Cone avoid result under certain combinatorial condition

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Introduction

The first goal of this talk is to provide a framework to prove the following kind of result.

Theorem

Given an instance of problem P, namely I, and an instance of problem Q, namely J, if the set of solutions of J is complex enough, then there exists a "non trivial" solution of I that does not "computes" the set of solutions of J.

For example,

Theorem 1 ([7])

Given a set A that is not effectively compressible, and a computable binary tree $\mathcal{T}$, if $[\mathcal{T}]$ does not admit computable strong enumeration, then there exists an infinite subset of A, namely G, such that G does not compute a strong enumeration of $[\mathcal{T}]$.

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Notions

We begin with examples of some notions.

- Problem: (1)RT^n_k; (2)WWKL; (3) SUBSET;
- Instance I: (1)a coloring; (2)a tree defining a closed set of positive measure; (3)a set;
- Solutions of instance I, $\mathscr{I}^I$:(1)homogeneous set of the coloring; (2)a path; (3) a subset of the set;
- Non trivial: (1)infinite; (2) infinite long;(3)infinite;

Intuition

If $\mathscr{I}^I$ forces φ and $\mathscr{I}^J$ forces ψ then $\mathscr{I}^I \mathscr{I}^J$ forces $\psi \wedge \varphi$. (Where $\mathscr{I}^I \mathscr{I}^J$ is short for $\mathscr{I}^I \bigcap \mathscr{I}^J$. And $\mathscr{I}^I$ forces φ means every $Y \in \mathscr{I}^I$ satisfy $\varphi(Y)$.)

Forcing condition

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In this method, it is defined by
some $\rho_1, \cdots, \rho_k \in 2^{<\omega}$,
some closed sets of instance of P, namely $\mathscr{P}_1, \cdots, \mathscr{P}_m$,
some $\mathcal{B} \subseteq \mathcal{P}(\{1, 2, \cdots, m\})$,

$$\bigcup_{I_1 \in \mathscr{P}_1} \bigcup_{I_2 \in \mathscr{P}_2} \cdots \bigcup_{I_m \in \mathscr{P}_m} \tag{1}$$

$$\mathscr{I}^{I_{r_1}} \mathscr{I}^{I_{r_2}} \ldots \mathscr{I}^{I_{r_j}} \mathscr{I}^{\rho_1} + \mathscr{I}^{I_{r_1}} \mathscr{I}^{I_{r_2}} \ldots \mathscr{I}^{I_{r_j}} \mathscr{I}^{\rho_2}$$
$$+ \cdots + \mathscr{I}^{I_{r_1}} \mathscr{I}^{I_{r_2}} \ldots \mathscr{I}^{I_{r_j}} \mathscr{I}^{\rho_k}$$
$$+ \mathscr{I}^{I_{t_1}} \mathscr{I}^{I_{t_2}} \ldots \mathscr{I}^{I_{t_i}} \mathscr{I}^{\rho_1} + \mathscr{I}^{I_{t_1}} \mathscr{I}^{I_{t_2}} \ldots \mathscr{I}^{I_{t_i}} \mathscr{I}^{\rho_2}$$
$$+ \cdots + \mathscr{I}^{I_{t_1}} \mathscr{I}^{I_{t_2}} \ldots \mathscr{I}^{I_{t_i}} \mathscr{I}^{\rho_k}$$

$\ldots$

Where $\{r_1, \cdots, r_j\}, \{t_1, \cdots, t_i\}, \cdots$ are all elements of $\mathcal{B}$.

Forcing condition

Or equivalently,

$$\bigcup_{I_1\in\mathscr{P}_1}\bigcup_{I_2\in\mathscr{P}_2}\cdots\bigcup_{I_m\in\mathscr{P}_m}\sum_{j\leq k}\sum_{B\in\mathcal{B}}\mathscr{I}^{\rho_j}\prod_{i\in B}\mathscr{I}^{I_i} \tag{2}$$

As usual, the forcing condition is a set of candidates.

Forcing condition

We assume that,

Assumption 2

- $\mathscr{I}^I$ is effectively closed in I.
- The mathematical problem as a function from instance to solution set $I \mapsto \mathscr{I}^I$ is continuous.

The purpose is: solutions encoded by a forcing condition is an effectively closed set provided every $\mathscr{P}_i$ is effectively closed.

How to extend the forcing condition

Definition 3 (Type 1 extension)

Type 1 extension simply extends some ρ_i to some $\tau \succ \rho_i$ such that the forcing condition still encode "sufficiently many" solutions while preserving all other components of the forcing condition.

