Brown's lemma in reverse mathematics

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Brown's lemma asserts that piecewise syndetic sets are partition regular, that is whenever we partition a piecewise syndetic set into finitely many sets, at least one set must be piecewise syndetic.

Other examples of partition regular classes are the sets of positive upper density (Szemerédi's theorem) and sets containing arbitrarily long arithmetic progressions (Van der Waerden's theorem).

This lemma was established by T.C. Brown in the context of locally finite semigroups and generalized by Justin in order to obtain an extension of van der Waerden's theorem to semigroups.

There are various applications of Brown's lemma (locally finite semigroups, tiling theory).

Theorem (Brown's Lemma)

Every finite coloring $f : \mathbb{N} \to r$ has an f-homogeneous piecewise syndetic set.

Definition (RCA_0)

A set X is **syndetic** if there exists $d \in \mathbb{N}$ such that X has gaps bounded by d.

A set $X \subseteq \mathbb{N}$ is **piecewise syndetic** if there exists $d \in \mathbb{N}$ such that X contains arbitrarily large finite sets with gaps bounded by d.

We say that X is d-**piecewise syndetic** if d witnesses that X is piecewise syndetic.

For instance, the 2-coloring

$$0100110001110000111 1 \ldots$$

has 1-piecewise syndetic homogeneous sets for both colors but no syndetic homogeneous sets.

The 2-coloring

$$01010101010101 \ldots$$

has 2-syndetic homogeneous sets in both colors but no 1-piecewise syndetic homogeneous set.

The 2-coloring

$$0011001100110011 \ldots$$

has 3-syndetic homogeneous sets but no 2-piecewise syndetic homogeneous sets.

The classical proof of Brown's lemma is elementary and easily formalizable in second-order arithmetic. However, it is based on the following variant of König's lemma, which is easily seen to be equivalent to ACA_0.

Lemma

If $S \subseteq r^{<\mathbb{N}}$ is infinite, then there exists $g : \mathbb{N} \to r$ such that for all n there is $\sigma \in S$ with $g \upharpoonright n \subseteq \sigma$.

Proposition

ACA_0 *is equivalent to* Σ^0_1-WKL_0.

Theorem (F)

Over RCA_0 *(even* RCA_0^**), Brown's Lemma is equivalent to* $\mathsf{I}\Sigma^0_2$*.*

Piecewise syndetic sets are Σ_3-definable and closed under supersets. Brown's lemma does not actually assert the existence of a set. It is a Π^1_1-statement. A partition lemma of this form has no computational strength but only inductive strength.

We conjecture that a statement of the form

† Every finite coloring $f\colon \mathbb{N} \to r$ has an f-homogeneous *large* set,

where *large* is a property about sets closed under supersets, is equivalent to

- $\mathsf{I}\Sigma^0_n$ if *large* is Σ_{n+1}-definable,
- $\mathsf{B}\Sigma^0_n$ if *large* is Π_n-definable.

Evidence

Brown's lemma and the infinite pigeonhole principle (RT^1) serve as an example for $n = 2$.

The following is equivalent to $\mathsf{I}\Sigma^0_2$ (Hirst):

- For every finite coloring $f : \mathbb{N} \to r$ there is b such that for all $x > a$ there is $y > x$ such that $f(x) = f(y)$.

The following version of Folkman's theorem is equivalent to $\mathsf{B}\Sigma^0_2$:

Theorem

Every $f : \mathbb{N} \to r$ has an f-homogeneous set X such that for every k there exists a set $F \subseteq X$ of size k with $FS(F) \subseteq X$.

The proof uses the finite version of Folkman's theorem.

Brown's Lemma vs van der Waerden's Theorem

In "Ramsey theory on the integers" page 279:

> *Comparing this statement to the statements given in Theorem 2.5, we see that Brown's lemma is very reminiscent of van der Waerden's theorem. However, it is known that Brown's lemma neither implies, nor is implied by, van der Waerden's theorem*

I asked the authors what that means.

> *By this we mean that assuming only BL, we cannot prove VDW (as you state you need VDW finite version to show infinite BL implies infinite VDW) and assuming only VDW, we cannot prove BL. So, even through infinite BL may be stronger than infinite VDW, the infinite BL does not by itself imply infinite VDW (or even finite VDW).*

Here VDW is the the finite one.

Theorem (Van Der Waerden's Theorem)

Every finite coloring $f : \mathbb{N} \to r$ has an f-homogenous set with arbitrarily long arithmetic progressions.

Let $N = \{0, 1, \ldots, N-1\}$.

