Ramsey Properties of Partial Orderings

Challenges in
Reverse Mathematatcis

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results include joint work
with Jared Corduan
and with Jared Corduan and Joseph Mileti

For this talk, I'm interested in the questions.

CHM: Chubb, Hirst, and McNicholl, *Journal of Symbolic Logic*.

CGM: Corduan, Groszek, and Mileti, *Journal of Symbolic Logic*.

G: Groszek, *Electronic Journal of Combinatorics*.

CG: Corduan and Groszek, *Notre Dame Journal of Formal Logic*, to appear.

Structural Ramsey Theory: Color copies of a structure $\mathbf{B}$ embedded in a structure $\mathcal{A}$. Look for homogeneous (monochromatic) substructures of $\mathcal{A}$.

Definition: If $\mathcal{A}$ is a countable infinite structure, and $\mathbf{B}$ is a finite (potential) substructure, then $\mathcal{A}$ is $\mathbf{B}$-Ramsey if:

> For every coloring of the copies of $\mathbf{B}$ in $\mathcal{A}$ with finitely many colors, there is a monochromatic substructure of $\mathcal{A}$ that is isomorphic to $\mathcal{A}$.

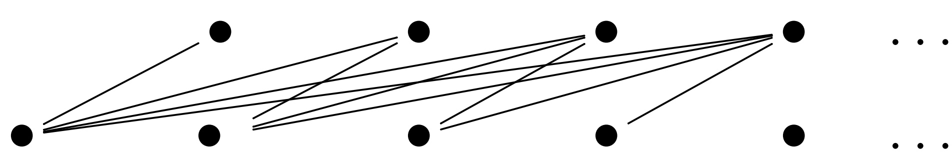
Definition: Structures $\mathcal{A}$ and $\mathcal{C}$ are *biembeddable* if each contains a copy of the other.

Observation: Biembeddable structures have the same Ramsey properties.

Example: If ω (ordered as usual) is viewed as a linear ordering, and $n = \{0, 1, \ldots, n-1\}$ as an n-element linear ordering, then ω is n-Ramsey.

Example: The complete countably infinite graph is K_n-Ramsey for every n.

Example: Let $\mathcal{A}$ be the bipartite graph with bottom level $\{a_0, a_1, a_2, a_3 \ldots\}$, with top level $\{b_1, b_2, b_3 \ldots\}$, and with an edge between a_m and b_n iff $m < n$. Let $\mathbf{B}$ denote the graph consisting of two vertices connected by a single edge. Then $\mathcal{A}$ is $\mathbf{B}$-Ramsey ($\mathcal{A}$ is edge-Ramsey). This example will come up again.



These are all consequences of the usual infinitary Ramsey's theorem (in the last case, for colorings of pairs).

Definition: In the context of partial orderings, n denotes an n-element linear ordering.

Definition: A partial ordering $\mathbb{P}$ is chain-Ramsey if it is n-Ramsey for every n.

Definition: $2^{<\omega}$ denotes the complete binary tree.

Fact: $2^{<\omega}$ is chain-Ramsey.

Chubb, Hirst, and McNicholl (CHM) investigated the reverse mathematics of this.

Definition (CHM): TT^n is the statement that $2^{<\omega}$ is n-Ramsey.

Results of CHM (over RCA_0):

For standard $n \geq 3$
$ACA_0 \Longleftrightarrow (TT^n)$ $[\Longleftrightarrow (RT^n)]$

$ACA_0 \Longrightarrow (TT^2) \Longrightarrow (RT^2)$

$I\mathbf{\Sigma}^0_2 \Longrightarrow (TT^1) \Longrightarrow B\mathbf{\Sigma}^0_2$ $[\Longleftrightarrow (RT^1)]$

Questions of CHM (over RCA_0):

What is the precise strength of TT^1? Is it equivalent to $B\Sigma^0_2$? To $I\Sigma^0_2$?

[Partial answer: It is strictly stronger than $B\Sigma^0_2$.]

What is the precise strength of TT^2? Is it equivalent to RT^2? To ACA_0?

What about other partial orderings?

[Partial answer ahead.]

Is there a partial ordering $\mathbb{P}$ such that "$\mathbb{P}$ is 2-Ramsey" is equivalent to ACA_0?

[Answer: Yes.]

Ramsey properties of trees (CGM)

Definition: By a tree we mean a countably infinite rooted tree.

