Strong reductions for extended formulations

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Semidefinite & Matrix Methods in Optimization & Communication
Singapore
Feb 2, 2016

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Semidefinite programming (SDP):

- generalizes linear programming
- also efficient in theory
- covers much of tractable convex optimization
- better approximations for hard problems!

Approximation algorithms using LPs and SDPs

VertexCover	LP
SetCover	LP
FacilityLocation	LP
MaxCut	SDP
SparsestCut	SDP
MaxCSP	Sum-of-squares (SDP)
⋮	

Unique Games Conjecture $\Rightarrow$ SDP is optimal algorithm for MaxCSPs

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- We know due to Chan et al. (2013) and Lee et al. (2014) that the Sherali-Adams and Lasserre hierarchies are optimal for CSPs.
- However the picture isn't so clear for other classes of problems...

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- Braun et al. (2015) came up with the notion of affine reductions to show for example a $\frac{3}{2} - \varepsilon$ LP inapproximability for VERTEXCOVER by reducing from MAXCUT.
- Bazzi et al. (2015) improved this to $2 - \varepsilon$ by reducing from 1F-CSP, together with intermediate Sherali-Adams reductions to show hardness of 1F-CSP.

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- Generalize the reductions from Braun et al. (2015) to drop the dependency on affineness.
- Generalize the reductions to fractional optimization problems such as e.g., SPARSESTCUT.
- Use this to prove new LP and SDP hardness results as well as some old ones.

Summary of Results

Problem	Factor	Source	Paradigm
MaxCut	$\frac{15}{16} + \varepsilon$	Max-3-XOR/0	SDP
SparsestCut, tw(supply) $= O(1)$	$2 - \varepsilon$	MaxCut	LP
SparsestCut, tw(supply) $= O(1)$	$\frac{16}{15} - \varepsilon$	MaxCut	SDP
BalancedSeparator, tw(demand) $= O(1)$	$\omega(1)$	UniqueGames	LP
IndependentSet	$\omega(n^{1-\varepsilon})$	Max-k-CSP	Lasserre $O(n^{\varepsilon})$ rounds
Matching, 3-regular	$1 + \varepsilon/n^2$	Matching	LP
1F-CSP Q-$\neq$-CSP	$\omega(1)$	UniqueGames	LP

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- Inapproximability of 1F-CSP proves a $2 - \varepsilon$ inapproximability for VertexCover.
- Inapproximability of Q-$\neq$-CSP proves a $Q - \varepsilon$ inapproximability for Q-regular hypergraph cover.

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- Inapproximability of 1F-CSP proves a $2 - \varepsilon$ inapproximability for VERTEXCOVER.
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- Lasserre relaxations are suboptimal for INDEPENDENTSET: there is an LP of linear size with a $2\sqrt{n}$ approximation factor.

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- Inapproximability of Q-$\neq$-CSP proves a $Q - \varepsilon$ inapproximability for Q-regular hypergraph cover.
- Lasserre relaxations are suboptimal for INDEPENDENTSET: there is an LP of linear size with a $2\sqrt{n}$ approximation factor.
- LPs on bounded treewidth graphs is easy: we show the existence of uniform LPs of size $O(n^k)$ for MATCHING, INDEPENDENTSET, VERTEXCOVER, MAXCUT and UNIQUEGAMES on graphs of treewidth k.

An abstract view of optimization problems

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An *optimization problem* $\mathcal{P} = (\mathcal{S}, \mathfrak{I}, \text{val})$ consists of

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- and a real valued objective $\text{val}\colon \mathfrak{I} \times \mathcal{S} \to \mathbb{R}$.

- $\text{val}_{\mathcal{I}}(s)$: quality of a solution $s \in \mathcal{S}$ w.r.t instance $\mathcal{I} \in \mathfrak{I}$
- $\text{OPT}(\mathcal{I}) := \min_{s \in \mathcal{S}} \text{val}_{\mathcal{I}}(s)$

Example: VERTEXCOVER

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- Instances: all induced subgraphs H of G;
- Feasible solutions: all vertex covers X of G;

Example: VertexCover

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Instances all induced subgraphs H of G;

Feasible solutions all vertex covers X of G;

Measure $\mathrm{val}_H(X) \coloneqq |X \cap V(H)|$.

