Rado Path Decomposition

Peter Cholak

January, 2016

Singapore

Joint with Greg Igusa, Ludovic Patey, and Mariya Soskova.

Monochromatic paths

Definition

Let $c : [\omega]^2 \to r$. A *monochromatic path* of color j is an ordered listing (possibly finite or empty) of integers $a_0, a_1, a_2 \ldots$ such that, for all $i \geq 0$, if a_{i+1} exists then $c(\{a_i, a_{i+1}\}) = j$.

An empty listing can be a path of any color. A singleton can be a path of any color. The color is determined for paths of more than one node. Paths might be finite or infinite.

3 colors is more canonical than 2 color.

Rado's Theorem

Improving on a result of Edrős, Rado published a theorem which implies:

Theorem (Rado Path Decomposition or RPD$_r$)

Let $c : [\omega]^2 \to r$. Then, for each $j < r$, there is a monochromatic path of color j such that these r paths (as sets) partition ω (so they are pairwise disjoint sets and their union is everything).

Richard Rado. Monochromatic paths in graphs. *Ann. Discrete Math.*, 3:191-194, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

Ultrafilter Proof, Part I

Let neighbors of m with color i be

$$N(m,i) = \{n : c(\{m,n\}) = i\}.$$

Fix a non-principal ultrafilter on ω. For all m, for some unique $j < r$, $N(m,j)$ is *large* (in the ultrafilter). Let $A_j = \{m : N(m,j) \text{ is large}\}$. The A_j partition ω.

Think of $m \in A_j$ as having color j. Each m has a unique color.

Ultrafilter Proof, Part II

For any pair of points $m < n$ in A_j, $N(m, j) \cap N(n, j)$ is large. So there are infinitely many $v \in N(m, j) \cap N(n, j)$. For all such v, $c(m, v) = c(v, n) = j$. Note that any such v is likely much larger than m and n and not necessary the same color.

Stagewise build finite paths such that the current end of the path of color j has color j and at stage s use a v like above to add s to the path of it's color (the path and s have the same color).

Ramsey Theorem and our Ultrafilter Proof

One of the A_j must be large. We can *thin* A_j to get a homogenous set of color j. Given $a_i \in A_j$ choose a_{i+1} in $A_j \cap \bigcap_{k \le i} N(a_k, j)$.

Questions, I

The existence of an ultrafilter cannot be shown in ZF. But, by independent results by Enayat, Kreuzer, and Towsner, adding a non-principal ultrafilter is conservative over ACA_0. So RPD_r follows from ACA_0.

Question

Does RPD_r imply ACA_0?

Question

Does RPD_r imply RT^2_r?

Computable paths are not enough

It is well known that there is a computable linear order, $(\omega, <_L)$ of type of $\omega + \omega^*$ with no computable ascending or descending sequence. For $x < y$, color the pair (x, y) red iff $x \leq_L y$. Blue otherwise. A computable red (blue) path is an ascending (descending) sequence.

This can be improved to show that there is no uniform Δ_2^0 path decomposition for 2-computable colorings.

Question

Is there an r-coloring without a Δ_2^0 path decomposition?

Cohesive Proof

Recall $N(m,i) = \{n : c(\{m,n\}) = i\}$ is the neighbors of m with color i. Let C be *cohesive* w.r.t. to all $N(m,i)$, so C is infinite and, for all m, i, either $C \subseteq^* N(m,i)$ or $C \subseteq^* \overline{N(m,i)}$.

Now a set X is *large* iff $C \subseteq^* X$ and repeat ultrafilter proof with this notion of largeness.

PA over $0'$

A careful analysis of the last proof shows that the path decomposition is computable in C'.

Why the jump? Exactly one $N(m, j)$ is large (in our cohesive set C) . It is Δ_2^C to determine which one.

By Jockusch and Stephan, $\mathbf{d} \gg 0'$ iff there is an r-cohesive set C such that $C' \leq_T \mathbf{d}$.

