A Tutorial on Weihrauch Complexity

Vasco Brattka

Universität der Bundeswehr München, Germany

University of Cape Town, South Africa

New Challenges in Reverse Mathematics

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Outline

Some History on Weihrauch Reducibility

- 1992 Klaus Weihrauch introduced the concept of his reducibility for single-valued functions $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and for sets of such functions (in two unpublished technical reports).
- 1989-2007 he supervised 6 MSc/PhD theses on this topic, mostly unpublished (von Stein, Mylatz, B., Hertling, Pauly).
- The reducibility was also considered for single-valued functions $f :\subseteq X \to Y$ on other topological/represented spaces.
- 2008 Guido Gherardi and Alberto Marcone noticed that this reducibility for multi-valued functions can be used to classify the computational content of Π_2 theorems.
- 2009 Akitoshi Kawamura (and Stephen Cook) rediscovered a polynomial-time version of Weihrauch reducibility and used it for the study of uniform computational time complexity.
- 2012 Dorais, Dzhafarov, Hirst, Mileti, Shafer rediscovered Weihrauch reducibility directly for the special case of Π^1_2 statements (work extended by Hirschfeldt and Jockusch).

Bibliography http://cca-net.de/publications/weibib.php

Bibliography on Weihrauch Complexity

Bibliography styles: alphabetically, chronologically, by publication type. BibTeX file: wei.bib

Chronologically

2016

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Currently there are 89 entries in this bibliography. Please help to update it!

A Calculus of Mathematical Problems

Mathematical Problems and Solutions

Definition

A mathematical problem is a partial multi-valued $f :\subseteq X \rightrightarrows Y$.

- There are a certain sets of potential inputs X and outputs Y.
- $D = \mathrm{dom}(f)$ contains the valid instances of the problem.
- $f(x)$ is the set of solutions of the problem f for instance x.

Definition

$g :\subseteq X \rightrightarrows Y$ solves $f :\subseteq X \rightrightarrows Y$, if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g(x) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$. We write $g \sqsubseteq f$ in this situation.

Definition

For $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Y \rightrightarrows Z$ we define the composition $g \circ f :\subseteq X \rightrightarrows Z$ by

$$(g \circ f)(x) := \{z \in Z : (\exists y \in Y)\ y \in f(x) \text{ and } z \in g(y)\}$$

and $\mathrm{dom}(g \circ f) := \{x \in X : f(x) \subseteq \mathrm{dom}(g)\}$.

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Examples of Mathematical Problems

- The Zero Problem $Z_X :\subseteq \mathcal{C}(X) \rightrightarrows X, h \mapsto h^{-1}\{0\}$.
- The Limit Problem is the mathematical problem

$$\lim :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, \langle p_0, p_1, \ldots\rangle \mapsto \lim_{i\to\infty} p_i$$

with $\mathrm{dom}(\lim) := \{\langle p_0, p_1, \ldots\rangle : (p_i)_i \text{ is convergent}\}$.

- Martin-Löf Randomness is the mathematical problem $\mathrm{MLR} : 2^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ with

$$\mathrm{MLR}(x) := \{y \in 2^\mathbb{N} : y \text{ is Martin-Löf random relative to } x\}.$$

- The Cohesiveness Problem is the mathematical problem $\mathrm{COH} : (2^\mathbb{N})^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ where $\mathrm{COH}(R_i)$ contains all infinite $X \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$ one of the sets

$$X \cap R_i \text{ or } X \cap (\mathbb{N} \setminus R_i)$$

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Theorems as Problems

Definition

Any theorem T of the Π_2 form

$$(\forall x \in X)(x \in D \Longrightarrow (\exists y \in Y)\ P(x,y))$$

is identified with $F :\subseteq X \rightrightarrows Y$ with $\mathrm{dom}(F) := D$ and

$$F(x) := \{y \in Y : P(x,y)\}.$$

Examples: Weak Weak Kőnig's Lemma is the mathematical problem

$$\mathsf{WWKL} :\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$$

with $\mathrm{dom}(\mathsf{WWKL}) := \{T \in \mathrm{Tr} : \mu([T]) > 0\}$.

The Intermediate Value Theorem is the mathematical problem

$$\mathsf{IVT} :\subseteq \mathcal{C}[0,1] \rightrightarrows \mathbb{R}, f \mapsto f^{-1}\{0\}$$

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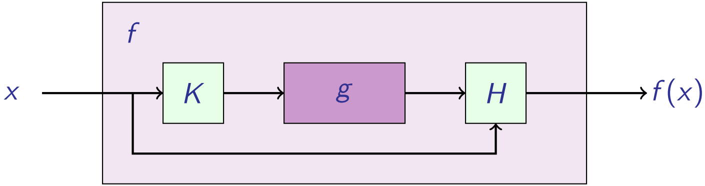
Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be two mathematical problems.



- f is Weihrauch reducible to g, $f \leq_{\mathrm{W}} g$, if there are computable $H :\subseteq X \times W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $H(\mathrm{id}_X, gK) \sqsubseteq f$.
- f is strongly Weihrauch reducible to g, $f \leq_{\mathrm{sW}} g$, if there are computable $H :\subseteq W \rightrightarrows Y$, $K :\subseteq X \rightrightarrows Z$ such that $HgK \sqsubseteq f$.
- Equivalences $f \equiv_{\mathrm{W}} g$ and $f \equiv_{\mathrm{sW}} g$ are defined as usual.

Theorem (Tavana and Weihrauch 2011)

$f \leq_{\mathrm{W}} g \iff$ *there is a Turing machine that computes f and uses g as an oracle exactly once during its infinite computation.*

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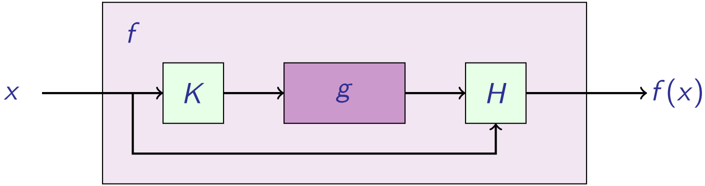

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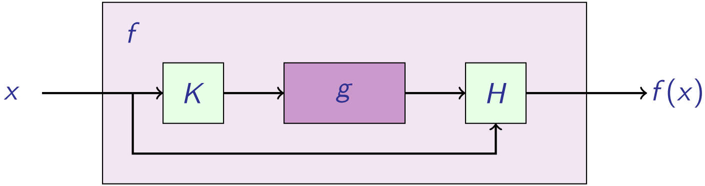

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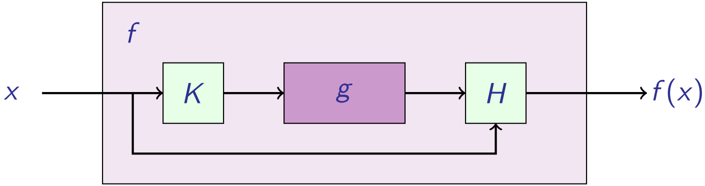

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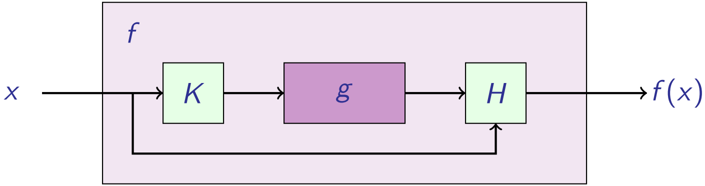

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Realizers and Representations

- A representation of X is a surjective map $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$.
- $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer of $f :\subseteq X \rightrightarrows Y$, in symbols $F \vdash f$, if $\delta_Y F(p) \in f\delta_X(p)$ for all $p \in \mathrm{dom}(f\delta_X)$.

$$\begin{tikzcd}
\mathbb{N}^\mathbb{N} \arrow[r, "F"] \arrow[d, "\delta_X"'] & \mathbb{N}^\mathbb{N} \arrow[d, "\delta_Y"] \\
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- f is continuous, computable, polynomial-time computable or Borel measurable, if it admits a corresponding realizer F.
- $f \leq_\mathrm{W} g \iff$ there are computable $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $H\langle \mathrm{id}, GK\rangle \vdash f$ whenever $G \vdash g$.

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Computable Metric Spaces and Cauchy Representations

Definition

(X, d, α) is called computable metric space if

1. $d : X \times X \to \mathbb{R}$ is a metric on X,
2. $\alpha : \mathbb{N} \to X$ is a sequence with a dense range,
3. $d \circ (\alpha \times \alpha) : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is computable.

Definition

$\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is called Cauchy representation, if

$$\delta_X(p) = x :\iff (\forall k)\ d(\alpha p(k), x) < 2^{-k}.$$

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Realizer Version of Problems

Definition

Let (X, δ_X) and (Y, δ_Y) be represented spaces and $f :\subseteq X \rightrightarrows Y$ a mathematical problem. Then we define the realizer version $f^{\mathrm{r}} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ of f by $f^{\mathrm{r}} := \delta_Y^{-1} \circ f \circ \delta_X$.

Proposition

$f \equiv_{\mathrm{sW}} f^{\mathrm{r}}$.

- This means that properties of $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ can be studied by considering only problems of type $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$.
- Arbitrary represented spaces X, Y are used as types in order to classify practical problems and theorems, which are most naturally expressed in such types.

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Cylinders and Strong Weihrauch Reducibility

By $\mathrm{id}:\mathbb{N}^\mathbb{N}\to\mathbb{N}^\mathbb{N}$ we denote the identity of Baire space $\mathbb{N}^\mathbb{N}$. We always have $f\le_{\mathrm{sW}}\mathrm{id}\times f$, the inverse is not necessarily true.

Definition

$f:\subseteq X\rightrightarrows Y$ is called a cylinder if $\mathrm{id}\times f\equiv_{\mathrm{sW}} f$ and $\mathrm{id}\times f$ is called the cylindrification of f.

Examples: $\mathsf{lim}, \mathsf{WKL}$ are cylinders, $\mathsf{WWKL}, \mathsf{COH}, \mathsf{MLR}$ are not.

Proposition (B. and Gherardi 2011)

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Remark: The relation between strong and ordinary Weihrauch reducibility has formal similarities to the relation between one-one and many-one reducibility.

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Algebraic Operations in the Weihrauch Lattice

Definition

Let f, g be two mathematical problems. We consider:

- $f \times g$: both problems are available in parallel (Product)
- $f \sqcup g$: both problems are available, but for each instance one has to choose which one is used (Coproduct)
- $f \sqcap g$: given an instance of f and g, only one of the solutions will be provided (Sum)
- $f * g$: f and g can be used consecutively (Comp. Product)
- $g \rightarrow f$: this is the simplest problem h such that f can be reduced to $g * h$ (Implication)
- f^*: f can be used any given finite number of times in parallel (Star)
- $\widehat{f}$: f can be used countably many times in parallel (Parallelization)
- f': f can be used on the limit of the input (Jump)

Definitions of Algebraic Operations

Definition

For $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ we define:

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z$, $(x, w) \mapsto f(x) \times g(w)$ (Product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z$, $z \mapsto \begin{cases} f(z) \text{ if } z \in X \\ g(z) \text{ if } z \in W \end{cases}$ (Coproduct)
- $f \sqcap g :\subseteq X \times W \rightrightarrows Y \sqcup Z$, $(x, w) \mapsto f(x) \sqcup g(w)$ (Sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*$, $f^* = \bigsqcup_{i=0}^{\infty} f^i$ (Star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$, $\widehat{f} = \mathsf{X}_{i=0}^{\infty} f$ (Parallelization)

Here

- $Y \times Z$ denotes the usual Caresian product,
- $Y \sqcup Z := (\{0\} \times Y) \cup (\{1\} \times Z)$ denotes the disjoint union,
- $X^* := \{f : \mathbb{N} \to X : \mathrm{dom}(f) = n$ for some $n \in \mathbb{N}\}$ denotes the set of words over X, where $n = \{0, ..., n-1\}$,
- $X^{\mathbb{N}} := \{f : \mathbb{N} \to X\}$ denotes the set of sequences over X.

The Algebraic Structure of the Weihrauch Lattice

Proposition (B., Gherardi 2011, Pauly 2010)

Weihrauch reducibility induces a distributive lattice with the coproduct $\sqcup$ as supremum and $\sqcap$ as infimum. Parallelization $\widehat{\ }$ and star operation * are closure operators in the Weihrauch lattice.

- With $\sqcup, \times, ^*$ one obtains a Kleene algebra (B., Pauly).
- The Weihrauch lattice is neither a Brouwer nor a Heyting algebra (Higuchi und Pauly 2012).