(C, S)-approximations of optimization problems

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How to measure the quality of approximations to a problem $\mathcal{P} = (\mathcal{S}, \mathfrak{I}, \text{val})$?

- $C : \mathfrak{I} \to \mathbb{R}$, called the *completeness guarantee*
- $S : \mathfrak{I} \to \mathbb{R}$, called the *soundness guarantee*
- $\text{OPT}\,(\mathcal{I}) \geq S(\mathcal{I}) \Rightarrow$ optimum over the LP or SDP relaxation is bounded below by $C(\mathcal{I})$.
- $\mathfrak{I}^S := \{\mathcal{I} \mid \mathcal{I} \in \mathfrak{I}, \text{OPT}\,(\mathcal{I}) \geq S(\mathcal{I})\}$ is the set of sound instances.
- Approximation ratio: C/S

(C, S)-approximate LP formulation

A linear program $Ax \leq b$ with $x \in \mathbb{R}^r$ s.t.

Feasible solutions vectors $x^s \in \mathbb{R}^r$ for every $s \in \mathcal{S}$ satisfying

$$Ax^s \leq b,$$

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Instances as affine functions $w_{\mathcal{I}} \colon \mathbb{R}^r \to \mathbb{R}$ for all $\mathcal{I} \in \mathfrak{I}^S$ s.t.

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Achieving (C, S) guarantee

$$\mathcal{I} \in \mathfrak{I}^S \Rightarrow \min \{ w_{\mathcal{I}}(x) \mid Ax \leq b \} \geq C(\mathcal{I})$$

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- *Size* of the formulation: number of inequalities in $Ax \leq b$
- *LP formulation complexity*, $\mathrm{fc}_{\mathrm{LP}}(\mathcal{P}, C, S)$: min size of all formulations.

(C, S)-approximate SDP formulation

Semidefinite program $\{X \in \mathbb{S}_+^r \mid \mathcal{A}(X) = b\}$ and:

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- *Size* of the formulation: r
- *SDP formulation complexity*, $\mathrm{fc}_{\mathrm{SDP}}(\mathcal{P}, \mathcal{C}, \mathcal{S})$: min size of all formulations.

Example of an LP formulation

Example

Recall the following LP for the VertexCover problem:

$$\min \sum_i x_i \qquad \text{s.t.}$$
$$x_i + x_j \geq 1 \quad \forall \{i, j\} \in E(G)$$
$$1 \geq x_i \geq 0$$

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- Every vertex cover X of $G \to \mathbb{1}_X$ of the LP
- Every instance H (induced subgraph of G) corresponds to the affine function $\langle \mathbb{1}_H, . \rangle$
- $C/S = \frac{1}{2}$ for this LP

Slack matrix

Definition

The (C, S)-approximate *slack matrix* of an optimization problem $\mathcal{P}$ is the $\mathfrak{I}^S \times \mathcal{S}$ matrix $M_{\mathcal{P},C,S}(\mathcal{I}, s) := \mathrm{val}_{\mathcal{I}}(s) - C(\mathcal{I})$.

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- **Nonnegative factorization of size** r: $M_{\mathcal{P},C,S} = \sum_{i=1}^{r} M_i$, $M_i \geq 0$ and $\mathrm{rk}\, M_i = 1$
- **PSD factorization of size** r: $M_{\mathcal{P},C,S}(\mathcal{I}, s) = \mathrm{Tr}[A_{\mathcal{I}} B_s]$, $A_{\mathcal{I}}, B_s \in \mathbb{S}_+^r$

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- **Nonnegative rank,** $\text{rk}_+ M_{\mathcal{P},C,S}$: min size of nonnegative factorization of $M_{\mathcal{P},C,S}$
- **PSD rank,** $\text{rk}_{\text{psd}} M_{\mathcal{P},C,S}$: min size of psd factorization of $M_{\mathcal{P},C,S}$

Why should we care?