Results over RCA_0:

A tree is 1-Ramsey only if it has height ω and either is a single branch or contains a copy of $2^{<\omega}$. Thus, up to biembeddability, there are at most two chain-Ramsey trees: a single branch, and the complete binary tree.

$B\boldsymbol{\Sigma}^0_2 \not\Longrightarrow (TT^1)$. Thus the pigeonhole principle for the complete binary tree is stronger than the usual pigeonhole principle.

(Precise result: If a theory $\mathcal{T}$ extends $RCA_0 + B\boldsymbol{\Sigma}^0_2$ by the addition of Π^1_1 axioms, then

$$(\mathcal{T} \vdash TT^1) \iff (\mathcal{T} \vdash I\boldsymbol{\Sigma}^0_2).$$

In particular, $RCA_0 + B\boldsymbol{\Sigma}^0_2 \not\Longrightarrow (TT^1)$.)

What about other partial orderings?

Scheme: Prove a general theorem about chain-Ramsey partial orderings. Then investigate its reverse mathematical properties.

Fact: If $\mathbb{P}$ is a countable partial ordering with at least one three-element chain, and $\mathbb{P}$ is 2-Ramsey, then either $\mathbb{P}$ or $\mathbb{P}^*$ (upside-down $\mathbb{P}$) is well-founded of height ω.

Observation: $\mathbb{P}$ and $\mathbb{P}^*$ have the same n-Ramsey properties.

Convention: From now on, partial ordering will mean a countably infinite partial ordering of height ω with least element.

Definition: A partial ordering $\mathbb{P}$ is densely self-embeddable if for every element $p \in \mathbb{P}$, there is an embedding of $\mathbb{P}$ into itself above $\mathbb{P}$.

Remark: For the partial orderings we will consider, $\mathbb{P}$ being biembeddable with a densely self-embeddable partial ordering is analogous to a tree containing a copy of $2^{<\omega}$.

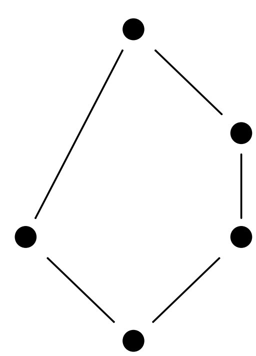
Observation: If $\mathbb{P}$ (a partial ordering with least element) is 1-Ramsey, then $\mathbb{P}$ is biembeddable with a densely self-embeddable partial ordering.

Remark: The proof colors elements $p \in \mathbb{P}$ according to whether $\mathbb{P}$ can be embedded in itself above p.

Definition: The n^{th} level of $\mathbb{P}$ consists of all elements of $\mathbb{P}$ of height n in the partial ordering.

$\mathbb{P}$ is finite-level if every level of $\mathbb{P}$ is finite.

Proposition (G): If $\mathbb{P}$ is finite-level and n-Ramsey for $n = 1, 2, 3$, then $\mathbb{P}$ omits N_5 (pictured) as a substructure.



Consequences:

Incomparable nodes with a common successor must have the same predecessors.

The ordering on $\mathbb{P}$ is the transitive closure of the ordering between adjacent levels.

Definition: A countably infinite partial ordering is weakly proto-Ramsey if it has a least element, is well-founded of height ω, and omits N_5.

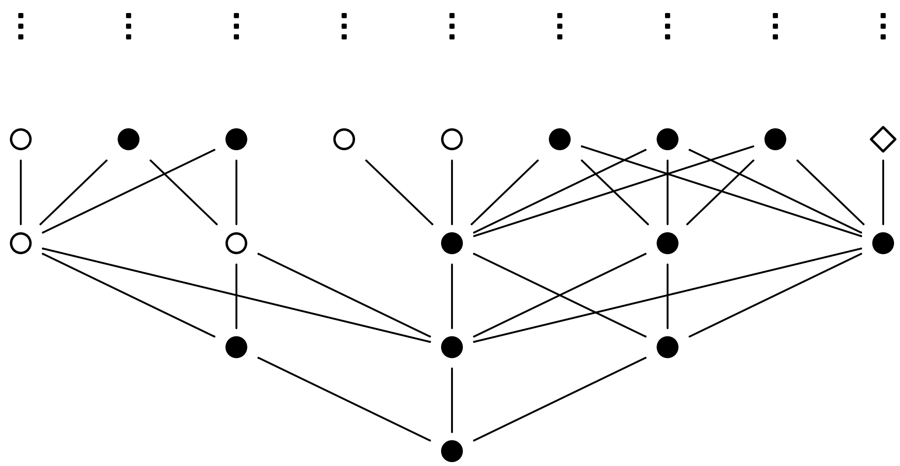
It is proto-Ramsey if it is also densely self-embeddable.