Theorem (Van Der Waerden's Theorem, finite)

For all r, l there exists N such that every coloring $C : N \to r$ has a C-homogeneous arithmetic progression of length l.

Shelah provided primitive recursive upper bounds for the van der Waerden numbers as a byproduct of a new elementary proof of Hales-Jewett theorem that uses Σ_1-induction.

Theorem (Folklore)

The finite version of van der Waerden's theorem is provable in RCA_0.

Shelah's proof of Hales-Jewett theorem avoids the use of double induction which results in Ackermannian upper bounds. Gowers in his celebrated work on Szemerédi's theorem obtained elementary recursive upper bounds for the van der Waerden's numbers. Gowers' bound is the following:

$$W(r,l) \leq 2^{2^{f(r,l)}}, \text{ where } f(r,l) = r^{2^{2^{l+9}}}.$$

However, the proof is far from elementary and so we can only conjecture that van der Waerden's theorem is provable in EFA (Elementary Function Arithmetic). On the other hand, the lower bounds for the van der Waerden's numbers are exponential, and hence van der Waerden's theorem is not provable in bounded arithmetic.

By the finite van der Waerden's theorem, every piecewise syndetic set contains arbitrarily long arithmetic progressions and this implication can be proved within RCA_0. Therefore it is not surprising that within RCA_0 Brown's lemma implies van der Waerden's theorem.

We establish this by showing:

Proposition

Over RCA_0 *(even* RCA_0^**), van der Waerden's theorem is equivalent to* $\mathrm{B}\Sigma^0_2$*.*

Notice that the proof in $\mathrm{B}\Sigma^0_2$ uses the finite van der Waerden's theorem.

Corollary

Over RCA_0 *(even* RCA_0^**), Brown's lemma implies van der Waerden's theorem.*

A proof in $\mathrm{I}\Sigma^0_3$

Theorem

Over RCA_0, $\mathrm{I}\Sigma^0_3$ *implies Brown's Lemma.*

Proof.

Let $f \colon \mathbb{N} \to r$ be a finite coloring. By bounded Σ^0_3-comprehension let:

$$C = \{c < r \colon \{x \in \mathbb{N} \colon f(x) \geq c\} \text{ is piecewise syndetic}\}.$$

Let c be the maximum element of C. If $c + 1 = r$, we are done. Suppose $c + 1 < r$.

Let $d \in \mathbb{N}$ be such that $\{x \in \mathbb{N} \colon f(x) \geq c\}$ is d-ps. By the choice of c, $\{x \in \mathbb{N} \colon f(x) \geq c + 1\}$ is not ps, and in particular is not d-ps. Then there must be an $e \in \mathbb{N}$ such that every finite set of size e and gaps bounded by d contains an x such that $f(x) \leq c$. Show that $\{x \in \mathbb{N} \colon f(x) = c\}$ is $d \cdot e$-ps. □

From Brown's Lemma to $\mathsf{I}\Sigma^0_2$

The number d does not depend on r.

Lemma (RCA_0, Diagonalization Lemma)

There exists a function $g \colon \mathbb{N} \times \mathbb{N} \to 2$ such that for all d the finite 2-coloring $f(x) = g(d, x)$ has no f-homogeneous d-piecewise syndetic sets.

Proof.

For $d = 3$ let $000111000111000\ldots$. □

Theorem

Over RCA_0 (even RCA^*_0), Brown's Lemma implies $\mathsf{I}\Sigma^0_2$.

Proof.

We aim to prove $\mathsf{S}\Pi^0_1$, strong collection for Π^0_1-formulas, that is:

$$(\forall a)(\exists b)(\forall n < a)\big(\exists m \theta(n, m) \to (\exists m < b)\theta(n, m)\big),$$

where θ is Π^0_1.

Define $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $h(n) = \lim_{s\to\infty} h(n,s) =$ the least m such that $\theta(n,m)$, provided that $\exists m \theta(n,m)$.

Suppose for a contradiction that there is an a with no b. Then for every b there is an $n < a$ such that $\exists m \theta(n,m)$ and $h(n) \geq b$.

Define a coloring $f : \mathbb{N} \to 2^a$ as follows. Let $g : \mathbb{N} \times \mathbb{N} \to 2$ be as in the Diagonalization Lemma and let $f(x) = \langle g(h(n,x),x) \colon n < a \rangle$.

By Brown's Lemma, there exists $c \in 2^a$ such that $\{x \in \mathbb{N} \colon f(x) = c\}$ is ps. Let e be a witness. For such an e, there exists $n < a$ such that $h(n)$ exists and $d = h(n) \geq e$.