Because of the following well known theorem due to Yannakakis (1988):

Theorem

Let $\mathcal{P} = (\mathcal{S}, \mathfrak{I}, \mathrm{val})$ *be an optimization problem with completeness guarantee* C *and soundness guarantee* S*.Then*

$$\mathrm{fc}_{\mathrm{LP}}(\mathcal{P}, C, S) = \mathrm{rk}_{+} M_{\mathcal{P},C,S},$$
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Common strategy to bound the formulation complexity is to bound the corresponding ranks by

- Rectangle covering arguments
- Common information
- (Quantum) Communication complexity
- ...

Reductions v1.0

Definition (Braun et al. (2015))

A reduction from $(\mathcal{P}_1, C_1, S_1)$ to $(\mathcal{P}_2, C_2, S_2)$ consists of two maps $*$ from $\mathfrak{I}_1^{S_1} \to \mathfrak{I}_2^{S_2}$ and $* : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\mathrm{val}_{\mathcal{I}_1}(s_1) = \mathrm{val}_{\mathcal{I}_1^*}(s_1^*) + \mu(\mathcal{I}_1),$$
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– Note the affine relationship between the objective values of the two problems.

Reductions with "distortion"

Definition

A reduction from $\mathcal{P}_1$ to $\mathcal{P}_2$ consists of

- Two maps from $* : \mathfrak{I}_1 \to \mathfrak{I}_2$ and $* : \mathcal{S}_1 \to \mathcal{S}_2$
- Two nonnegative $\mathfrak{I}_1 \times \mathcal{S}_1$ matrices M_1 and M_2

such that

$$\mathrm{val}_{\mathcal{I}_1}(s_1) - C_1(\mathcal{I}_1) = \left(\mathrm{val}_{\mathcal{I}_1^*}(s_1^*) - C_2(\mathcal{I}_1^*)\right) M_1(\mathcal{I}_1, s_1) + M_2(\mathcal{I}_1, s_1)$$

and OPT $(\mathcal{I}_1^*) \geq S_2(\mathcal{I}_1^*)$ whenever OPT $(\mathcal{I}_1) \geq S_1(\mathcal{I}_1)$.

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The matrices M_1 and M_2

- encode additional "computation" in the reduction
- can be an arbitrary function of the solutions and instances

The price of distortion

Since there are no free lunches...

Theorem

Let $(\mathcal{P}_1, C_1, S_1)$ *and* $(\mathcal{P}_2, C_2, S_2)$ *be two problems with a reduction from* $\mathcal{P}_1$ *to* $\mathcal{P}_2$*. Then*

$$\mathrm{fc}_{\mathrm{LP}}(\mathcal{P}_1) \leq \mathrm{rk}_+(M_2) + \mathrm{rk}_+(M_1) + \mathrm{rk}_+(M_1) \cdot \mathrm{fc}_{\mathrm{LP}}(\mathcal{P}_2) + O(1),$$

$$\mathrm{fc}_{\mathrm{SDP}}(\mathcal{P}_1) \leq \mathrm{rk}_{\mathrm{psd}}(M_2) + \mathrm{rk}_{\mathrm{psd}}(M_1) + \mathrm{rk}_{\mathrm{psd}}(M_1) \cdot \mathrm{fc}_{\mathrm{SDP}}(\mathcal{P}_2) + O(1),$$

where M_1 *and* M_2 *are the matrices in the reduction.*

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where M_1 and M_2 are the matrices in the reduction.

- Clearly the matrices M_1 and M_2 should have low complexity to obtain useful reductions.