Open Problem

Does the strong Weihrauch reducibility induce a lattice structure?

- It is known that $\sqcap$ is an infimum for $\leq_{\mathrm{sW}}$ and hence one obtains a lower semi-lattice (B., Gherardi).
- One can show that $\sqcup$ fails as supremum for $\leq_{\mathrm{sW}}$.

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Constants of the Weihrauch Lattice

- $\mathbf{0} :=$ the equivalence class of the nowhere defined problems is the bottom element of the Weihrauch lattice, and a neutral element with respect to $\sqcup$. It acts like a zero with respect to $\times$ and $*$.
- $\mathbf{1} :=$ the equivalence class of the identity $\mathrm{id} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a neutral element with respect to $\times$ and $*$.
- $\mathbf{0}^* \equiv_{\mathrm{W}} \mathbf{1}$.
- $\infty :=$ the equivalence class of all problems without realizer is the top element of the Weihrauch lattice and a neutral element with respect to $\sqcap$.
- ∞ exists if and only if the Axiom of Choice does not hold for Baire space $\mathbb{N}^{\mathbb{N}}$.
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Compositional Product and Implication

The Weihrauch lattice is not complete and infinite suprema and infima do not always exist. There are some known existent ones.

Theorem (B. and Pauly 2013)

For two mathematical problems f, g the following exist:

- $f * g := \max\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f \text{ and } g_0 \leq_{\mathrm{W}} g\}$ *and*
- $g \to f := \min\{h : f \leq_{\mathrm{W}} g * h\}$.

The maximum and minimum is understood with respect to $\leq_{\mathrm{W}}$.

Proof. (Sketch) For every $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ we consider the transpose $f^{\mathrm{t}} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ defined by

$$f^{\mathrm{t}}\langle p, q\rangle := \eta_p \circ f(q),$$

where η is a standard representation of all continuous functions $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. For arbitrary f, g we obtain

$$f * g \equiv_{\mathrm{W}} f^{\mathrm{rt}} \circ g^{\mathrm{rt}}.$$

The case of $g \to f$ can be treated similarly. $\square$

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Relations Between Algebraic Operations

f pointed $:\iff \mathbf{1} \leq_{\mathrm{W}} f \iff (\exists x \in \mathrm{dom}(f))\ x$ computable.

Proposition

For pointed f, g we obtain

$$f \sqcap g \leq_{\mathrm{W}} f \sqcup g \leq_{\mathrm{W}} f \times g \leq_{\mathrm{W}} f * g,$$

where pointedness is needed only for $f \sqcup g \leq_{\mathrm{W}} f \times g$.

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Algebraic Closure Properties

- f is called idempotent if $f \times f \equiv_{\mathrm{W}} f$,
  for pointed f this holds if and only if $f^* \equiv_{\mathrm{W}} f$.
- **Examples**: $\lim, \mathsf{WKL}, \mathsf{WWKL}, \mathsf{MLR}$ are idempotent IVT is not.
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$\lim, \mathsf{WKL}, \mathsf{MLR}$ **parallelizable** → **idempotent** ← $\mathsf{WKL}, \mathsf{WWKL}, \mathsf{MLR}$ **closed under composition**

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Closure Operators and Reducibilities

Remark

There is a vague analogy between versions of Weihrauch reducibilities induced by closure operators and computability theoretic reducibilities:

Closure operation	**Reducibility**
$f \leq_{\mathrm{sW}} g$	*one-one reducibility*
$f \leq_{\mathrm{W}} g$	*many-one reducibility*
$f \leq_{\mathrm{W}} g^*$	*weak truth-table reducibility*
$f \leq_{\mathrm{W}} \widehat{g}$	*Turing reducibility*
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Question

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Embedding of the Medvedev Lattice

Proposition (B. and Gherardi 2011)

$A \leq_{\mathrm{M}} B \iff c_A \leq_{\mathrm{W}} c_B \iff \mathrm{id}|_B \leq_{\mathrm{W}} \mathrm{id}|_A$ *for* $A, B \subseteq \mathbb{N}^{\mathbb{N}}$.

- $c_A : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, p \mapsto A$ is the constant multi-valued function.
- By $\mathrm{id}|_A :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ denotes the identity restricted to A.
- We note that $\mathrm{id}|_A \leq_{\mathrm{W}} \mathbf{1} \leq_{\mathrm{W}} c_A$.
- $p \leq_{\mathrm{T}} q \iff \{p\} \leq_{\mathrm{M}} \{q\}$, hence also the Turing semi-lattice embeds into the Weihrauch lattice.

Proposition (B. and Gherardi 2011)

- $c_{A \oplus B} \equiv_{\mathrm{W}} c_A \times c_B \equiv_{\mathrm{W}} (c_A \sqcup c_B)^* \equiv_{\mathrm{W}} \widehat{c_A \sqcup c_B}$,
- $c_{A \otimes B} \equiv_{\mathrm{W}} c_A \sqcap c_B$,
- $\mathrm{id}|_{A \oplus B} \equiv_{\mathrm{W}} \mathrm{id}|_A \times \mathrm{id}|_B$,
- $\mathrm{id}|_{A \otimes B} \equiv_{\mathrm{W}} \mathrm{id}|_A \sqcup \mathrm{id}|_B$.

Here $A \oplus B = \langle A \times B \rangle$, $A \otimes B = 0A \cup 1B$ for $A, B \subseteq \mathbb{N}^{\mathbb{N}}$.

Embedding of the Medvedev Lattice

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$A \leq_{\mathrm{M}} B \iff c_A \leq_{\mathrm{W}} c_B \iff \mathrm{id}|_B \leq_{\mathrm{W}} \mathrm{id}|_A$ *for* $A, B \subseteq \mathbb{N}^{\mathbb{N}}$.

- $c_A : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, p \mapsto A$ is the constant multi-valued function.
- By $\mathrm{id}|_A :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ denotes the identity restricted to A.
- We note that $\mathrm{id}|_A \leq_{\mathrm{W}} \mathbf{1} \leq_{\mathrm{W}} c_A$.
- $p \leq_{\mathrm{T}} q \iff \{p\} \leq_{\mathrm{M}} \{q\}$, hence also the Turing semi-lattice embeds into the Weihrauch lattice.

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- $c_{A \oplus B} \equiv_{\mathrm{W}} c_A \times c_B \equiv_{\mathrm{W}} (c_A \sqcup c_B)^* \equiv_{\mathrm{W}} \widehat{c_A \sqcup c_B}$,
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Here $A \oplus B = \langle A \times B \rangle$, $A \otimes B = 0A \cup 1B$ for $A, B \subseteq \mathbb{N}^{\mathbb{N}}$.

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Weihrauch Reducibility and Medvedev Reducibility

Lemma (B., Hendtlass and Kreuzer 2015)

$$f \leq_{\mathrm{W}} g$$
$$\Longrightarrow (\forall \textit{ computable } p \in \mathrm{dom}(f))(\exists \textit{ computable } q \in \mathrm{dom}(g))$$
$$f(p) \leq_{\mathrm{M}} g(q)$$

for $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$.

- Hence, Weihrauch reducibility can be seen as a parameterized version of Medvedev reducibiltiy.
- Computability theoretic problems such as MLR, where the input is just an oracle, can and have also been studied in the Medvedev lattice (for computable inputs).
- As long as the proofs relativize, one obtains corresponding results in the Weihrauch lattice.
- Other problems such as WKL, WWKL depend on inputs in a relevant way and can be compared to problems such as MLR in the Weihrauch lattice.

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- As long as the proofs relativize, one obtains corresponding results in the Weihrauch lattice.
- Other problems such as $\mathsf{WKL}, \mathsf{WWKL}$ depend on inputs in a relevant way and can be compared to problems such as MLR in the Weihrauch lattice.

Zoo of Reducibilities

uniform
resource sensitive

Weihrauch complexity
(computable)

refines →

non-uniform
closed under composition

Reverse mathematics
(over RCA_0)

$$\begin{tikzcd}
f \leq_{\mathrm{sW}} g \arrow[r] \arrow[dr] & f \leq_{\mathrm{W}} g \arrow[r] \arrow[dr] & f \leq_{\mathrm{gW}} g \arrow[dr] & \\
& f \leq_{\mathrm{sc}} g \arrow[r] & f \leq_{\mathrm{c}} g \arrow[r] & f \leq_{\omega} g
\end{tikzcd}$$

Diagram based on: Hirschfeldt and Jockusch, On Notions of Computability Theoretic Reduction Between Π^1_2 Principles, preprint 2015.

Logical Interpretation

uniform
resource sensitive

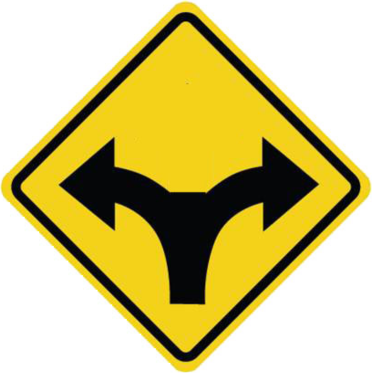
non-uniform
closed under composition

$$\begin{array}{ccc}\textbf{Weihrauch complexity} & & \textbf{Reverse mathematics} \\ \text{(computable)} & \xrightarrow{\text{refines}} & \text{(over } \mathsf{RCA}_0\text{)}\end{array}$$

Question

Can the slogan "Weihrauch complexity is a kind of a model of reverse mathematics with some form of (intuitionistic) linear logic" be converted into a theorem?

Choice

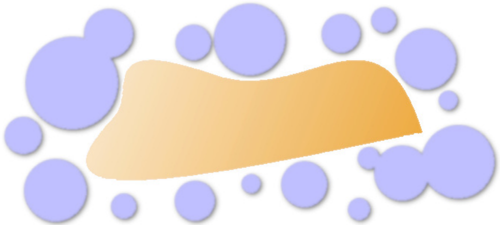


Co-c.e. Closed Sets in Computable Metric Spaces

- Let (X, d, α) be a computable metric space and $A \subseteq X$ closed.
- By $B_{\langle n,k\rangle} := B(\alpha(n), \overline{k})$ we denote the ball with center $\alpha(n)$ and rational radius $\overline{k}$. Here $\overline{\langle a, b, c\rangle} := \frac{a-b}{c+1}$.

Then the following are equivalent to each other:

- A is co-c.e. closed,
- $X \setminus A = \bigcup_{i=0}^{\infty} B_{n_i}$ for a computable sequence $(n_i)_i$ of natural and numbers,
- $A = f^{-1}\{0\}$ for a computable function $f : X \to \mathbb{R}$.



Representing Closed Subsets by Negative Information

- We define a representation $\psi_- : \mathbb{N}^\mathbb{N} \to \mathcal{A}_-(X)$ of the set $\mathcal{A}_-(X)$ of all closed subsets of X by

$$\psi_-(p) := X \setminus \bigcup_{i=0}^{\infty} B_{p(i)}.$$

- The computable points in the represented space $\mathcal{A}_-(X)$ are exactly the co-c.e. closed subsets $A \subseteq X$.
- There is also a natural representation of the set $\mathcal{C}(X)$ of continuous functions $f : X \to \mathbb{R}$.

Proposition

$P : \mathcal{C}(X) \to \mathcal{A}_-(X), f \mapsto f^{-1}\{0\}$ is a computable isomorphism in the sense that P and P^{-1} are computable.

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Choice

Definition

$\mathrm{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ is called the choice problem of a computable metric space X.

This is the problem that corresponds to the statement:

- Every non-empty closed set $A \subseteq X$ has a point $x \in A$.

Corollary

$\mathrm{C}_X \equiv_{\mathrm{sW}} \mathrm{Z}_X$ for every computable metric space X.

The choice problem is equivalent to the zero problem of finding a solution $x \in X$ of the equation

$$f(x) = 0$$

for a continuous function $f : X \to \mathbb{R}$. Formally, we consider the zero problem as $\mathrm{Z}_X :\subseteq \mathcal{C}(X) \rightrightarrows X, f \mapsto f^{-1}\{0\}$.

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Mind-Changes and Choice

Proposition

$\mathsf{C}_0 \equiv_{\mathrm{W}} \mathbf{0}$, $\mathsf{C}_1 \equiv_{\mathrm{W}} \mathbf{1}$, $\mathsf{C}_2 \equiv_{\mathrm{W}} \mathsf{LLPO}$, $\mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \lim_{\mathbb{N}}$.

Proposition (B. and Gherardi 2011)

Let $f \leq_{\mathrm{W}} g$. If g is computable with n mind changes, then so is f.