How to extend the forcing condition

Try to force $\Phi_e^G(n)$ to be a wrong description. (For each n a description of $[\mathcal{T}]$ lies within a finite set $\mathcal{V} = \{a_1, \cdots, a_N\}$.)

This can be done if there exists $\tau \succ \rho_i$ which is a solution of the given instance I of P such that the forcing condition after the type 1 extension still encode "sufficiently many" solutions.

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1. For "sufficiently many" instance J including the actual instance I, $\Phi_e(n)^{-1}(W)$ has non empty intersection with

$$\sum_{i \leq k} \mathscr{I}^{\rho_i} \sum_{B \in \mathcal{B}} \prod_{j \in B} \mathscr{I}^{I_j}$$

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2. Similar to item 1 but I is not included.
3. Contrary to item 1, $\Phi_e(n)^{-1}(W)$ does not cover "sufficiently many" instance, sets of answers $V \subseteq \mathcal{W}$ that $\Phi_e(n)^{-1}(V^c)$ does not cover "sufficiently many" instance are not so "diverse".

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4. Similar to item 3 but sets of answers $V \subseteq \mathcal{W}$ that $\Phi_e(n)^{-1}(V^c)$ does not cover "sufficiently many" instance are "diverse".

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- Case (3). The collection of sets of answer that is not disagreed is "concentrated" and include the set of correct answers. We can compute a "description" of $[\mathcal{T}]$, a contradiction.

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- Case (4). In this case we apply type 2 extension defined as following.

How to extend the forcing condition

To force a Π_1^0 requirement ψ, say $\Phi_e^G(n)\uparrow$, consider the sets of answers that is not disagreed, i.e., $V \subseteq \mathcal{V}$ s.t.,

$$[T_V^c] =^{def} \{I : \text{for every solution } Y \text{ in } \mathscr{I}^I \cap c, \Phi_e^Y(n)\uparrow \vee \Phi_e^Y(n) \in V\} \neq \emptyset \quad (3)$$

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Definition 4

A type 2 extension of c induced by $\mathscr{P}_{m+1}, \mathscr{P}_{m+2}, \cdots, \mathscr{P}_{m+n}$, $\mathcal{K} \subseteq \mathcal{P}(\{m+1, \cdots, m+n\})$ is,

$$c \bigcap \bigcup_{I_{m+1}\in\mathscr{P}_{m+1}} \cdots \bigcup_{I_{m+n}\in\mathscr{P}_{m+n}} \sum_{K\in\mathcal{K}} \prod_{j\in K} \mathscr{I}^{I_j} \tag{4}$$

$$= \bigcup_{I_1\in\mathscr{P}_1} \cdots \bigcup_{I_{m+n}\in\mathscr{P}_{m+n}} \sum_{i\leq k} \mathscr{I}^{\rho_i} \sum_{K\in\mathcal{K}, B\in\mathcal{B}} \prod_{j\in K\cup B} \mathscr{I}^{I_j}$$

How to extend the forcing condition

Note that,

- if a collection of set of answers $\{V_i\}, i \in K$ has empty intersection, then $\mathscr{I}^{\rho_i} \prod_{j \in K \cup B} \mathscr{I}^{l_j}$ forces $\Phi_e^G(n) \uparrow$ (for any $B \in \mathcal{B}$).
- $[T_V^c]$ is effectively closed.

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The key point is,

Lemma 5

If the collection of the set of answers that is not disagreed, $V_1, \cdots, V_w$, are not so "diverse", and $[T_{V_m}^c]$ contain "sufficiently many " instances, let $\mathcal{K} = \{K \subseteq \{1, \cdots, w\} : \bigcap_{j \in K} V_j = \emptyset\}$, then type 2 extension of c induced by $\mathcal{K}$, $[T_{V_j}^c], j \le w$ still contains "sufficiently many" solutions.

What role does combinatorics play

Assumption 6

For any forcing condition c encoding "sufficiently many" solutions, let E be a set of initial segment of solutions, if whenever for some instance J $\mathscr{I}^E$ has non empty intersection with

$$\mathscr{I}^J \sum_{i \le k} \mathscr{I}^{\rho_i} \sum_{B \in \mathcal{B}} \prod_{j \in B} \mathscr{I}^{I_j}$$

for "sufficiently many" $I_i \in \mathscr{P}_i, i \le k$, then there exists $\gamma \in E$, $i \le k$ such that $\mathscr{I}^\gamma \cap \mathscr{I}^J \neq \emptyset$ and type 1 extension $\gamma \succ \rho_i$ of the forcing condition still encode "sufficiently many" solutions.