Proof Sketch

Reformulate the reduction relationship in terms of matrices:

$$M_{\mathcal{P}_1, C_1, S_1} = \left(F_{\mathfrak{I}} M_{\mathcal{P}_2, C_2, S_2} F_{\mathcal{S}}\right) \circ M_1 + M_2$$

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- $F_{\mathfrak{I}}$ is a $\mathfrak{I}_1 \times \mathfrak{I}_2$ matrix, encoding $* : \mathfrak{I}_1 \to \mathfrak{I}_2$
- $F_{\mathcal{S}}$ is a $\mathcal{S}_2 \times \mathcal{S}_1$ matrix, encoding $* : \mathcal{S}_1 \to \mathcal{S}_2$

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Further simplify the matrix relationship:

$$M_{\mathcal{P}_1,C_1,S_1} = \left(F_{\mathfrak{I}} \widetilde{M}_{\mathcal{P}_2,C_2,S_2} F_{\mathcal{S}}\right) \circ M_1 + \operatorname{diag}(F_{\mathfrak{I}} a) \cdot M_1 + M_2$$

where $M_{\mathcal{P}_2,C_2,S_2} = \widetilde{M}_{\mathcal{P}_2,C_2,S_2} + a\mathbb{1}$ (incurs the $O(1)$ factors) and use the identities

- $\operatorname{rk}_+(A \circ B) \leq (\operatorname{rk}_+ A) \cdot (\operatorname{rk}_+ B)$
- $\operatorname{rk}_+(ABC) \leq \operatorname{rk}_+ B$
- $\operatorname{rk}_+(A + B) \leq \operatorname{rk}_+ A + \operatorname{rk}_+ B$

Fractional optimization problems

- An optimization problem where the objective $\mathrm{val}_{\mathcal{I}}$ is of the form $\mathrm{val}_{\mathcal{I}}^{n} / \mathrm{val}_{\mathcal{I}}^{d}$.
- Efficient LP based algorithms are used to find an optimal value of a linear combination of $\mathrm{val}_{\mathcal{I}}^{n}$ and $\mathrm{val}_{\mathcal{I}}^{d}$.

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Example

The SPARSESTCUT problem of

- $c : E(K_n) \to \mathbb{R}_{\geq 0}$, called the *capacity* function
- $d : E(K_n) \to \mathbb{R}_{\geq 0}$ called the *demand* function

The objective function for a cut s is to minimize $\frac{\sum_{i \in s, j \notin s} c(i,j)}{\sum_{i \in s, j \notin s} d(i,j)}$.

LP formulation for a fractional problem

A linear program $Ax \leq b$ with $x \in \mathbb{R}^r$ s.t.:

Feasible solutions as vectors $x^s \in \mathbb{R}^r$ for every $s \in \mathcal{S}$ satisfying

$$Ax^s \leq b \qquad \text{for all } s \in \mathcal{S},$$

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Instances as a pair of affine functions $w_{\mathcal{I}}^n, w_{\mathcal{I}}^d \colon \mathbb{R}^r \to \mathbb{R}$ for all $\mathcal{I} \in \mathfrak{I}^S$ satisfying

$$w_{\mathcal{I}}^n(x^s) = \mathrm{val}_{\mathcal{I}}^n(s)$$
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$$w^d_{\mathcal{I}}(x^s) = \text{val}^d_{\mathcal{I}}(s)$$

Achieving (C, S) approximation guarantee If $\mathcal{I} \in \mathfrak{I}^S$

$$Ax \leq b \Rightarrow \begin{cases} w^d_{\mathcal{I}}(x) \geq 0 \\ w^n_{\mathcal{I}}(x) \geq C(\mathcal{I}) w^d_{\mathcal{I}}(x) \end{cases}$$

Example of an LP formulation for a fractional problem

Example

A common LP relaxation for the SPARSESTCUT problem with *capacity* function c and *demand* function d is the following

$$\min \sum_{i,j} c(i,j) x_{ij} \qquad \text{s.t.}$$

$$\sum_{i,j} d(i,j) x_{ij} \geq \alpha \sum_{i,j} d(i,j)$$

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$$1 \geq x_{ij} \geq 0$$

– This is the LP used by Gupta et al. (2013); the value of α is found by binary search.

