $\mathbf{I\Sigma_n} \iff$ Every $\mathbf{\Sigma_n}$ set is regular.

Lemma

$\mathbf{I\Sigma_1} \iff$ *Every ω-r.e. set is regular.*

Lemma (Formalized Superlow Basis Theorem)

If $\mathcal{M} \models \mathrm{RCA}_0$ then every infinite $\mathbf{\Delta_1}$ binary tree has an infinite path P such that $\mathcal{M}[P]$ has no new ω-r.e. sets.

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The Turing jump

The *jump* of a set $X \subseteq M$ is $\{e : \Phi_e^X(e) \text{ converges}\}$.

The system RCA_0^* is like RCA_0 with $\mathbf{B\Sigma_1}$ in place of $\mathbf{I\Sigma_1}$.

Over RCA_0^*:

- $\mathcal{M} \models \mathbf{I\Sigma_1} \iff A'$ is regular for every $\mathbf{\Delta_1}$ set A.
- $\mathcal{M} \models \mathbf{I\Sigma_{n+1}} \iff \mathcal{M}[A'] \models \mathbf{I\Sigma_n}$ for every $\mathbf{\Delta_1}$ set A.
- Similarly for $\mathbf{B\Sigma_{n+1}}$.
- If $\mathcal{M} \models \mathbf{B\Sigma_2}$ then a set Y is $\mathbf{\Delta_2}$ iff it is $\mathbf{\Delta_1}$ in $\mathcal{M}[A']$ for some $\mathbf{\Delta_1}$ set A.

A jump inversion theorem

Theorem (Friedberg jump theorem)

In the true natural numbers ω, if X Turing-computes $\emptyset'$, there is a Y such that Y' is Turing-equivalent to X.

Lemma (Formalized version. B. (Cf Towsner 2015))

If $\mathcal{M} \models \mathrm{RCA}_0^* + \mathbf{B\Sigma_{n+1}}$ is topped by A and $\mathcal{M}[A' \oplus X] \models \mathbf{B\Sigma_n}$, then there is a Y such that $\mathcal{M}[Y] \models \mathbf{B\Sigma_{n+1}}$ and such that $\mathcal{M}[Y'] = \mathcal{M}[A' \oplus X]$. Similarly for $\mathbf{I\Sigma_{n+1}}$.

An application

COH is the statement: If $\langle R_0, R_1, \ldots \rangle$ is a uniformly $\mathbf{\Delta_1}$ sequence of sets, there is an infinite set C satisfying

$$(\forall k)[\text{either } C \cap R_k \text{ or } C \cap \overline{R_k} \text{ is finite}].$$

Theorem (B)

$\mathrm{RCA}_0 + \mathbf{B\Sigma_2} \vdash \mathrm{COH} \iff$ *Every infinite $\mathbf{\Delta_2}$ binary tree has an infinite $\mathbf{\Delta_2}$ path.*

Corollary (Chong, Slaman, Yang; new proof)

COH *is Π^1_1 conservative over* $\mathrm{RCA}_0 + \mathbf{B\Sigma_2}$.

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Extending non-topped models?

Theorem (Towsner 2015)

If $\mathcal{M} \models \mathrm{RCA}_0 + \mathbf{I\Sigma_n}$ is countable and $X \subseteq M$ is any set at all, there is an extension $\mathcal{M}[Y] \models \mathrm{RCA}_0 + \mathbf{I\Sigma_n}$ in which X is $\mathbf{\Delta_{n+1}}$.

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Proof uses 'exact pair' forcing with blocking and jump control. With jump inversion, proves the $n = 1$ case of Towsner.

Theorem (Unverified)

Similarly for each $\mathbf{I\Sigma_n}$ and for full PA. (Perhaps with some technical hypotheses about $\mathcal{M}$.)

This would prove Π^1_1 conservation for ACA_0^+.

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