A Survey on Q-degrees

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Reductions: from m-reduction to Turing reduction

- m-degrees
- truth-table degrees
- weak truth-table degrees
- Turing degrees

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☐ Degree structures

☐ Model-theoretic properties

Post's problem and Post's approach

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Simple sets and hypersimple sets - Post's first two tries

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☐ Effectively simple sets are Turing complete. (Martin, 1966)

From immune sets to hyper immune sets - a direct generalization

- A set is hyperimmune if it is infinite and there is no disjoint strong array intersecting it.
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- Effectively hypersimple sets are all effectively simple, and hence are Turing complete.

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What does "Q-complete" mean here?

Q-reduction and Q-complete sets

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- reflexive and transitive
- ω, forget it.
- If $A \leq_Q B$ and B is a Π_n set, then so is A.
- There exists a least Q-degree, i.e. the degree of Π_1 sets.

- $A \leq_Q K$ if and only if A is Π_2.

- When A, B are both c.e. sets, $A \leq_Q B$ implies $A \leq_T B$.

  So, on c.e. sets, Q-reduction is strictly stronger than Turing reduction.

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Semirecursive sets

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This concept was proposed by Jockusch in 1968.

- Computable sets are semirecursive.
- The complement of a semirecursive set is also semirecursive.
- If a simple set A is also semirecursive, then A is hypersimple.
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Theorem [Marchenkov 1976]

If a c.e. set A is semirecursive and not Q-complete, then A is not T-complete.

Martin's work

So, try to find a semirecursive, hyperhypersimple set, whose existence will provide a solution to Post's problem.

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Theorem [Martin 1976]

No hyperhypersimple set is semirecursive.

Ershov's approach

Ershov's definition

1. An equivalence relation η is positive if η is c.e.
2. A set A is η-closed if it consists of equivalence classes w.r.t. η.
3. The η-closure $[A]_\eta$ is the smallest η-closed set containing A.
4. An η-closed set A is η-finite or η-infinite, if it consists of finite or infinite number of equivalence classes of A.
5. η-simple, η-hypersimple, η-hyperhypersimple sets.
6. η-hyperhypersimple sets are not Q-complete. (Marchenkov 1976)
7. There exists an incomputable, semirecursive, η-hyperhypersimple c.e. set.

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You know it: Friedberg-Muchnik's work.

Downey-LaForte-Nies' work on c.e. Q-degrees

- Density.
- Minimal pairs.
- Nonbranching degrees.
- The largest c.e. Q-degree does not split.
- Above any incomplete c.e. Q-degree, there is a splittable c.e. Q-degree.
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Questions

D.c.e. Q-degrees and isolation in Q-degrees

- Isolated degrees
- Nonisolated degrees
- Pseudo-isolated degrees

Work in progress.

Thanks!