Proposition (B., de Brecht and Pauly 2012)

$f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \iff f$ *is computable with finitely many mind changes.*

Corollary

$\mathsf{C}_n <_{\mathrm{W}} \mathsf{C}_{n+1} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ *for all* $n \in \mathbb{N}$.

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Choice on Cantor Space and Weak Kőnig's Lemma

Theorem (B. and Gherardi 2011)

$\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2}$.

Proof. The equivalence $\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}}$ follows since the map

$$[\,] : \mathsf{Tr} \to \mathcal{A}_-(2^{\mathbb{N}}), T \mapsto [T]$$

which maps a binary tree to the set of its infinite paths is computable and has a computable right inverse. The equivalence proof for $\mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2}$ exploits the fact that for finding an infinite path it is sufficient to make countably many binary decisions (regarding the question which subtree is infinite) and vice versa. □

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Positive Choice

- The positive choice problem $\mathsf{PC}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ of a computable metric space X with a Borel measure μ is the restriction of C_X to $\mathrm{dom}(\mathsf{PC}_X) := \{A \subseteq X : \mu(A) > 0\}$.
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Proposition (B., Gherardi and Hölzl 2015)

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- We have $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \not\leq_{\mathrm{sW}} \mathsf{WWKL}$. Hence WWKL is not a cylinder.
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Basic Complexity Classes

$$\begin{tikzcd}
 & \mathsf{C}_{\mathbb{N}^\mathbb{N}} \arrow[d] & \\
 & \lim \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_\mathbb{N}} \arrow[d] & \\
 & \mathsf{C}_\mathbb{R} \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N} \times \mathsf{C}_{2^\mathbb{N}} \arrow[dl] \arrow[dr] & \\
\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^\mathbb{N}} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2} \arrow[d] & & \lim_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{N} \arrow[ddl] \\
\mathsf{WWKL} \equiv_{\mathrm{sW}} \mathsf{PC}_{2^\mathbb{N}} \arrow[dr] & & \\
 & \mathsf{K}_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{C}_2^* \arrow[d] & \\
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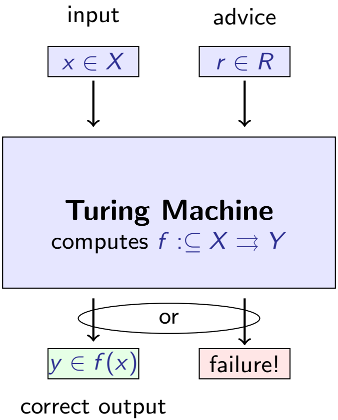
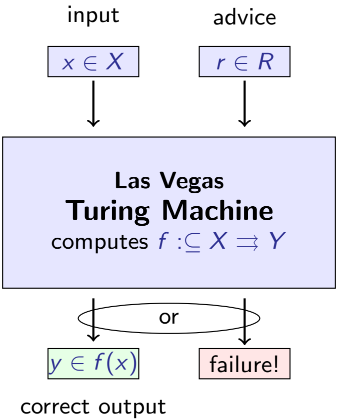
\end{tikzcd}$$

Turing Machines with Advice



Condition: $(\forall x \in \mathrm{dom}(f))\ \{r \in R : r \text{ does not fail with } x\} \neq \emptyset$

Las Vegas Turing Machines



Condition: $(\forall x \in \mathrm{dom}(f))\ \mu\{r \in R : r \text{ does not fail with } x\} > 0$

Calibrating Computability with Choice

Theorem (B., de Brecht and Pauly 2012)

For $R \subseteq \mathbb{N}^{\mathbb{N}}$ and $f :\subseteq X \rightrightarrows Y$ the following are equivalent:

- $f \leq_{\mathrm{W}} \mathsf{C}_R$,
- f *is computable on a Turing machine with advice from* R.

Corollary

- $f \leq \mathsf{C}_{\{0\}} \iff f$ *is computable,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \iff f$ *comp. with finitely many mind changes,*
- $f \leq_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} \iff f$ *is non-deterministically computable,*
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In the last case f is single-valued on computable Polish spaces.

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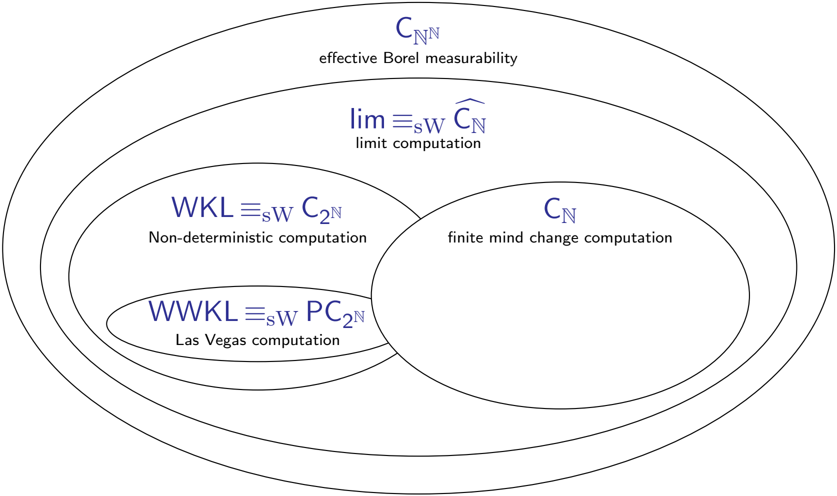
Computational Classes



Independent Choice Theorem

Theorem (B., de Brecht and Pauly 2012)

$\mathsf{C}_R * \mathsf{C}_S \leq_{\mathrm{W}} \mathsf{C}_{R\times S}$ *for all* $R, S \subseteq \mathbb{N}^{\mathbb{N}}$.

Proof. Run a Turing machine that simulates upon advice (r, s) two consecutive machines with advice r and s, respectively. □

Proposition

If $s : R \to S$ *is a computable surjection, then* $\mathsf{C}_S \leq_{\mathrm{W}} \mathsf{C}_R$.

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$\mathsf{PC}_R * \mathsf{PC}_S \leq_{\mathrm{W}} \mathsf{PC}_{R \times S}$ *for* $R, S \subseteq \mathbb{N}^{\mathbb{N}}$ *with* σ*–finite Borel measures and their product measure.*

Proof. (Sketch) The proof proceeds along the lines of the case for closed choice plus an additional invocation of Fubini's Theorem. □

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PC_R *is closed under composition for* $R \in \{\mathbb{N}, 2^{\mathbb{N}}, \mathbb{N} \times 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}\}$.

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Choice for Computable Polish Spaces

Theorem (B., de Brecht and Pauly 2012)

Let X be a computable Polish space. Then

- $\mathsf{C}_X \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$,
- $\mathsf{C}_X \leq_{\mathrm{sW}} \mathsf{C}_{2^\mathbb{N}}$ *if X is computably compact,*
- $\mathsf{C}_{2^\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{C}_X$ *if X is perfect,*
- $\mathsf{C}_X \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}\times 2^\mathbb{N}}$ *if X is a computable K_σ–space,*
- $\mathsf{C}_X \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *with respect to some oracle, if X is not K_σ.*

Corollary

For all $n \geq 1$:

- $\mathsf{C}_{[0,1]^n} \equiv_{\mathrm{sW}} \mathsf{C}_{2^\mathbb{N}}$
- $\mathsf{C}_{\mathbb{R}^n} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}\times 2^\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{N} \times \mathsf{C}_{2^\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{N} * \mathsf{C}_{2^\mathbb{N}}$
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\mathsf{C}_{2^\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]^n} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]^\mathbb{N}} \arrow[ddr] & & \\
 & & \mathsf{C}_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{Z} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{Q} \arrow[dl] \\
 & \vdots \arrow[d] & \\
 & \mathsf{C}_3 \arrow[d] & \\
 & \mathsf{C}_2 \arrow[d] & \\
 & \mathsf{C}_1 &
\end{tikzcd}
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perfect non locally compact

perfect locally compact

perfect compact

countable discrete

finite

Choice Elimination for Choice on Cantor Space

The following result is reminiscent of certain conservation results.

Theorem (B., de Brecht and Pauly 2012)

$f \leq_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} * g \Longrightarrow f \leq_{\mathrm{W}} g$

for single-valued $f :\subseteq X \to Y$ on computable metric spaces X, Y.

Proof. (Idea.) A non-deterministic computation that yields a unique result cannot really exploit the advice $r \in 2^{\mathbb{N}}$. The compact set of successful advices can be systematically searched in order to find a successful advice. $\square$

Corollary

$f \leq_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} \Longrightarrow f$ *computable (for f as above).*

Corollary

$\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}}$.

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Choice Elimination for Choice on Natural Numbers

- f is called a fractal if there is a $F :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ with $F \equiv_{\mathrm{W}} f$ and $F|_U \equiv_{\mathrm{W}} f$ for every open $U \subseteq \mathbb{N}^\mathbb{N}$ with $U \cap \mathrm{dom}(F) \neq \emptyset$.
- f is called a total fractal if there is a total F as above.
- strong (total) fractals are defined analogously with $\equiv_{\mathrm{sW}}$.

Theorem (Le Roux and Pauly 2015)

$f \leq_{\mathrm{W}} \mathsf{C}_\mathbb{N} * g \Longrightarrow f \leq_{\mathrm{W}} g$ for total fractals f.

Proof. (Idea.) Replace f by a total fractal an apply the Baire Category Theorem to the sets A_n of inputs to F for which $\mathsf{C}_\mathbb{N}$ yields the number n as a possible result. Then $\mathbb{N}^\mathbb{N} = \bigcup_{n=0}^{\infty} A_n$ and one of the sets A_n is somewhere dense. The fractality condition yields the desired reduction. □

Corollary (B. and Gherardi 2011)

$\mathsf{IVT} \not\leq_{\mathrm{W}} \mathsf{C}_\mathbb{N}$ and hence $\mathsf{IVT} \,|_{\mathrm{W}}\, \mathsf{C}_\mathbb{N}$.

It is clear that also $\mathsf{PC}_{2^\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{C}_\mathbb{N}$.

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Basic Complexity Classes

$$
\begin{tikzcd}
 & \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \arrow[d] & \\
 & \lim \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_{\mathbb{N}}} \arrow[d] & \\
 & \mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \times \mathsf{C}_{2^{\mathbb{N}}} \arrow[dl] \arrow[dr] & \\
\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2} \arrow[d] & & \lim_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \arrow[ddl] \\
\mathsf{WWKL} \equiv_{\mathrm{sW}} \mathsf{PC}_{2^{\mathbb{N}}} \arrow[dr] & & \\
 & \mathsf{K}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_2^* \arrow[d] & \\
 & \mathsf{LLPO} \equiv_{\mathrm{sW}} \mathsf{C}_2 &
\end{tikzcd}
$$

Join Irreducibility

For $g_n :\subseteq X \rightrightarrows Y$ we define
$\bigsqcup_{n=0}^{\infty} g_n :\subseteq \mathbb{N} \times X \rightrightarrows Y, (n, x) \mapsto g_n(x)$.

Definition

f is called join irreducible, if one of the following equivalent conditions hold:

- $f \equiv_{\mathrm{W}} \bigsqcup_{n=0}^{\infty} g_n \Longrightarrow (\exists n)\ f \equiv_{\mathrm{W}} g_n$,
- $f \leq_{\mathrm{W}} \bigsqcup_{n=0}^{\infty} g_n \Longrightarrow (\exists n)\ f \leq_{\mathrm{W}} g_n$.

Equivalence follows since the Weihrauch lattice is distributive.

Proposition (B., de Brecht and Pauly 2012)

Every fractal f is join irreducible.

Corollary

$\mathrm{C}_{\mathbb{N}} \sqcup \mathrm{C}_{2^{\mathbb{N}}} <_{\mathrm{W}} \mathrm{C}_{\mathbb{N}} \times \mathrm{C}_{2^{\mathbb{N}}}$.

$\mathrm{C}_{\mathbb{N}} \times \mathrm{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathrm{C}_{\mathbb{R}}$ is a fractal.

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Minima versus Maxima and Boundedness versus Induction

We consider:

- min : $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}, p \mapsto \min\{p(n) : n \in \mathbb{N}\}$,
- max $:\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, p \mapsto \max\{p(n) : n \in \mathbb{N}\}$.

Finding a minimum is simpler because the first element in the sequence is already an upper bound on the result and hence the search space is finite.

Proposition

$\max \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$ *and* $\min \equiv_{\mathrm{sW}} \mathsf{K}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_2^*$.