Slack matrix for fractional optimization problems

Definition

The (C, S)-approximate slack matrix for a fractional optimization problem $\mathcal{P}$ is the $2\mathfrak{I}^S \times \mathcal{S}$ matrix of the form

$$M_{\mathcal{P},C,S} = \begin{bmatrix} M^{(d)}_{\mathcal{P},C,S} \\ M^{(n)}_{\mathcal{P},C,S} \end{bmatrix}$$

where $M^{(d)}_{\mathcal{P},C,S}, M^{(n)}_{\mathcal{P},C,S}$ are nonnegative $\mathfrak{I}^S \times \mathcal{S}$ matrices with entries

$$M^{(d)}_{\mathcal{P},C,S}(\mathcal{I}, s) := \mathrm{val}^d_{\mathcal{I}}(s)$$
$$M^{(n)}_{\mathcal{P},C,S}(\mathcal{I}, s) := \mathrm{val}^n_{\mathcal{I}}(s) - C(\mathcal{I})\,\mathrm{val}^d_{\mathcal{I}}(s).$$

Reductions between fractional problems

Definition

A reduction from $\mathcal{P}_1$ to $\mathcal{P}_2$ consists of

- Two maps $* : \mathfrak{I}_1 \to \mathfrak{I}_2$ and $* : \mathcal{S}_1 \to \mathcal{S}_2$
- Four nonnegative $\mathfrak{I}_1 \times \mathcal{S}_1$ matrices $M_1^{(n)}, M_1^{(d)}, M_2^{(n)}, M_2^{(d)}$

such that

$$M^{(n)}_{\mathcal{P}_1, C_1, S_1}(\mathcal{I}_1, S_1) = M^{(n)}_{\mathcal{P}_2, C_2, S_2}(\mathcal{I}_1^*, S_1^*) M_1^{(n)}(\mathcal{I}_1, s_1) + M_2^{(n)}(\mathcal{I}_1, s_1),$$

$$\mathrm{val}^d_{\mathcal{I}_1}(s_1) = \mathrm{val}^d_{\mathcal{I}_1^*}(s_1^*) M_1^{(d)}(\mathcal{I}_1, S_1) + M_2^{(d)}(\mathcal{I}_2, S_2),$$

$$\mathrm{OPT}\,(\mathcal{I}_1) \geq S_1(\mathcal{I}_1) \Rightarrow \mathrm{OPT}\,(\mathcal{I}_1^*) \geq S_2(\mathcal{I}_1^*)$$

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$$\mathrm{OPT}\,(\mathcal{I}_1) \geq S_1(\mathcal{I}_1) \Rightarrow \mathrm{OPT}\,(\mathcal{I}_1^*) \geq S_2(\mathcal{I}_1^*)$$

- As before, the matrices $M_1^{(n)}, M_1^{(d)}, M_2^{(n)}, M_2^{(d)}$ encode additional "computation" in the reduction.

Price of distortion - Part II

Theorem

Let $\mathcal{P}_1, C_1, S_1)$ and $\mathcal{P}_2, C_2, S_2)$ be two fractional problems with a reduction from $\mathcal{P}_1$ to $\mathcal{P}_2$. Then

$$\mathrm{fc}_{\mathrm{LP}}(\mathcal{P}_1) \leq \mathrm{rk}_{\mathrm{LP}} \begin{bmatrix} M_2^{(n)} \\ M_2^{(d)} \end{bmatrix} + \mathrm{rk}_{\mathrm{LP}} \begin{bmatrix} M_1^{(n)} \\ M_1^{(d)} \end{bmatrix} + \mathrm{rk}_{+} \begin{bmatrix} M_1^{(n)} \\ M_1^{(d)} \end{bmatrix} \cdot \mathrm{fc}_{\mathrm{LP}}(\mathcal{P}_2),$$

$$\mathrm{fc}_{\mathrm{SDP}}(\mathcal{P}_1) \leq \mathrm{rk}_{\mathrm{SDP}} \begin{bmatrix} M_2^{(n)} \\ M_2^{(d)} \end{bmatrix} + \mathrm{rk}_{\mathrm{SDP}} \begin{bmatrix} M_1^{(n)} \\ M_1^{(d)} \end{bmatrix} + \mathrm{rk}_{\mathrm{psd}} \begin{bmatrix} M_1^{(n)} \\ M_1^{(d)} \end{bmatrix} \cdot \mathrm{fc}_{\mathrm{SDP}}(\mathcal{P}_2),$$

where $M_1^{(n)}, M_2^{(d)}, M_2^{(n)}, M_2^{(d)}$ are the matrices in the reduction.

