Higher reverse mathematics

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Since then, substantial work has been done in the system RCA_0^ω:

- Uniform versions of classical principles (Kohlenbach, Sakamoto/Yamazaki, Sanders)
- Topology and measure theory (Hunter, Kreuzer)
- Ultrafilters (Kreuzer, Towsner*)
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Today: interactions between higher reverse math and set theory (S., Hachtman)

Higher reverse math: an informal example

Theorem (Grilliot's Trick)

The following are "effectively equivalent":

- *The jump functional* $x \mapsto x'$ *exists.*
- *"Uniform Weak Konig's Lemma": There is a functional* F *such that, if* T *is an infinite binary tree, then* $F(T)$ *is a path through* T*.*

"Proof".

Let

- $T_n^0 = \{\sigma \in 2^\omega : (\forall i(\sigma(i) = 0)) \vee (|\sigma| < n \wedge \forall i(\sigma(i) = 1))\}$
- $T_n^1 = \{\sigma \in 2^\omega : (\forall i(\sigma(i) = 0)) \vee (|\sigma| < n \wedge \forall i(\sigma(i) = 1))\}$
- $T_\infty = \{\sigma \in 2^\omega : \forall i, j(\sigma(i) = \sigma(j))\}$.

Then either $F(T_\infty)$ goes left (zero) or right (one).

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Then either $F(T_\infty)$ goes left (zero) or right (one). Suppose $F(T_\infty)$ goes left. Then, given real x and natural e, let $T_{x,e}$ consist of "all ones" branch + every all-zeroes node of length s such that $\varphi_e^x(e)[s]\uparrow$.

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Base theories

Kohlenbach: introduced RCA_0^ω, a conservative extension of RCA_0 for all finite types. Different appearance from RCA_0.
S.: base theory RCA_0^3 for types $0, 1, 2$; similar form to RCA_0.

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RCA_0^ω is a conservative extension of RCA_0^3.

- Language: arithmetic, application symbols "$F(x)$", and **coding**: $^\frown, *$
- Ordered semiring axioms $+$ Σ^0_1 induction
- Δ^0_1-comprehension for reals and functionals (with arbitrary parameters) in the language
- Coding operations defined as:
  - $n^\frown(a_0, a_1, a_2, ...) = (n, a_0, a_1, ...)$,
  - $F * r = (F(0^\frown r), F(1^\frown r), F(2^\frown r), ...)$

Δ^0_1 comprehension

RCA^3_0: ordered semiring axioms, Σ^0_1 induction, extensionality, and versions of Δ^0_1-comprehension for reals and functionals in the language of third-order arithmetic + "coding operations"

- "Σ^0_1" has usual meaning: existential quantifier over naturals, matrix has bounded quantifiers over naturals only (and equality for naturals only)
- A Δ^0_1-definition of a *real* is a Σ^0_1 formula $\varphi(x^{\mathbb{N}}, y^{\mathbb{N}})$ such that

$$\forall x \exists ! y \varphi(x, y).$$

- A Δ^0_1-definition of a *functional* is a Σ^0_1 formula $\varphi(x^{\mathbb{R}}, y^{\mathbb{N}})$ such that

$$\forall x \exists ! y \varphi(x, y).$$

Note: arbitrary type *parameters* are allowed in Σ^0_1 formulas.

Models of RCA_0^3

A model of RCA_0^3 has form

$$(Nat, Rea, Fun; +, \times, 0, 1, ^\frown, *, app)$$

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Example

If $X \subseteq \omega^\omega$ is a Turing ideal, there is a smallest model of RCA_0^3 with second-order part X:

$$(\omega, X, \{s \mapsto \Phi_e^{t \oplus s} : t \in X, \Phi_e^{t \oplus -} \text{ total on } X\})$$

Example

Other models: $(\omega, \mathbb{R}, \text{continuous functions})$ and $(\omega, \mathbb{R}, \text{Borel functions})$

Reverse math of determinacy

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Theorem (Steel)

Over RCA_0, the following are equivalent:

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Question

Is this the only reason?

Determinacy on reals

Since ω-sequences of reals can be coded by reals, "$T \subseteq (\omega^\omega)^{<\omega}$ is well-founded" is Π^1_1.

Definition

- Open determinacy for reals ($\Sigma^{\mathbb{R}}_1$-Det): "Any open game of length ω on $\mathbb{R}$ is determined." (Game tree $\subseteq \mathbb{R}^{<\omega}$, I wins iff play leaves tree.)
- Clopen determinacy for reals ($\Delta^{\mathbb{R}}_1$-Det): "Any clopen game of length ω on $\mathbb{R}$ is determined." (Game tree *wellfounded* $\subseteq \mathbb{R}^{<\omega}$, first to leave tree loses.)