This suggests the following correspondence:

- $\mathsf{B}\Sigma_1^0$ (= boundedness for Σ_1^0 formulas) corresponds to $\mathsf{K}_{\mathbb{N}}$,
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- $\mathsf{I}\Sigma^0_1$ (= induction for Σ^0_1 formulas) corresponds to $\mathsf{C}_\mathbb{N}$.

Basic Complexity Classes and Reverse Mathematics

$$
\begin{tikzcd}
 & \mathsf{C}_{\mathbb{N}^\mathbb{N}} \arrow[d] & & \mathsf{ATR}_0 \\
 & \lim \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_\mathbb{N}} \arrow[d] & & \mathsf{ACA}_0 \\
 & \mathsf{C}_\mathbb{R} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{N} \times \mathsf{C}_{2^\mathbb{N}} \arrow[dl] \arrow[dddr] & & \mathsf{WKL}_0 + \mathsf{I}\Sigma^0_1 \\
\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^\mathbb{N}} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2} \arrow[d] & & & \mathsf{WKL}_0 \\
\mathsf{WWKL} \equiv_{\mathrm{sW}} \mathsf{PC}_{2^\mathbb{N}} \arrow[ddr] & & & \mathsf{WWKL}_0 \\
 & & \lim_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{C}_\mathbb{N} \arrow[dl] & \mathsf{I}\Sigma^0_1 \\
 & \mathsf{K}_\mathbb{N} \equiv_{\mathrm{sW}} \mathsf{C}_2^* \arrow[d] & & \mathsf{B}\Sigma^0_1 \\
 & \mathsf{C}_1 & & \mathsf{RCA}_0
\end{tikzcd}
$$

The Classification of Theorems

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Choice on Natural Numbers

Theorem (B. and Gherardi 2012)

The following problems and theorems are Weihrauch equivalent:

- *The choice problem* $\mathsf{C}_{\mathbb{N}}$ *on natural numbers.*
- *The Baire Category Theorem* BCT_1.
- *The Banach Inverse Mapping Theorem* IMT.
- *The Open Mapping Theorem.*
- *The Closed Graph Theorem.*
- *The Uniform Boundedness Theorem.*

All for infinite dimensional computable normed spaces (in case of BCT_1 *even for all perfect computable metric spaces).*

All members of the equivalence class share the following features:

- All members map computable inputs to (some) computable outputs.
- All members are not uniformly computable.
- All members are computable with finitely many mind changes.
- All members have parallelizations that are equivalent to the limit map and they are closed under composition.

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Theorem (Baire Category Theorem)

Every complete metric space X cannot be written as a countable union $X = \bigcup_{i=0}^{\infty} A_i$ of nowhere dense closed sets $A_i \subseteq X$.

For perfect computable complete metric space X we define:

- $\mathrm{BCT}_0 :\subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows X, (A_i)_{i\in\mathbb{N}} \mapsto X \setminus \bigcup_{i=0}^{\infty} A_i$ with $\mathrm{dom}(\mathrm{BCT}_0) = \{(A_i)_{i\in\mathbb{N}} : A_i^\circ = \emptyset\}$.
- $\mathrm{BCT}_1 :\subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows \mathbb{N}, (A_i)_{i\in\mathbb{N}} \mapsto \{n \in \mathbb{N} : A_n^\circ \neq \emptyset\}$ with $\mathrm{dom}(\mathrm{BCT}_1) = \{(A_i)_{i\in\mathbb{N}} : X = \bigcup_{i=0}^{\infty} A_i\}$.

The strong Weihrauch equivalence class does not depend on the underlying space, but on the logical form.

Theorem (B. and Gherardi 2011)

$\mathrm{BCT}_1 \equiv_{\mathrm{sW}} \mathrm{C}_{\mathbb{N}}$ *and* $\mathrm{BCT}_0 \equiv_{\mathrm{W}} \mathrm{id}$.

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Proof.

Proof idea for $\mathsf{BCT}_1 \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.
"$\mathsf{BCT}_1 \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$" Given (A_i), the set

$$\{\langle k, n\rangle : \emptyset \neq B_k \subseteq A_n\}$$

is co-c.e. in all parameters. Hence one can find a number $\langle k, n\rangle$ in this set using $\mathsf{C}_{\mathbb{N}}$. In this case $n \in \mathsf{BCT}_1(A_i)$.
"$\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{BCT}_1$" Given a sequence $(n_i)_{i\in\mathbb{N}}$ that enumerates a set of natural numbers, we compute the sequence (A_i) of closed subsets $A_i \subseteq X$ with

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This sequence is computable in (n_i) and each $n \in \mathsf{BCT}_1(n_i)$ has the property that n does not appear in (n_i). $\square$

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Theorem (Banach's Inverse Mapping Theorem)

Every bijective bounded linear operator $T : X \to Y$ on Banach spaces X, Y has a bounded inverse $T^{-1} : Y \to X$.

For computable Banach spaces X, Y we define

- $\mathsf{IMT} :\subseteq \mathcal{C}(X, Y) \to \mathcal{C}(Y, X), T \mapsto T^{-1}$ with $\mathrm{dom}(\mathsf{IMT}) = \{T : T \text{ linear}\}$.

The strong Weihrauch equivalence depends on the underlying spaces.

Theorem (B. and Gherardi 2011)

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Choice on Cantor Space

Theorem

The following problems and theorems are Weihrauch equivalent:

- *The choice problem* $\mathsf{C}_{2^{\mathbb{N}}}$ *on Cantor space* $2^{\mathbb{N}}$.
- *Weak Kőnig's Lemma* WKL.
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All members of the equivalence class share the following features:

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The Heine-Borel Theorem

Theorem (Heine-Borel)

Every countable open cover $(U_i)_i$ of the unit interval $[0,1]$ has a finite subcover.

Two different logical formalizations:

- $\mathsf{HB}_0 :\subseteq \mathcal{O}([0,1])^{\mathbb{N}} \rightrightarrows \mathbb{N}, (U_i)_i \mapsto \{n \in \mathbb{N} : [0,1] \subseteq \bigcup_{i=0}^{n} U_i\}$, $\mathrm{dom}(\mathsf{HB}_0) := \{(U_i)_i : [0,1] \subseteq \bigcup_{i=0}^{\infty} U_i\}$.
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The set $\mathcal{O}(X)$ of open subsets of X is represented as $\mathcal{A}_-(X)$, using complements.

Proposition

$\mathsf{HB}_0 \equiv_{\mathrm{W}} \mathrm{id}$ *is computable* $\mathsf{HB}_1 \equiv_{\mathrm{W}} \mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}}$.

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The Brouwer Fixed Point Theorem

Theorem (Brouwer Fixed Point Theorem)

Every continuous map $f : [0,1]^n \to [0,1]^n$ has a fixed point $x \in [0,1]^n$, i.e., $f(x) = x$.

- $\mathrm{BFT}_n : \mathcal{C}([0,1]^n, [0,1]^n) \rightrightarrows [0,1]^n, f \mapsto \{x : f(x) = x\}$.
- Connected Choice $\mathrm{CC}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ is the restriction of closed choice C_X to connected sets.

Theorem (B., Le Roux and Pauly 2012)

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Connected Choice versus Closed Choice

Proposition (B., Le Roux and Pauly 2012)

The map

$$A \mapsto (A \times [0,1] \times \{0\}) \cup (A \times A \times [0,1]) \cup ([0,1] \times A \times \{1\})$$

is computable and maps any *non-empty closed* $A \subseteq [0,1]$ *to a* connected *non-empty closed* $A \subseteq [0,1]^3$.

Theorem (B., Le Roux, J.S. Miller and Pauly 2012)

$\mathsf{CC}_{[0,1]^n} \equiv_{\mathrm{sW}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}}$ *for all* $n \geq 2$.

The proof for $n \geq 3$ follows from the Proposition, but the case $n = 2$ needs a more involved and completely different construction due to J.S. Miller.

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Corollary (Baigger 1985, Orevkov 1963)

There exists a computable function $f : [0,1]^n \to [0,1]^n$ *that has no computable fixed point* $x \in [0,1]^n$ *for every* $n \geq 2$.

However, there is always a low fixed point.

Corollary

There exists a non-empty connected co-c.e. closed subset $A \subseteq [0,1]^n$ *without computable point for every* $n \geq 2$.

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$\widehat{\mathsf{IVT}} \equiv_{\mathrm{sW}} \mathsf{WKL}$.

Corollary (Pour-El and Richards 1989)

There are computable $f_n : [0,1] \to \mathbb{R}$ with $f_n(0) \cdot f_n(1) < 0$ and without computable $x_n \in [0,1]$ such that $f_n(x_n) = 0$ for all $n \in \mathbb{N}$.

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"The Baire Category Theorem proves that the Baire Category Theorem does not compute the Intermediate Value Theorem."

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Choice on Euclidean Space

Frostman's Lemma is a result from geometric measure theory that guarantees the existence of certain measures that are supported on a given closed set.

Theorem (Fouché and Pauly 2015)

The following problems and theorems are Weihrauch equivalent:

- *The choice problem* $\mathsf{C}_{\mathbb{R}}$ *on Euclidean space* $\mathbb{R}$.
- *Frostman's Lemma.*

All members of the equivalence class share the following features:

- All members map computable inputs to (some) low outputs.
- All members are neither uniformly nor non-uniformly computable.
- All members are non-deterministically computable with finite mind changes.
- All members are closed under composition and not parallelizable.

Problem

Suggest other natural theorems equivalent to $\mathsf{C}_{\mathbb{R}}$*!*

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Unique Choice

- Unique Choice $\mathsf{UC}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X$ is the restriction of closed choice C_X to

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$\mathsf{UC}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathrm{id}$ *and* $\mathsf{UC}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$.

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All or Unique Choice and Robust Division

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$$\mathsf{RDIV} : [0,1] \times [0,1] \rightrightarrows [0,1], (x,y) \mapsto \begin{cases} \left\{\frac{x}{\max(x,y)}\right\} & \text{if } y \neq 0 \\ [0,1] & \text{if } y = 0 \end{cases}$$

- Robust division RDIV can be used to solve linear equations in compact domains: $ax = b$.
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Nash Equilibria

- A bi-matrix game is a pair $A, B \in \mathbb{R}^{m \times n}$ of $m \times n$–matrices.
- A vector $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ with $s_i \geq 0$ for all $i = 1, \ldots, m$ and $\sum_{j=1}^{m} s_j = 1$ is called a mixed strategy.
- By S^m we denote the set of mixed strategies of dimension m.
- A Nash equilibrium is a pair $(x, y) \in S^n \times S^m$ such that $(\forall w \in S^n)\ x^{\mathrm{T}} A y \geq w^{\mathrm{T}} A y$ and $(\forall z \in S^m)\ x^{\mathrm{T}} B y \geq x^{\mathrm{T}} B z$.

Theorem (Nash 1951)

Every bi-matrix game admits a Nash equilibrium.

- $\mathrm{NASH}_{n,m} : \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$, where $(A, B) \mapsto \{(x, y) : (x, y) \text{ is a Nash equilibrium for } (A, B)\}$.
- $\mathrm{NASH} := \bigsqcup_{n,m \in \mathbb{N}} \mathrm{NASH}_{n,m}$.

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A Las Vegas Algorithm for Robust Division

Proposition

Robust division RDIV is Las Vegas computable.

1. Given $x, y \in [0,1]$ and a random advice $r \in [0,1]$, we aim to compute the fraction $z = \frac{x}{\max(x,y)}$.
2. We guess that r is a correct solution, i.e., $r = z$ if $y > 0$, and we produce approximations of r (rational intervals $(a,b) \ni r$).
3. Simultaneously, we try to find out whether $y > 0$, which we will eventually recognize, if this is correct.
4. If we find that $y > 0$, then we can compute the true result $z = \frac{x}{\max(x,y)}$ and produce approximations of it.
5. If at some stage we find that the best approximation (a,b) of r that was already produced as output is incompatible with z, i.e., if $z \notin (a,b)$, then we indicate a failure.

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$\mathrm{NASH} \equiv_{\mathrm{W}} \mathrm{RDIV}^* \leq_{\mathrm{W}} \mathrm{WWKL}$.

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A Probabilistic Algorithm for Zero Finding

1. A continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) \cdot f(1) < 0$ is given as input.
2. Guess a binary sequence or, equivalently, a bit $b \in \{0, 1\}$ and a point $x \in [0, 1]$.
3. Interpret the guess $b = 1$ such that the zero set $f^{-1}\{0\}$ contains no open intervals and use the trisection method to compute a zero $z \in [0, 1]$ with $f(z) = 0$ in this case (disregarding x).
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Warning: This is not a Las Vegas algorithm! But it yields:

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There is no Las Vegas Algorithm for Zero Finding

Theorem

IVT $\not\leq_{\mathrm{W}}$ WWKL.