Remark: We do not assume $\mathbb{P}$ is finite-level in these definitions.

For purposes of proving a general theorem about finite-level chain-Ramsey partial orderings (with least element) in the real world (ZFC), we can restrict our attention to (biembeddability classes of) proto-Ramsey partial orderings.

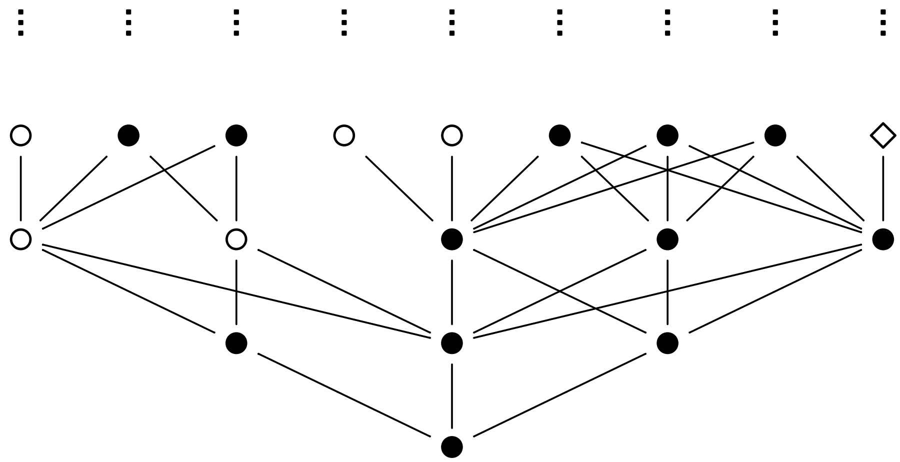
The distinction between proto-Ramsey and weakly proto-Ramsey is significant when it comes to reverse mathematics.

A (weakly) proto-Ramsey partial ordering:

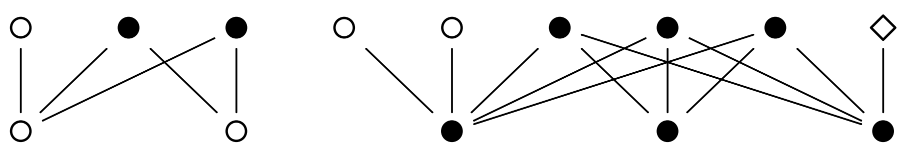


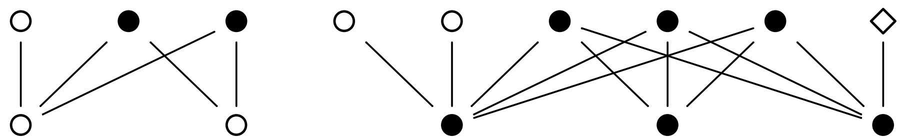
Note that on level 2, the $\circ$ nodes and the $\bullet$ nodes have different predecessors, so cannot have any common successor. This introduces branching.

From the structure of $\mathbb{P}$ we obtain a collection of bipartite graphs as follows:

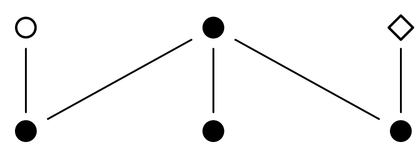


Consider each pair of adjacent levels (for examples, levels 2 and 3 above) as a bipartite graph with edges between comparable points:





Take each connected component (for example, the right-hand one) and identify points on the top level that have the same predecessors:



The collection of bipartite graphs (with distinguished top and bottom parts) obtained is $\mathcal{G}(\mathbb{P})$.

If $\mathbb{P}$ is finite-level, then all elements of $\mathcal{G}(\mathbb{P})$ are finite. Furthermore, if $\mathbb{P}$ is proto-Ramsey and all elements of $\mathcal{G}(\mathbb{P})$ are finite, then $\mathbb{P}$ is biembeddable with a finite-level proto-Ramsey partial ordering.