For computable graphs a path decomposition is computable in $\mathbf{d}$ if $\mathbf{d} \gg 0'$.

Question

Can this be improved?

Generic Path Decompositions

Consider $(\tau_0, \tau_1 \ldots \tau_{r-1}, X)$ such that X is infinite and if $\tau_j = \sigma\hat{\ }m$ then $X \subseteq^* N(m, j)$ (so m has color j) as our forcing conditions. A generic G for this forcing is a path decomposition. Forcing Σ_1^G statements (like does $\Phi^G(w)\downarrow$) is Σ_2^X. So this forcing cannot be used for cone avoidance.

Stable Colorings

A coloring c is *stable* iff for all m, $\lim_n c(m,n)$ exists. Fix a stable coloring and now let *large* mean almost all and repeat our ultrafilter proof.

Stable computable colorings have Δ^0_2 path decompositions.

A path decomposition restricted for the coloring $c : [X]^2 \to r$ (where $X \subset \omega$) does not help find a path decomposition for the coloring $c : [\omega]^2 \to r$. So COH does not help reduce the problem.

Finite versions of RPD_r

Pokrovskiy showed that given any $r > 2$ and M there is an r-coloring of $[M]^2$ (this graph is just K_M) which does not partition into r many paths, one of each color. For $r = 3$, 3 paths is enough but two of them might have the same color. $r = 2$ is special and will be dealt with shortly.

The normal proof of the finite version from the infinite version using compactness breaks down because the paths linking numbers below M might also involve some very large numbers.

Proof for 2-colorings of K_M

Assume the colors are RED and BLUE. Inductively assume we have two paths of color RED and BLUE. Let x be the least integer not in any path. Let x_r be the end of RED path and similarly with x_b.

If there is any RED path between x_r and x avoiding our partially constructed paths, add that path to the end of the RED path. (Since finite, this is a computable question.) Similarly for BLUE.

Otherwise look at the color of (x_r, x_b). If this is RED add x_b, x (in that order) to the end of the RED path and remove x_b from the end of the BLUE path. So x_b *switches to RED*. If this is BLUE add x_r, x (in that order) to the end of the BLUE path and remove x_r from the end of the RED path. Since there are only finitely many x's we settle on our final paths.

This proof fails for $r = 3$.

Path Decompositions for 2-colorings

Theorem

If $c : [\omega]^2 \to 2$ is computable then there is a Δ^0_2 Path Decomposition and the proof is nonuniform.

A Key Observation about Switching

Assume x_b switches to RED. Only the ends of the paths switch so if x_b switches again back to BLUE x also must switch back to BLUE but there no BLUE path from x_b to x. If x_b switches from the blue path to the red path, it cannot switch again.

If there are infinitely many BLUE and RED switches then both paths stabilized and are infinte. If there are only finitely many switches then again the path stabilized but one might be finite.

But otherwise our algorithm breaks down. We used this failure to create another algorithm which works within the environment of this failure. Hence the end result is nonuniform.

Questions, II

Question

Let $\mathcal{I}$ be a Turing ideal where RPD_r holds. Is $0' \in \mathcal{I}$? Does RT_r^2 hold in $\mathcal{I}$? WKL? ...

Question

What is the Medvedev degree of RPD_r? The Muchnik degree?

Take a path decomposition for an r-coloring. A coloring restricted to k of the paths is not necessarily a k coloring.

Question

Does RPD_r imply RPD_{r+1}? Is this computably true?

Some More References

Daniel T Soukup.
Decompositions of edge-coloured infinite complete graphs into monochromatic paths II.
arXiv.org, July 2015.

Henry Towsner.
Ultrafilters in reverse mathematics.
J. Math. Log., 14(1), 11, 2014.

Alexey Pokrovskiy.
Partitioning edge-coloured complete graphs into monochromatic cycles and paths.
J. Combin. Theory Ser. B, 106:70–97, 2014.