Proof. (Idea) The proof is based on a finite extension construction: under the assumption that there is an algorithm for the reduction, one can create an instance (a function f) by finite extension that forces the reduction to translate this function into a tree that has measure zero. $\square$

Corollary

IVT $|_{\mathrm{W}}$ WWKL.

The inverse result WWKL $\not\leq_{\mathrm{W}}$ IVT is easy to see: IVT maps computable inputs to computable outputs, WWKL does not.

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Idempotency of Restricted Choice Principles

Proposition (B., Gherardi and Hölzl 2015)

$\mathsf{C}_2 \times \mathsf{AUC}_{[0,1]} \not\leq_{\mathrm{W}} \mathsf{CC}_{[0,1]}$.

Corollary (B., Le Roux and Pauly 2012)

$\mathsf{CC}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{IVT}$ *is not idempotent.*

Also $\mathsf{AUC}_{[0,1]}$ is not idempotent. Since $\mathsf{C}_2 \times \mathsf{AUC}_{[0,1]} \leq_{\mathrm{W}} \mathsf{AUC}^*_{[0,1]}$:

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$$
\begin{tikzcd}
 & \mathsf{lim} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_{\mathbb{N}}} \arrow[d] & \\
 & \mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \times \mathsf{C}_{2^{\mathbb{N}}} \arrow[dl] \arrow[dr] & \\
\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}} \arrow[dd] \arrow[dr] & & \mathsf{BCT}_1 \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \arrow[dd, dashed] \\
 & \mathsf{WWKL} \equiv_{\mathrm{sW}} \mathsf{PC}_{2^{\mathbb{N}}} \arrow[dr, dashed] & \\
\mathsf{IVT} \equiv_{\mathrm{sW}} \mathsf{CC}_{[0,1]} \arrow[d] \arrow[drr] & & \mathsf{NASH} \equiv_{\mathrm{sW}} \mathsf{AUC}^*_{[0,1]} \arrow[d] \arrow[dll] \\
\mathsf{K}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_2^* \arrow[dr] & & \mathsf{RDIV} \equiv_{\mathrm{sW}} \mathsf{AUC}_{[0,1]} \arrow[dl] \\
 & \mathsf{LLPO} \equiv_{\mathrm{sW}} \mathsf{C}_2 \arrow[d] & \\
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All or Co-Unique Choice and Diagonal Non-Computability

- All-or-Co-Unique Choice $\mathsf{ACC}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$ is the restriction of closed choice C_X to
$$\mathrm{dom}(\mathsf{ACC}_X) := \{A \subseteq X : A = X \text{ or } |X \setminus A| = 1\}.$$
- $\mathsf{ACC}_X \equiv_{\mathrm{sW}} \mathrm{id}$ for perfect computable metric spaces.
- $\mathsf{ACC}_2 = \mathsf{C}_2$ and $\mathsf{ACC}_n \equiv_{\mathrm{sW}} \mathsf{LLPO}_n$ for $n \geq 2$.

Proposition (Weihrauch 1992)

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PA, Diagonal Non-Computability and WKL

- $\mathrm{PA} : \mathcal{D} \rightrightarrows \mathcal{D}, \mathbf{a} \mapsto \{\mathbf{b} : \mathbf{b} \gg \mathbf{a}\}$ is the problem of Peano arithmetic.

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- $\mathrm{WKL}_n :\subseteq \mathrm{Tr}_n \rightrightarrows n^{\mathbb{N}}, T \mapsto [T]$ denotes Weak Kőnig's Lemma for big n–ary trees.
- A tree $T \subseteq n^* = \{0, 1, ..., n-1\}^*$ is called big, if it satisfies the following condition: if w is a node of T which is on an infinite path, then all but at most one successor nodes are on an infinite path of T too.

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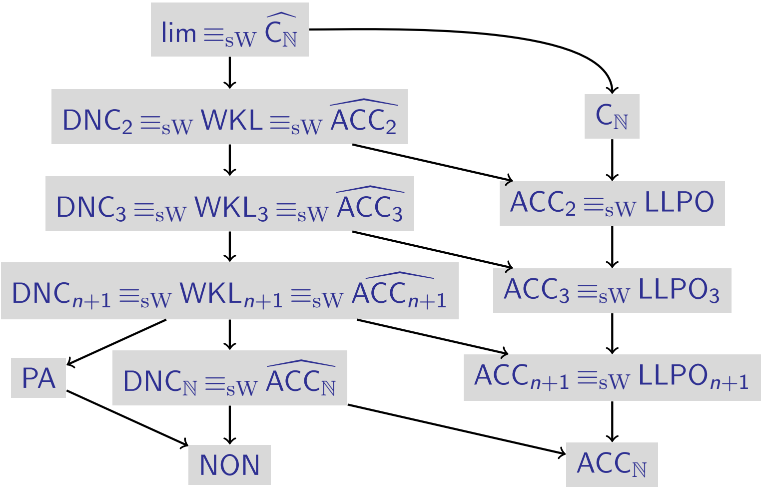

Dense Realization and All or Co-Unique Choice

- $f :\subseteq X \rightrightarrows Y$ is called densely realized, if $f^{\mathrm{r}}(p)$ is dense in $\mathrm{dom}(\delta_Y)$ for every $p \in \mathrm{dom}(f\delta_X)$.
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Jumps

$$
\begin{tikzcd}
 & \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \arrow[d] & & \mathsf{ATR}_0 \\
 & \lim \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_{\mathbb{N}}} \arrow[d] & & \mathsf{ACA}_0 \\
 & \mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \times \mathsf{C}_{2^{\mathbb{N}}} \arrow[dl] \arrow[dddr] & & \mathsf{WKL}_0 + \mathsf{I}\Sigma^0_1 \\
 \mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2} \arrow[d] & & & \mathsf{WKL}_0 \\
 \mathsf{WWKL} \equiv_{\mathrm{sW}} \mathsf{PC}_{2^{\mathbb{N}}} \arrow[ddr] & & & \mathsf{WWKL}_0 \\
 & & \lim_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \arrow[dl] & \mathsf{I}\Sigma^0_1 \\
 & \mathsf{K}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{C}_2^* \arrow[d] & & \mathsf{B}\Sigma^0_1 \\
 & \mathsf{C}_1 & & \mathsf{RCA}_0
\end{tikzcd}
$$

Limits and LPO

- $\lim_X :\subseteq X^{\mathbb{N}} \to X, (x_n)_n \mapsto \lim_{n\to\infty} x_n$ denotes the limit operation of a Hausdorff space X.
- $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, \langle p_0, p_1, p_2, ...\rangle \to \lim_{n\to\infty} p_n$ denotes the limit operation of Baire space $\mathbb{N}^{\mathbb{N}}$ with encoded input.

Proposition (B. 2005)

$\lim \equiv_{\mathrm{sW}} \lim_X$ *for all perfect computable metric spaces* X.

- $\mathrm{LPO} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, p \mapsto \begin{cases} 1 & \text{if } (\forall n)\ p(n) = 0 \\ 0 & \text{otherwise} \end{cases}$
  denotes the limited principle of omniscience.
- $\mathrm{C}_2 \equiv_{\mathrm{sW}} \mathrm{LLPO} \leq_{\mathrm{W}} \mathrm{RDIV} \leq_{\mathrm{W}} \mathrm{LPO} \leq_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$.

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- $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, \langle p_0, p_1, p_2, ...\rangle \to \lim_{n\to\infty} p_n$ denotes the limit operation of Baire space $\mathbb{N}^{\mathbb{N}}$ with encoded input.

Proposition (B. 2005)

$\lim \equiv_{\mathrm{sW}} \lim_X$ *for all perfect computable metric spaces* X.

- $\mathrm{LPO} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}, p \mapsto \begin{cases} 1 & \text{if } (\forall n)\ p(n) = 0 \\ 0 & \text{otherwise} \end{cases}$
  denotes the limited principle of omniscience.
- $\mathrm{C}_2 \equiv_{\mathrm{sW}} \mathrm{LLPO} \leq_{\mathrm{W}} \mathrm{RDIV} \leq_{\mathrm{W}} \mathrm{LPO} \leq_{\mathrm{W}} \mathrm{C}_{\mathbb{N}}$.

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Parallelized Choice on Natural Numbers

Theorem

The following problems and theorems are Weihrauch equivalent:

- *The parallelization* $\widehat{\mathsf{C}_{\mathbb{N}}}$ *of the choice problem on natural numbers.*
- *The limit problem* $\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \langle p_0, p_1, p_2, ...\rangle \mapsto \lim_{n\to\infty} p_n$.
- *The differentiability problem* $\mathrm{d} :\subseteq \mathcal{C}[0,1] \to \mathcal{C}[0,1], f \mapsto f'$ *(von Stein 1989).*
- *The Monotone Convergence Theorem* MCT.
- *The Fréchet-Riesz Theorem for Hilbert spaces. (follows from B. and Yoshikawa 2006)*
- *The Radon-Nikodym Theorem. (Hoyrup, Rojas, Weihrauch 2012)*

All members of the equivalence class share the following features:

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A Dichotomy for Linear Operators

Theorem (B. 1999)

Let X, Y be computable Banach spaces and $T :\subseteq X \to Y$ a densely defined linear operator with a c.e. closed graph. Then:

- $T \leq_W \mathrm{id} \iff T$ *computable* $\iff T$ *bounded.*
- $\lim \leq_W T \iff T$ *unbounded.*

Corollary (von Stein 1992)

$\mathrm{d} \equiv_W \lim$, *where* $\mathrm{d} :\subseteq \mathcal{C}[0,1] \to \mathcal{C}[0,1], f \mapsto f'$

Corollary (First Main Theorem of Pour-El and Richards 1989)

An unbounded $T :\subseteq X \to Y$ as above admits a computable $x \in \mathrm{dom}(T)$ such that $T(x)$ is not computable.

Corollary (Myhill 1971)

There exists a computable and continuously differentiable $f : [0,1] \to \mathbb{R}$ such that f' is not computable.

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Jumps

- For every representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ we define the jump $\delta' :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ by $\delta' := \delta \circ \lim$.
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Jumps and the Algebraic Structure

Proposition (B., Gherardi and Marcone 2011)

- $(f \circ g)' = f \circ g'$
- $(f \times g)' \equiv_{\mathrm{sW}} f' \times g'$
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- f *strongly idempotent* $\Longrightarrow f'$ *strongly idempotent,*
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The Weihrauch Lattice refines the Borel Hierarchy

- $f^{(0)} := f$ and $f^{(n+1)} := (f^{(n)})'$ for all $n \in \mathbb{N}$.

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$f \leq_{\mathrm{W}} \mathrm{id}^{(n)} \iff f$ is effectively $\mathbf{\Sigma}^0_{n+1}$–measurable for all $n \in \mathbb{N}$.

reducibility	**hierarchy**
many-one	arithmetical
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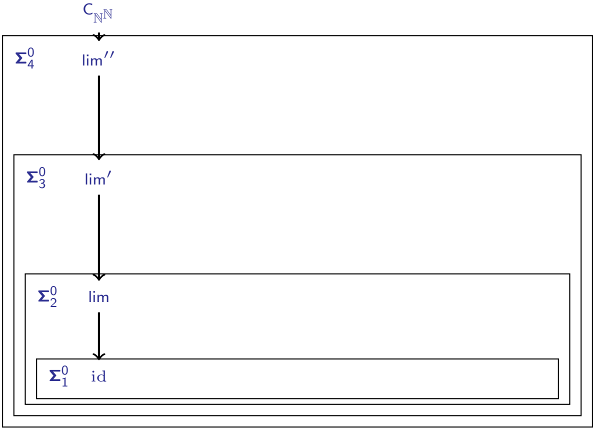

The Cluster Point Problem

- $\mathsf{CL}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X, (x_n)_n \mapsto \{x : x \text{ is a cluster point of } (x_n)_n\}$ is called the cluster point problem of a topological space X.

Theorem (B., Gherardi and Marcone 2011)

$\mathsf{CL}_X \equiv_{\mathrm{sW}} \mathsf{C}'_X$ *for every computable metric space* X.