Theorem (G): Let $\mathbb{P}$ be a proto-Ramsey partial ordering. Then $\mathbb{P}$ is chain-Ramsey if and only of $\mathcal{G}(\mathbb{P})$ has the following two properties:

Joint embedding property: For any two elements G and H, there is an element K in which both G and H can be embedded.

Edge-Ramsey (as a collection of structures): For any element G and number m, there is an element H such that if the edges of H are colored in m colors, then H contains a monochromatic copy of G.

Corollary (G): Let $\mathbb{P}$ be a weakly proto-Ramsey partial ordering. Then $\mathbb{P}$ is chain-Ramsey if and only if $\mathbb{P}$ is biembeddable with a proto-Ramsey partial ordering, and $\mathcal{G}(\mathbb{P})$ has the above two properties.

The reverse mathematics (CG):

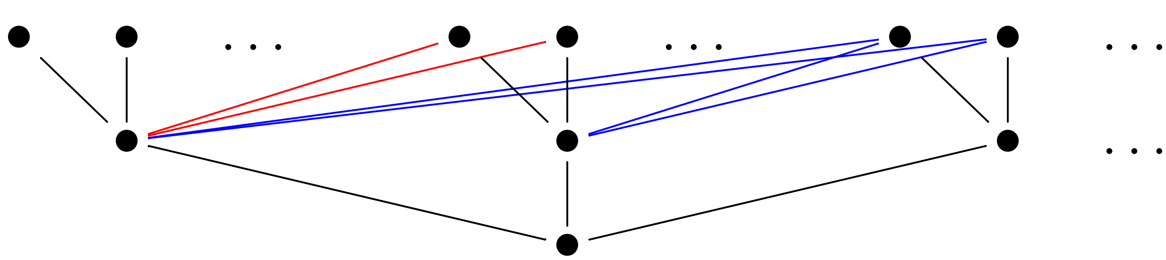
Essentially (modulo choosing the right way to phrase things), for a fixed finite number $n \geq 3$ of colors, the theorem for the finite-level case is equivalent to ACA_0 over RCA_0, and the corollary for the finite-level case is provable in ATR_0.

Question: For trees, the essential part of the corollary is actually provable in RCA_0. How much strength do we actually need for the general finite-level case?

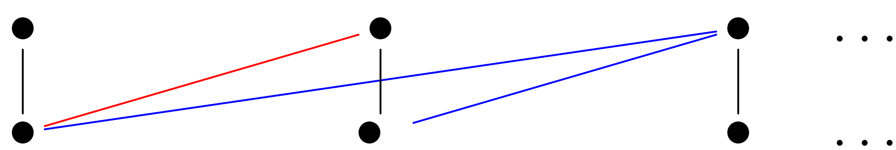
Question: What about the reverse mathematics of the infinite-level case?

Example: Let $\mathbb{P}$ be the partial ordering whose elements are finite sequences of natural numbers, with the following ordering:
$\langle i_0, i_1, \ldots, i_m \rangle \leq \langle j_0, j_1, \ldots, j_n \rangle$ if and only if
(1.) $m < n$,
(2.) $i_k = j_k$ for $0 \leq k < m$, and
(3.) $i_m \leq j_m$.

Picture $\mathbb{P}$ by beginning with the complete ω-branching tree $\omega^{<\omega}$ and putting in extra connections between adjacent levels by connecting $\sigma^\frown m$ to all immediate successors of $\sigma^\frown n$ whenever $m < n$.



$\mathcal{G}(\mathbb{P})$ contains two graphs, a single edge, and the edge-Ramsey infinite bipartite graph described at the beginning of the talk.



Theorem (CG): "$\mathbb{P}$ is 2-Ramsey" is equivalent over RCA_0 to ACA_0.

Question: Are there different examples of partial orderings whose n-Ramsey properties are unexpectedly strong?

References:

Chubb, J., J. Hirst and T. McNicholl, Reverse mathematics, computability, and partitions of trees. *Journal of Symbolic Logic*, Volume 74, Issue 1 (2009), 201-215.

Corduan, J., M. Groszek and J. Mileti, Reverse mathematics and Ramsey's property for trees. *Journal of Symbolic Logic*, Volume 75, Issue 3 (2010), 945-954.

Corduan, J. and M. Groszek, Reverse mathematics and Ramsey properties of partial orderings. *Notre Dame Journal of Formal Logic*, to appear.

Groszek, M., Ramsey properties of countably infinite partial orderings, *Electronic Journal of Combinatorics*, Volume 20, Issue 1 (2013).