In another words, it is "easy" to apply type 1 extension without destroying the "sufficiently many" property.

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Some interesting point is,

- assumption 6 is purely combinatorial;
- to deal with problem P, it is not necessary to restrict on the coding of solutions given by $\mathscr{I}_P$;
- pre-choose a solution from the forcing condition if you are dealing with some problem (property) that WKL_0 does not imply (preserve).

A digression

What about results like,

Theorem 8

There exists instance I_Q of Q *such that for any instance of* P*, I_P, there exists "non trivial" solution G of I_P such that G does not compute any non trivial solution of I_Q.*

Theorem 9

for any instance of P*, I, there exists "non trivial" solution G that is generalized low;*

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Theorem 9

for any instance of P*, I, there exists "non trivial" solution G that is generalized low;*

Usually, it matters that whether $\Phi_e^\tau(n)$ halt but the outcome does not. So a deliberate Type 2 extension is not needed, but assumption 6 is still required.

Further discussion

The result suggest to characterize the power of a problem in terms of describing path through trees.

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An attempt looks like,

Definition 10

There exists a uniformly I-enumerable tree $T^I \subseteq \omega^{<\omega}$, such that

- for all $\rho \in \omega^{<\omega}$, $|\{\tau \in T^I : \tau \succ \rho\}| = \infty$ implies there exists some path extending τ;
- any path of $[T^I]$ computes a non trivial solution of I;
- any nontrivial solution of I computes a certain description of $[T^I]$.

A description of $[T]$ is simply a sequence of clopen set.

The point is study the combinatorics restriction of admissible description.

Further discussion

Question 11

Does there exists an instance of RT_3^1, I_3^1 such that for any instance of RT_2^1, I_2^1 and any solution of I_2^1, namely G, G does not compute a non trivial solution of I_3^1?

Question 12

Does there exists a 1-random X such that for any instance of RT_2^1, I_2^1 and any solution of I_2^1, namely G, G does not derandomize X?

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Definition 13

Let D_n be the canonical representation of finite set of $2^{<\omega}$.
An enumeration of $T \subseteq 2^{<\omega}$ is a $h : \omega \to \omega$ such that $(\forall n) D_{h(n)} \cap T \neq \emptyset$. Moreover, h is

- k-enumeration iff $(\forall n)|D_{h(n)}| \leq k$;
- non-trivial iff $(\forall n \forall \rho \in D_{h(n)})\ |\rho| = n$;
- strong iff it is a k-enumeration for some $k \in \mathbb{N}$;

Fix the SUBSET problem $\mathscr{I} : I \rightarrow \mathscr{I}^{I} = \{Y : Y \subseteq I\}$

Definition 14

- An effectively closed set of instance is said to contain *sufficiently many* instances if it contains both J, J^c for some instance J.
- A forcing condition c is said to contain sufficiently many solutions iff there exists $I_i \in \mathscr{P}_i, i \leq k$ with $(\forall i, j \leq k)\rho_i \subseteq I_j$ and

$$\bigcup_{B\in\mathcal{B}} (\bigcap_{j\in B} I_j) = \omega$$

Appendix

Requirements are,
$P_e : |G \cap A| > e$;
$R_e : \Phi_e^{G \cap A}$ is not a non-trivial strong $e-$enumeration of $[\mathcal{T}]$, i.e. one of the following holds:

1. $\Phi_e^{G \cap A}$ is not total;
2. $(\exists n)[\Phi_e^{G \cap A}(n)] \cap [\mathcal{T}] = \emptyset$;
3. $(\exists n)|\Phi_e^{G \cap A}(n)| > e$;
4. $(\exists n)\ \rho \in \Phi_e^{G \cap A}(n), |\rho| \neq n$

Appendix

Definition 15 (Diverse)

For a collection of sets $\mathcal{V} = \{V_1, \ldots, V_w\}$, $\mathcal{V}$ is K-disperse iff for all K-partitions of $\{V_1, \cdots, V_w\}$, $P_1 \cup P_2 \cup \cdots \cup P_K = \{V_1, \cdots, V_w\}$, there exists $k \le K$ such that $\bigcap_{V_i \in P_k} V_i = \emptyset$.

End

Thank you