Proof. (Idea) This can be proved by showing that the jump of ψ_- is equivalent to the cluster point representation of $\mathcal{A}_-(X)$. One direction follows since

$$X^{\mathbb{N}} \to \mathcal{A}_-(X), (x_n)_n \mapsto \{x : x \text{ is a cluster point of } (x_n)_n\}$$

is limit computable. The other direction is more involved. □

Example

- $\mathsf{C}'_2 \equiv_{\mathrm{sW}} \mathsf{CL}_2$ is the infinite pigeonhole principle,
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The Jump of Choice on Cantor Space

- $\mathsf{BWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X, (x_n)_n \mapsto \{x : x$ is a cluster point of $(x_n)_n\}$ is CL_X rest. to $\mathrm{dom}(\mathsf{BWT}_X) := \{(x_n)_n : \overline{\{x_n : n \in \mathbb{N}\}}$ is compact$\}$.

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The following problems and theorems are strongly Weihrauch equivalent:

- *The jump $\mathsf{C}'_{2^{\mathbb{N}}}$ of choice on Cantor space $2^{\mathbb{N}}$.*
- *The jump of Weak Kőnig's Lemma WKL'.*
- *Kőnig's Lemma KL. (B. and Rakotoniaina 2015)*
- *The Bolzano-Weierstraß Theorem $\mathsf{BWT}_{\mathbb{R}}$ on $\mathbb{R}$. (B., Gherardi, Marcone 2011)*

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- *The jump $\mathsf{C}'_{2^{\mathbb{N}}}$ of choice on Cantor space $2^{\mathbb{N}}$.*
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- *The Bolzano-Weierstraß Theorem $\mathsf{BWT}_{\mathbb{R}}$ on $\mathbb{R}$. (B., Gherardi, Marcone 2011)*

All members of the equivalence class share the following features:

- All members map computable inputs to (some) outputs that are low relative to the halting problem.
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- All members are closed parallelization, but not under composition.

The Jump of Choice on Cantor Space

- $\mathsf{BWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X, (x_n)_n \mapsto \{x : x$ is a cluster point of $(x_n)_n\}$ is CL_X rest. to $\mathrm{dom}(\mathsf{BWT}_X) := \{(x_n)_n : \overline{\{x_n : n \in \mathbb{N}\}}$ is compact$\}$.

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The Bolzano-Weierstraß Theorem

Proposition (B., Gherardi, Marcone 2011)

- $\mathsf{WKL}' \equiv_{\mathrm{sW}} \mathsf{BWT}_X$ *for perfect computable metric spaces* X.
- $\mathsf{K}'_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}}$.

Proposition (B. and Rakotoniaina 2015)

$\mathsf{K}_{\mathbb{N}}^{(n)} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}^{(n)} \leq_{\mathrm{sW}} \mathsf{K}_{\mathbb{N}}^{(n+1)}$ *for all* $n \in \mathbb{N}$.

Proof. (Idea) This follows from
$\mathsf{K}_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{sW}} \lim_{\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{BWT}_{\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{K}'_{\mathbb{N}}$. □

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$\mathsf{C}_2^{(n)}$ *is* $\boldsymbol{\Sigma}^0_{n+2}$*–measurable but not* $\boldsymbol{\Sigma}^0_{n+1}$*–measurable for all* $n \in \mathbb{N}$.

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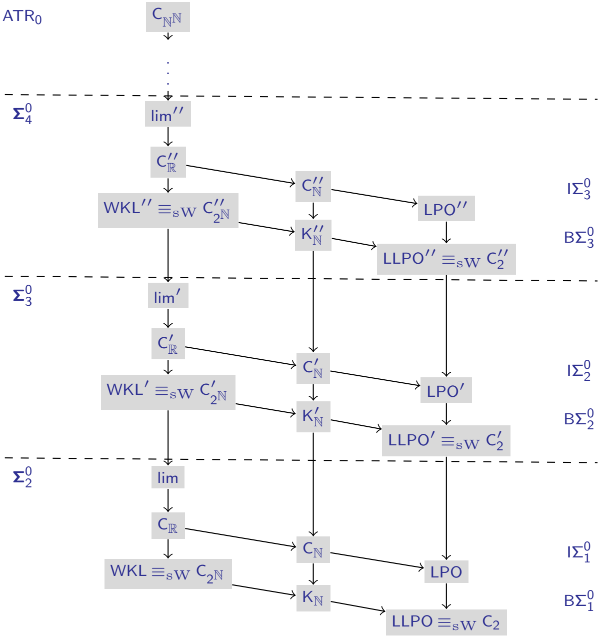
Higher Complexity Classes



The Cluster Point Problem in the Role of Induction

- We recall that $\mathrm{DNC}_{n+1} <_{\mathrm{W}} \mathrm{DNC}_n$ for all $n \geq 2$.
- R. Friedberg proved that non-uniformly the corresponding Turing degrees coincide.
- Dorais, Hirst and Shafer (2015) refined this construction and analyzed it in reverse mathematics.

Proposition (B., Hendtlass, Kreuzer 2015)

$\mathrm{DNC}_2 \leq_{\mathrm{W}} \mathrm{DNC}_n * \mathrm{C}'_{\mathbb{N}}$ *for all* $n \geq 2$.

- The proof is a uniform version of the construction of Dorais, Hirst and Shafer (2015).

Question

How can $(\mathrm{DNC}_{n+1} \to \mathrm{DNC}_n)$ *be characterized?*

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Cardinality Based Separation Techniques

- We define the cardinality $\#f$ as the supremum of all cardinalities $|M|$ of sets $M \subseteq \mathrm{dom}(f)$ such that the sets $f(x)$ with $x \in M$ are pairwise disjoint.

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$f \leq_{\mathrm{sW}} g \Longrightarrow \#f \leq \#g$.

Proposition

If $f :\subseteq X \rightrightarrows \mathbb{N}$ is a strong fractal and $\mathrm{range}(g)$ *compact, then* $f \leq_{\mathrm{W}} g \Longrightarrow f \leq_{\mathrm{sW}} g$.

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Cohesiveness and the Bolzano-Weierstraß Theorem

- $\mathrm{WBWT}_X :\subseteq X^{\mathbb{N}} \rightrightarrows X', (x_n)_n \mapsto \mathrm{BWT}_X$ is called the Weak Bolzano Weierstraß Theorem of X.
- $\mathrm{COH} : (2^{\mathbb{N}})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ where $\mathrm{COH}(R_i)$ contains all infinite $X \subseteq \mathbb{N}$ such that for all $i \in \mathbb{N}$ one of the sets $X \cap R_i$ or $X \cap (\mathbb{N} \setminus R_i)$ is finite is called the Cohesiveness Problem.

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On the Combinatorial "Core" of Problems

Problem	**Characterization**	**Core**
$\lim$	$\lim \equiv_{\mathrm{sW}} \widehat{\mathsf{LPO}}$	LPO
WKL	$\mathsf{WKL} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2}$	$\mathsf{C}_2 \equiv_{\mathrm{sW}} \mathsf{LLPO}$
KL	$\mathsf{KL} \equiv_{\mathrm{sW}} \widehat{\mathsf{C}_2'}$	$\mathsf{C}_2' \equiv_{\mathrm{sW}} \mathsf{IPP}$
COH	$\mathsf{COH} \equiv_{\mathrm{sW}} \widehat{\mathsf{WBWT}_2}$	WBWT_2
DNC_n	$\mathsf{DNC}_n \equiv_{\mathrm{sW}} \widehat{\mathsf{ACC}_n}$	$\mathsf{ACC}_n \equiv_{\mathrm{sW}} \mathsf{LLPO}_n$
NASH	$\mathsf{NASH} \equiv_{\mathrm{sW}} \mathsf{AUC}_{[0,1]}^*$	$\mathsf{AUC}_{[0,1]} \equiv_{\mathrm{sW}} \mathsf{RDIV}$
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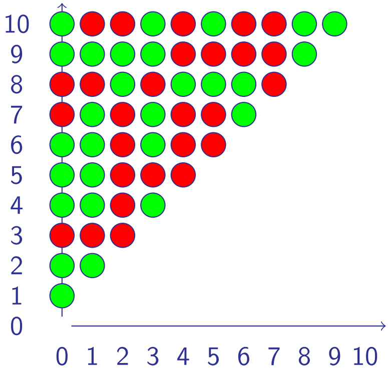
Ramsey's Theorem



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Theorem (Ramsey 1930)

Every coloring $c : [\mathbb{N}]^n \to k$ admits an infinite homogeneous set $M \subseteq \mathbb{N}$.

- Here $[M]^n$ denotes the set of n–element subsets of $M \subseteq \mathbb{N}$.
- We identify k with $\{0, 1, ..., k-1\}$ for all $k \in \mathbb{N}$.
- A set $M \subseteq \mathbb{N}$ is called homogeneous for the coloring c, if there is some $i \in k$ such that $c(A) = i$ for all $A \in [M]^n$.
- By $\mathcal{C}_{n,k}$ we denote the set of colorings $c : [\mathbb{N}]^n \to k$.
- By $\mathrm{RT}^n_k : \mathcal{C}_{n,k} \rightrightarrows 2^{\mathbb{N}}$ we denote the corresponding multi-valued function, where $\mathrm{RT}^n_k(c)$ contains exactly all infinite homogeneous sets $M \subseteq \mathbb{N}$ for c.
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Lower Bounds on Ramsey

Proposition (B. and Rakotoniaina 2015)

$\mathsf{C}_2^{(n)} \leq_{\mathrm{W}} \mathsf{RT}_2^n$ *for all* $n \geq 1$.

Proof.(Idea.) We note that $\mathsf{C}_2^{(n)} \equiv_{\mathrm{sW}} \mathsf{BWT}_2 \circ \lim_{2^{\mathbb{N}}}^{[n-1]}$. Let $p \in \mathrm{dom}(\mathsf{BWT}_2 \circ \lim_{2^{\mathbb{N}}}^{[n-1]})$ and $q := \lim_{2^{\mathbb{N}}}^{[n-1]}(p)$. Then

$$q(i_0) = \lim_{i_1 \to \infty} \lim_{i_2 \to \infty} \ldots \lim_{i_{n-1} \to \infty} p\langle i_{n-1}, \ldots, i_0 \rangle$$

for all $i_0 \in \mathbb{N}$. We compute the coloring $c : [\mathbb{N}]^n \to 2$ with

$$c\{i_0 < i_1 < \ldots. < i_{n-1}\} := p\langle i_{n-1}, i_{n-2}, \ldots, i_1, i_0 \rangle.$$

For $M \in \mathsf{RT}_2^n$ we obtain $c(M) \in \mathsf{BWT}_2(q)$. $\square$

Corollary

$\mathsf{WKL}^{(n)} \leq_{\mathrm{W}} \widehat{\mathsf{RT}_k^n}$ *for all* $n \geq 1$, $k \geq 2$.

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Products and Parallellization of Ramsey

Theorem (B. and Rakotoniaina 2015)

$\mathrm{RT}^n_{\mathbb{N}} \times \mathrm{RT}^{n+1}_k \leq_{\mathrm{sW}} \mathrm{RT}^{n+1}_{k+1}$ *for all* $n, k \geq 1$.

Proof. (Idea.) Given a coloring $c_1 : [\mathbb{N}]^n \to \mathbb{N}$ with finite range and a coloring $c_2 : [\mathbb{N}]^{n+1} \to k$ we construct a coloring $c^+ : [\mathbb{N}]^{n+1} \to k+1$ as follows:

$$c^+(A) := \begin{cases} c_2(A) & \text{if } A \text{ is homogeneous for } c_1 \\ k & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Then $\mathrm{RT}^{n+1}_2(c^+) \subseteq \mathrm{RT}^n_{\mathbb{N}}(c_1) \cap \mathrm{RT}^{n+1}_k(c_2)$ and hence the desired reduction follows. □

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$(\mathrm{RT}^n_k)^* \leq_{\mathrm{W}} \mathrm{RT}^{n+1}_2$ *for all* $n, k \geq 1$.

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Parallelization of Ramsey

Theorem (B. and Rakotoniaina 2015)

$\widehat{\mathrm{RT}^n_k} \leq_{\mathrm{sW}} \mathrm{RT}^{n+2}_2$ *for all* $n, k \geq 1$.