Show that $\{x \in \mathbb{N} \colon g(d,x) = c(n)\}$ is d-ps, for the desired contradiction. Let s be such that $h(n,x) = h(n) = d$ for all $x > s$. Fix a size k. Let G be a finite set of size $s+1+k$ and gaps bounded by e such that $f(x) = c$ for all $x \in G$.

Then G contains at least k elements $> s$. Let F consist of the last k elements of G. Clearly F has size k and gaps bounded by d. Moreover, for all $x \in F$ we have that $g(d, x) = g(h(n), x) = g(h(n, x), x) = f(x)(n) = c(n)$. □

A proof in $I\Sigma^0_2$.

Definition (RCA_0)

Let $H \subseteq \mathbb{N}$ be finite. Define the *gap size* of H, denoted $gs(H)$, as the largest difference between two consecutive elements of H. In other words, the gap size of H is the least d such that H has gaps bounded by d.

Theorem (Brown's Lemma, finite)

Let $k \colon \mathbb{N} \to \mathbb{N}$. Then for all r there exists N such that every $C \colon N \to r$ has a C-homogeneous set H of size $\geq k(gs(H))$.

Theorem

The finite version of Brown's Lemma is provable in RCA_0.

Lemma ($\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$)

Let $k\colon \mathbb{N} \to \mathbb{N}$ be a Σ^0_2-definable total function. Then for all r there exists N such that every $P\colon N \to r$ has a P-homogeneous set H of size $\geq k(gs(H))$.

Theorem

Over RCA_0, $\mathsf{I}\Sigma^0_2$ *implies Brown's Lemma.*

Proof.

Let $f\colon \mathbb{N} \to r$ be a finite coloring and suppose for a contradiction that $\{x \in \mathbb{N}\colon f(x) = c\}$ is not piecewise syndetic for every $c < r$. Then for all d and $c < r$ there exists a k such that no set of size k and gaps bounded by d is f-homogeneous for color c. By $\mathsf{B}\Sigma^0_2$, for all d there exists k such that no set of size k and gaps bounded by d is f-homogeneous.

There exists a Σ^0_2-definable (in fact Δ^0_2-definable) total function $k\colon \mathbb{N} \to \mathbb{N}$ such that for all d no set of size $k(d)$ and gaps bounded by d is f-homogeneous.

By the modified finite Brown's Lemma, let N be large enough be such that every $C\colon N \to r$ has a C-homogeneous set H of size $\geq k(gs(H))$. In particular there exists an f-homogeneous set H of size $\geq k(gs(H))$, for the desired contradiction. □

Questions

Let $B(r,k)$, for $r > 0$ and $k\colon \mathbb{N} \to \mathbb{N}$, be the least natural number N such that every r-coloring of N has a homogeneuos set F of size at least $k(gs(F))$. The proof of the finite Brown's lemma gives superexponential upper bounds $N(r,k)$ for $B(r,k)$. For instance, if $k(d) = 2^d$, then $N(r,k) \geq 2_r$, where

$$2_r = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}, \text{ a tower of } r \text{ twos.}$$

Question

Is finite Brown's lemma provable in RCA_0^**?*

We expect the answer to be positive.

As the finite van der Waerden's theorem is already provable in RCA_0 and presumably in RCA_0^*, the question whether Brown's lemma implies van der Waerden's theorem and vice versa can be settled only over a weak system of arithmetic.

Question

What is the relationship between (the finite versions of) Brown's lemma and van der Waerden's theorem over a suitable bounded fragment of second-order arithmetic?

Question (Mummert)

What is the Weihrauch degree of Brown's lemma?

$BL :\subseteq \{f : \mathbb{N} \to \mathbb{N}\} \rightrightarrows \mathbb{N}$, where
$BL(f) = \{c \colon f^{-1}(c) \text{ is piecewise syndetic}\}$.

$BL_r : \{f : \mathbb{N} \to r\} \rightrightarrows \mathbb{N}$, where
$BL_r(f) = \{c < r \colon f^{-1}(c) \text{ is piecewise syndetic}\}$.

$BL_r^* : \{f : \mathbb{N} \to r\} \rightrightarrows \mathbb{N}^2$, where
$BL_r^*(f) = \{(c, d) \colon f^{-1}(c) \text{ is } d\text{-piecewise syndetic}\}$.

- $\mathsf{RT}^1_r \leq_{sW} BL_r$.
- $C_{\mathbb{N}} \leq_{sW} BL_2$ (Mummert).

Tks