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Theorem (S.)

Over RCA^3_0, $\Delta^{\mathbb{R}}_1$-Det is strictly weaker than $\Sigma^{\mathbb{R}}_1$-Det.

Uses nontrivial countably closed higher-type forcings — counterpart of complexity of "clopenness" at second-order

Shortly afterwards: Hachtman, via analysis of Goedel's L (see later)

The game

Let α be an ordinal.

- C_α is clopen game "walk down α": Players I and II (independently) build decreasing sequences in α; first who cannot play, loses.
- O_α is open game "play C_α until I wins": Players I and II play ω-many games of C_α (in sequence). II wins iff she wins every game.

We let $\mathbb{T} \subseteq (\mathfrak{c}^+)^{<\omega}$ be game tree for $O_{\mathfrak{c}^+}$.

Actually, right game is slight variation on this.

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- Properties:
  - RCA_0^3, "$\mathcal{G}$ is undetermined" are easy
  - Clopen games of rank $< \mathfrak{c}^+$ determined via ranking argument
  - Countable closure: no clopen games of rank $\geq \mathfrak{c}^+$ in M

Subsequent analysis

Shortly afterwards, Sherwood Hachtman drew a connection with his work on the constructible universe L:

Definition (Hachtman)

θ is the least ordinal such that L_θ satisfies: "$\mathcal{P}(\omega)$ exists and for every height-ω tree T with no path, there is $\rho : T \to ON$ such that $x \supsetneq y \implies \rho(x) < \rho(y)$."

Theorem (Hachtman)

$(\omega, \omega^\omega \cap L_\theta, \omega^{\omega^\omega} \cap L_\theta)$ separates clopen and open determinacy for reals.

That is, Hachtman found a set-theoretic *canonical model* of the separation. This θ is also connected to Σ^0_4 determinacy on naturals, and reflection principles.

Question

Are there other canonical models? (Hyperanaytic functionals?)

Transfinite recursion principles, I/II: Choiceless versions

Definition

- *TR*: Σ^1_1 recursion along a well-ordering with domain $\subseteq \mathbb{R}$.
- *RR*: Σ^1_1 recursion along a well-founded tree with domain $\subseteq \mathbb{R}$.

Definition

- *WO*: The reals are well-orderable. (Role: Kleene-Brouwer ordering of tree)
- *SF*: Real-indexed families of nonempty sets of reals have choice functionals. (Role: quasistrategy$\rightarrow$ strategy)

Proposition (S.)

Over RCA^3_0, we have:

- *$RR + SF$ is equivalent to clopen determinacy for reals.*
- *$TR + WO + SF$ implies clopen determinacy for reals.*

Σ^2_1-Separation

Definition

Σ^2_1-Sep is the statement: "Given $\varphi, \psi \in \Sigma^2_1$, if at most one holds for each real r, then have separating functional."

Proposition (S.)

Over $RCA^3_0 + SF$, Σ^2_1-Sep implies clopen determinacy for reals.

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Proof sketch.

Suppose G is a clopen game. For each node $\sigma \in G$, at most one of the following hold:

- There is a witness to σ being a win for player I.
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Applying Σ^2_1-Sep yields a winning quasistrategy

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Applying Σ^2_1-Sep yields a winning quasistrategy. . . after analysis. □

What is the relationship between Σ^2_1-Sep and open determinacy for reals?

Comparing choice principles, I/II

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Two natural choice principles:

- SF = "Every family S_r ($r \in \mathbb{R}$) of nonempty sets of reals has a choice function"
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Proposition

SF does not imply WO over RCA_0^3.

Proofs.

- In $ZF + AD_{\mathbb{R}}$, projective functionals give separating model
- Over ZF, Truss 1978 provided a forcing argument
- Over RCA_0^3, set of continuous functionals is model of $SF + \neg WO$

$\square$

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What about other direction? Note that choice functions are *definable* from a well-ordering . . .

Comparing choice principles, II/II

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Two natural choice principles:

- *SF* = "Every family S_r ($r \in \mathbb{R}$) of nonempty sets of reals has a choice function"
- *WO* = "The reals are well-orderable"

Theorem (S.)

Over RCA_0^3, WO does not imply SF.

Proof sketch.

- Force with countable partial injections $\mathbb{R} \to \omega_1$; call generic induced ordering "$\prec_G$."
- Take functionals which are definable from $\prec_G$ via truth tables of "countable depth"
- Let $S_r = \{s : r \prec_G s\}$. This family has no choice function.

□

Thanks!