Proof. (Idea.) Given a sequence $(c_i)_i$ of colorings $c_i : [\mathbb{N}]^n \to k$, we compute a sequence $(d_m)_m$ of colorings $d_m \in \mathcal{C}_{n,k^m}$ that capture the products $(\mathrm{RT}^n_k)^m$ and a sequence $(d^+_m)_m$ of colorings $d^+_m : [\mathbb{N}]^{n+1} \to 2$ by

$$d^+_m(A) := \begin{cases} 0 & \text{if } A \text{ is homogeneous for } d_m \\ 1 & \text{otherwise} \end{cases}$$

for all $A \in [\mathbb{N}]^{n+1}$. Now, in a final step we compute a coloring $c : [\mathbb{N}]^{n+2} \to 2$ with

$$c(\{m\} \cup A) := d^+_m(A)$$

for all $A \in [\mathbb{N}]^{n+1}$ and $m < \min(A)$. Given an infinite homogeneous set $M \in \mathrm{RT}^{n+2}_2(c)$ we determine a sequence $(M_i)_i$ as follows: for each fixed $i \in \mathbb{N}$ we first search for a number $m > i$ in M and then we let $M_i := \{x \in M : x > m\}$. $\square$

Lower Bounds and Stability

Corollary

For all $n \geq 2$ we obtain:

- $\lim_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{SRT}^1_{\mathbb{N}}$
- $\lim \leq_{\mathrm{W}} \mathsf{SRT}^3_2$
- $\mathsf{WKL}' \leq_{\mathrm{W}} \mathsf{RT}^3_2$ *(Hirschfeldt and Jockusch 2015)*
- $\mathsf{WKL}^{(n)} \leq_{\mathrm{W}} \mathsf{SRT}^{n+2}_2$

- A coloring $c : [\mathbb{N}]^n \to k$ is called *stable*, if $\lim_{i \to \infty} c(A \cup \{i\})$ exists for all $A \in [\mathbb{N}]^{n-1}$.
- SRT^n_k is the restriction of RT^n_k to stable colorings.

Upper Bounds

Theorem (Cholak, Jockusch, Slaman 2009)

$\mathsf{RT}^n_k \leq_{\mathrm{W}} \mathsf{SRT}^n_k * \mathsf{COH}$ *for all* $n, k \geq 1$.

Theorem

$\mathsf{SRT}^{n+1}_k \leq_{\mathrm{W}} \mathsf{RT}^n_k * \lim$ *for all* $n, k \geq 1$.

Proof. (Idea.) In fact, we even proved $\mathsf{SRT}^{n+1}_k \equiv_{\mathrm{W}} (\mathsf{CRT}^n_k)'$. □

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$\mathsf{RT}^{n+1}_k \leq_{\mathrm{W}} \mathsf{RT}^n_k * \mathsf{WKL}'$ *for all* $n, k \geq 1$.

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Corollary

$\widehat{\mathsf{RT}^n_k} \equiv_{\mathrm{W}} \mathsf{WKL}^{(n)}$ *for all* $n \geq 1$, $k \geq 2$.

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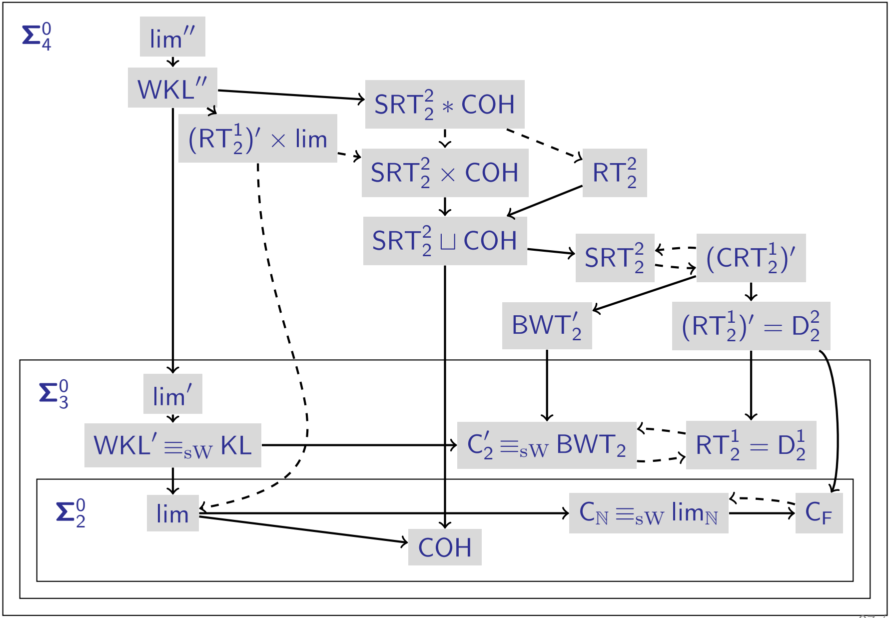
Ramsey's Theorem and Cohesiveness



The Squashing Theorem

Definition

$f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is called finitely tolerant if there is a computable $T :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $p, q \in \mathrm{dom}(f)$, $r \in \mathbb{N}^{\mathbb{N}}$, $k \in \mathbb{N}$: $((\forall n \geq k)(p(n) = q(n))$ and $r \in f(q)) \Longrightarrow T\langle r, k\rangle \in f(p)$.

- f finitely tolerant $\Longrightarrow f$ fractal.
- $\lim, \mathrm{BWT}_n, \mathrm{BWT}_{\mathbb{N}}, \mathrm{BWT}_{2^{\mathbb{N}}}, \mathrm{RT}^n_k, \mathrm{RT}^n_{\mathbb{N}}$ are finitely tolerant.

Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

Let $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and let f be finitely tolerant and total. Then $g \times f \leq_{\mathrm{W}} f \Longrightarrow \widehat{g} \leq_{\mathrm{W}} f$.

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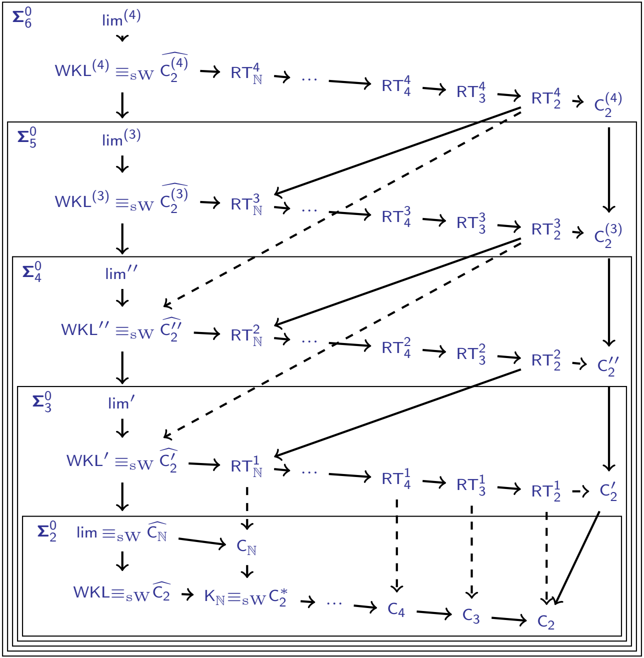
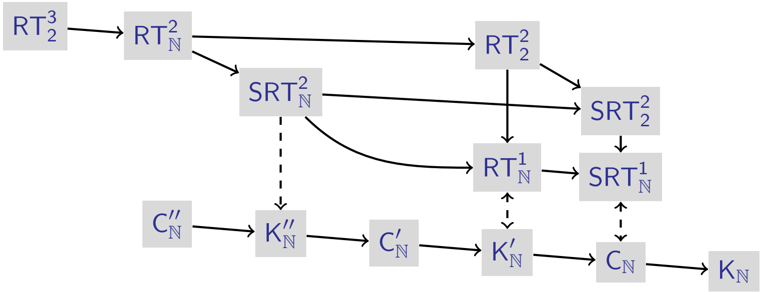
Ramsey's Theorem in the Weihrauch Lattice



Boundedness, Induction and Choice



Corollary (Jump of compact choice)

$\mathsf{K}'_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}$, $\mathsf{K}'_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{SRT}^2_2$, $\mathsf{K}'_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{SRT}^2_2 * \mathsf{SRT}^2_2$ and $\mathsf{K}^{(n)}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{SRT}^n_{\mathbb{N}}$ for $n \geq 2$.

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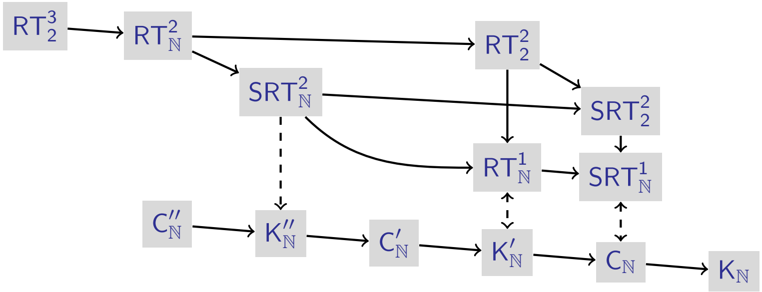

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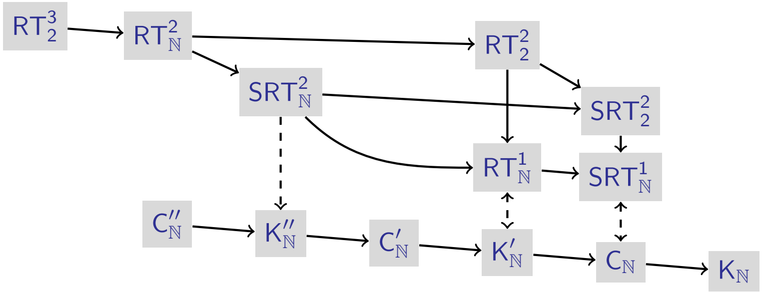

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Lowness

$$\boxed{\mathsf{L} = \mathsf{J}^{-1} \circ \mathsf{lim}}$$

The Uniform Low Basis Theorem

- $\mathsf{J} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, p \mapsto p'$ denotes the Turing jump.
- $\mathsf{J} \equiv_{\mathrm{sW}} \lim$ and $\mathsf{J}^{-1} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is computable.
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- $q \in \mathbb{N}^{\mathbb{N}}$ is low $:\iff q' \leq_{\mathrm{T}} \emptyset' \iff (\exists p \text{ comp.})\, \mathsf{L}(p) = q$.

Definition (B., de Brecht and Pauly 2011)

f is low $:\iff f \leq_{\mathrm{sW}} \mathsf{L}$.

- L is not a cylinder, hence $\leq_{\mathrm{sW}}$ cannot be replaced by $\leq_{\mathrm{W}}$.
- L is also not idempotent.

Theorem (B., de Brecht and Pauly 2011)

$\mathsf{C}_{\mathbb{R}} \leq_{\mathrm{sW}} \mathsf{L}$, *that is* $\mathsf{C}_{\mathbb{R}}$ *is low.*

This is a uniform version of the Low Basis Theorem.

Corollary

$\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^{\mathbb{N}}}$ *and* $\mathsf{BCT}_1 \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$ *are low.*

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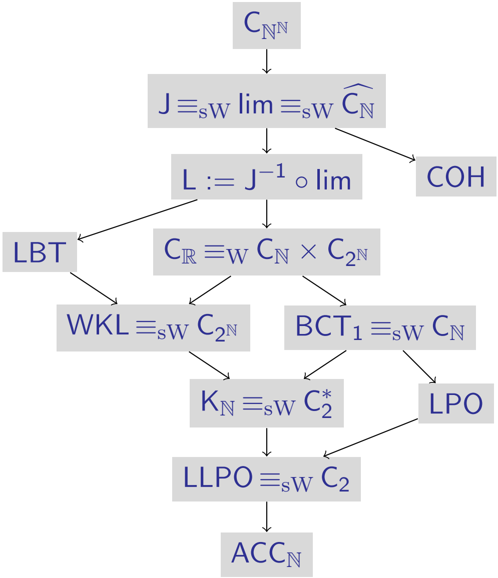
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- $f *_{\mathrm{s}} g := \sup\{f_0 \circ g_0 : f_0 \leq_{\mathrm{sW}} f$ and $g_0 \leq_{\mathrm{sW}} g\}$.
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- $\mathsf{L}_2 := \mathsf{J}^{-1} \circ \mathsf{J}^{-1} \circ \lim \circ \lim$ characterizes low$_2$ similarly as L characterizes lowness.
- f low$_2$ $:\iff f \leq_{\mathrm{sW}} \mathsf{L}_2$.

Theorem (B., Gherardi, Marcone 2012)

- f *low* $\iff f \leq_{\mathrm{sW}} \mathsf{L} \iff \lim *_{\mathrm{s}} f \leq_{\mathrm{W}} \lim$.
- f *low*$_2$ $\iff f \leq_{\mathrm{sW}} \mathsf{L}_2 \iff \lim' *_{\mathrm{s}} f \leq_{\mathrm{W}} \lim'$.

Theorem (B., Hendtlass and Kreuzer 2015)

COH *and* $\mathsf{WBWT}_{\mathbb{R}}$ *are low*$_2$ *but not low.*

The proof uses $\mathsf{WKL}' \equiv_{\mathrm{W}} \lim * \mathsf{COH}$ and the fact that WKL is low.

Genericity

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- $p \in \mathbb{N}^{\mathbb{N}}$ is 1–generic $:\iff$ p is a point of continuity of J.
- $\lim_{\mathsf{J}} := \mathsf{J}^{-1} \circ \lim \circ \mathsf{J}^{\mathbb{N}} = \mathsf{L} \circ \mathsf{J}^{\mathbb{N}}$ is the limit operator with respect to the jump topology (also called Π–topology).
- $p \in \mathbb{N}^{\mathbb{N}}$ is called limit computable in the jump $:\iff$ there is a computable sequence $(p_n)_n$ such that $\lim_{n\to\infty} \mathsf{J}(p_n) = \mathsf{J}(p)$.

Proposition (B., de Brecht and Pauly 2011)

- *$p \in \mathbb{N}^{\mathbb{N}}$ 1–generic and limit computable $\Longrightarrow$ p limit computable in the jump.*
- *$f \in \mathbb{N}^{\mathbb{N}}$ diagonally non-computable and $p \in \mathbb{N}^{\mathbb{N}}$ limit computable in the jump $\Longrightarrow f \not\leq_{\mathrm{T}} p$.*

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- 1-GEN : $2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, p \mapsto \{q : q$ is 1–generic in $p\}$.

Proposition (B., Hendtlass and Kreuzer 2015)

$f \leq_{\mathrm{W}} \lim_J$ if f has a limit computable realizer with only 1–generic points in its range.

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- 1-WGEN denotes the problem of weakly 1–generics (defined similarly as above).
- $\Pi_1^0\mathsf{G}$ denotes the so called Π_1^0–genericity problem studied in reverse mathematics (interpreted in the straightforward sense).
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$\mathsf{ACC}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{BCT}_0'$.

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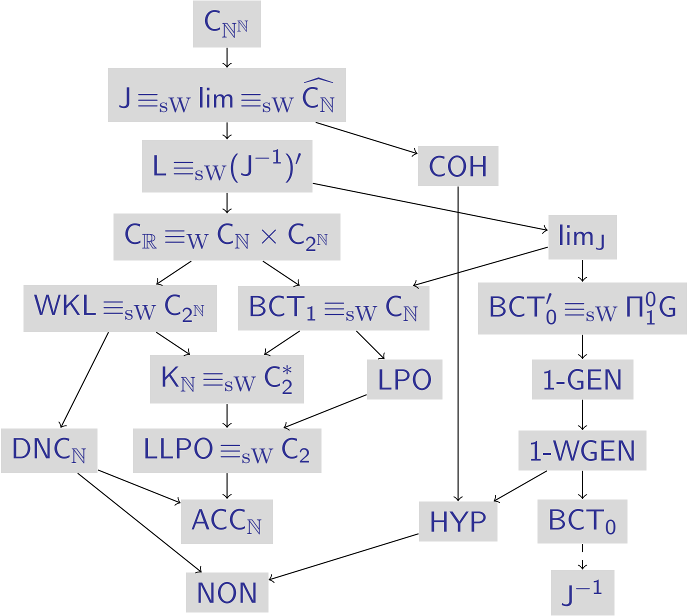
Genericity in the Weihrauch Lattice



Randomness

Martin-Löf Randomness

- $\mathsf{MLR} : 2^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$, the problem of Martin-Löf randomness is defined by $\mathsf{MLR}(p) := \{q \in 2^{\mathbb{N}} : q$ is Martin-Löf random relative to $p\}$.
- q is called Martin-Löf random relative to p, if for every sequence $(U_i)_i$ of open sets $U_i \subseteq 2^{\mathbb{N}}$ that is computable relative to p with $\mu(U_i) < 2^{-i}$, we obtain $p \notin \bigcap_{i=0}^{\infty} U_i$.
- MLR is densely realized, hence $\mathsf{C}_2 \not\leq_{\mathrm{W}} \mathsf{MLR}$.
- MLR is parallelizable and hence idempotent.

Proposition (B., Gherardi and Hölzl 2015)

$\mathsf{MLR} * \mathsf{MLR} \leq_{\mathrm{W}} \mathsf{MLR}$.

Follows from van Lambalgen's Theorem.

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Theorem (B. and Pauly 2013)

$\mathsf{MLR} \equiv_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$.

Proof. (Sketch.) $(\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL}) \leq_{\mathrm{W}} \mathsf{MLR}$: It suffices to prove $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * \mathsf{MLR}$. By Kučera's Lemma, every Martin-Löf random real p is a path in every infinite binary tree T of positive measure up to some finite prefix. Using $\mathsf{C}_{\mathbb{N}}$ we can cut away longer and longer prefixes of p until we find a path in T.

$\mathsf{MLR} \leq_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}} \to \mathsf{WWKL})$: Given some h with $\mathsf{WWKL} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} * h$ we need to prove that $\mathsf{MLR} \leq_{\mathrm{W}} h$. Given some universal Martin-Löf test $(U_i)_i$, the complement $A_0 := 2^{\mathbb{N}} \setminus U_0$ is a closed set of positive measure and given the corresponding tree T with $A = [T]$ the function h will deliver some sequence q that can be converted into a Martin-Löf random real by a finite mind change computation. This computation can be converted into a regular computation that yields a Martin-Löf random real. □

Characterization of Martin-Löf Randomness

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Quantitative Versions of WWKL

Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016)

By ε-WWKL $:\subseteq \mathrm{Tr} \rightrightarrows 2^{\mathbb{N}}$ we denote the restriction of WKL to $\mathrm{dom}(\varepsilon\text{-WWKL}) := \{T : \mu([T]) > \varepsilon\}$ for $\varepsilon \in \mathbb{R}$.

Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016 and B., Gherardi and Hölzl 2015)

ε-WWKL $\leq_{\mathrm{W}}$ δ-WWKL $\iff$ $\varepsilon \geq \delta$ for all $\varepsilon, \delta \in [0,1]$.

Proof. (Idea) "$\Longrightarrow$" Assume $\varepsilon < \delta$. Then there are positive integers a, b with $\varepsilon < \frac{a}{b} \leq \delta$. We consider

- $\mathsf{C}_{a,b}$ which is C_b restricted to sets $A \subseteq \{0, ..., b-1\}$ with $|A| \geq a$.

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Theorem (Dorais, Dzhafarov, Hirst, Mileti and Shafer 2016 and B., Gherardi and Hölzl 2015)

ε-WWKL $\leq_{\mathrm{W}} \delta$-WWKL $\iff \varepsilon \geq \delta$ *for all* $\varepsilon, \delta \in [0,1]$.

Proof. (Idea) "$\Longrightarrow$" Assume $\varepsilon < \delta$. Then there are positive integers a, b with $\varepsilon < \frac{a}{b} \leq \delta$. We consider

- $C_{a,b}$ which is C_b restricted to sets $A \subseteq \{0, ..., b-1\}$ with $|A| \geq a$.

Then $C_{a,b} \leq_{\mathrm{W}} \varepsilon$-WWKL and $C_{a,b} \not\leq_{\mathrm{W}} \delta$-WWKL. Hence ε-WWKL $\not\leq_{\mathrm{W}} \delta$-WWKL $\square$

Proposition (B., Hendtlass and Kreuzer 2015)

ε-WWKL *is not parallelizable for* $\varepsilon \in [0,1)$.

Quantitative Versions of WWKL

- $(1-*)\text{-WWKL} :\subseteq \mathrm{Tr}^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}, (T_i)_i \mapsto \bigsqcup_{i=0}^{\infty} (1-2^{-i})\text{-WWKL}(T_i)$

Theorem (B., Hendtlass and Kreuzer 2015)

$(1-*)$-WWKL *is parallelizable.*

Proposition (B., Hendtlass and Kreuzer 2015)

$\mathrm{ACC}_{\mathbb{N}} \leq_{\mathrm{W}} (1-*)\text{-WWKL}$.

Corollary

$\mathrm{DNC}_{\mathbb{N}} \leq_{\mathrm{W}} (1-*)\text{-WWKL}$.

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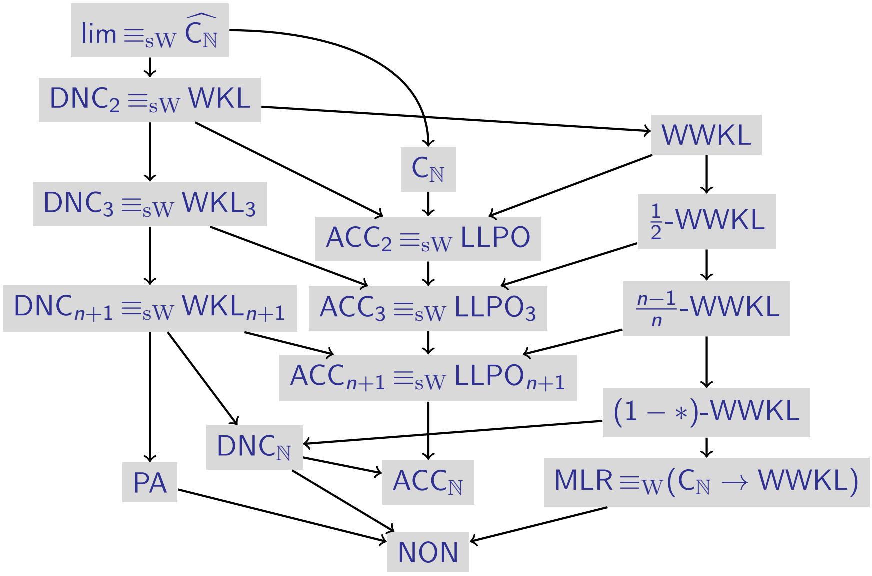
Theorem (B., Gherardi and Hölzl 2015)

MLR $<_{\mathrm{W}}$ $(1-*)$-WWKL.

Proof. (Sketch) We use a universal Martin-Löf test, which is a computable sequence $(U_i)_i$ of c.e. open sets $U_i \subseteq 2^{\mathbb{N}}$ such that $\mu(U_i) < 2^{-n}$ and $\bigcap_{i=0}^{\infty} U_i$ is exactly the set of all sequences which are not Martin-Löf random. Hence, $A_i := 2^{\mathbb{N}} \setminus U_i$ is a co-c.e. closed set with $\mu(A_i) > 1 - 2^{-n}$ and each A_i only contains Martin-Löf random sequences. Hence, we can compute a corresponding sequence $(T_i)_i$ of infinite binary trees with $[T_i] = A_i$. Upon input of this sequence $(1-*)$-WWKL yields a Martin-Löf random sequence. The entire argument can be relativized, i.e., it also works in presence of some oracle $p \in 2^{\mathbb{N}}$. This yields the reduction MLR $\leq_{\mathrm{W}}$ $(1-*)$-WWKL. In order to see that the reduction is strict, one has to take into account that MLR is densely realized. $\square$

From MLR to WWKL in the Weihrauch Lattice



Uniform Theorem of Kurtz

Theorem of Kurtz. Every 2–random computes a 1–generic.

Theorem (B., Hendtlass and Kreutzer 2015)

1-GEN $<_{\mathrm{W}} (1-*)$-WWKL$'$.

Proof. (Idea) We apply the "fireworks technique" of Rumyantsev and Shen to get a uniform reduction. □

Theorem (B., Hendtlass and Kreutzer 2015)

$\mathrm{BCT}_0' \not\leq_{\mathrm{W}} \mathrm{WWKL}^{(n)}$ *for all* $n \in \mathbb{N}$.

Proof. (Idea) There exists a co-c.e. comeager set $A \subseteq 2^{\mathbb{N}}$ such that no point of A is low for Ω. $\mathrm{WWKL}^{(n)}$ has a realizer that maps computable inputs to outputs that are low for Ω for $n \geq 1$. □

Corollary

$\mathrm{BCT}_0' \not\leq_{\mathrm{W}}$ 1-GEN.

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Summary on Weihrauch Complexity

- Weihrauch complexity is a uniform and resource sensitive computable version of reverse mathematics.
- It measures the amount of resources needed to compute certain realizers of theorems.
- Positive and negative results are directly constructed without any need for further models.
- Results have immediate interpretations in computable analysis.
- Many results from reverse mathematics are fully uniform with only one usage of the resource.
- Sometimes proofs can be transferred, sometimes completely new methods have to be developed.
- The Weihrauch lattice can be seen as a refinement of the Borel hierarchy for functions and hence methods of descriptive set theory and topology can be applied directly.
- Many complexity classes have direct computational interpretations.

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Bibliography styles: alphabetically, chronologically, by publication type. BibTeX file: wei.bib

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