From Sherali-Adams to general LP reductions

Definition

A *one-free bit* CSP (1F-CSP) is a CSP where every clause has *exactly* two satisfying assignments over its free variables.

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Theorem

With small numbers $\eta, \varepsilon, \delta > 0$ *positive integers* t, q, Δ *we have for any* $0 < \zeta < 1$ *and* n *large enough*

$$\mathrm{fc}_{\mathrm{LP}}(\textsc{UniqueGames}_{\Delta}(n, q), 1 - \zeta, \delta) - n\Delta^{t} q^{t+1} \leq \mathrm{fc}_{\mathrm{LP}}(\text{1F-CSP}, (1 - \varepsilon)(1 - \zeta t), \eta)$$

Definition (UNIQUEGAMES$_\Delta(n, q)$)

Let n, q and Δ be positive integer parameters. The maximization problem UNIQUEGAMES$_\Delta(n, q)$ consists of

instances All edge-weighted Δ-regular bipartite graphs (G, w) with partite sets $\{0\} \times [n]$ and $\{1\} \times [n]$ with every edge $\{i, j\}$ labeled with a permutation $\pi_{i,j} \colon [q] \to [q]$ such that $\pi_{i,j} = \pi_{j,i}^{-1}$.

feasible solutions All functions $s \colon \{0, 1\} \times [n] \to [q]$ called *labelings* of the vertices.

measure The weighted fraction of correctly labeled edges, i.e., edges $\{i, j\}$ with $s(i) = \pi_{i,j}(s(j))$:

$$\mathrm{val}_{(G,w)}(s) := \frac{\sum_{\substack{\{i,j\} \in E(G) \\ s(i)=\pi_{i,j}(s(j))}} w(i, j)}{\sum_{\{i,j\} \in E(G)} w(i, j)}$$

Reducing UNIQUEGAMES to 1F-CSP

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- Let $v \in V(G)$ and $u_1, \ldots, u_t$ adjacent to v.
- There is a clause $C(v, u_1, \ldots, u_t, x, S)$ for any $x \in \{-1, 1\}^q$ and $S \subseteq [q]$ of size $q(1 - \varepsilon)$ that is an "approximate local test" of a correct labeling.

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- Feasible solutions are translated via the long code of the labeling s, i.e. $s^*(\langle v, z\rangle) := z_{s(v)}$.

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- Feasible solutions are translated via the long code of the labeling s, i.e. $s^*(\langle v, z\rangle) := z_{s(v)}$.
- Define the matrix

$$M_{v,u_1,\ldots,u_t}((G, w, \pi), s) := \mathbb{E}_{\mathbf{x},\mathbf{S}}\left[C(v, u_1, \ldots, u_t, \mathbf{x}, \mathbf{S})[s^*]\right] - (1-\varepsilon)\left(\sum_{i\in[t]} \chi[s(v) = \pi_{v,u_i}(s(u_i))] - t + 1\right)$$

Reducing UNIQUEGAMES to 1F-CSP

– It turns out that the matrix M_2 in the reduction is $M_2((G, w, \pi), s) = \frac{1}{t(1-\varepsilon)} \mathbb{E}\left[M_{\mathbf{v},\mathbf{u_1},\dots,\mathbf{u_t}}((G, w, \pi), s)\right]$

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- We show that it has nonnegative rank at most $n\Delta^t q^{t+1}$ by "unrolling" the expectation.
- Note that we do not have to argue about Sherali-Adams solutions as in Bazzi et al. (2015); this is a simple LP reduction in our framework.
- The base LP hardness of UniqueGames is due to Charikar et al. (2009) and Chan et al. (2013).

Reducing UNIQUEGAMES to Q-$\neq$-CSP

Definition

A *not equal CSP* (Q-$\neq$-CSP for short) is a CSP with value set $\mathbb{Z}_Q$, the additive group of integers modulo Q, where every clause has the form $\bigwedge_{i=1}^{k} x_i \neq a_i$ for some constants a_i.

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Theorem

With small numbers $\eta, \varepsilon, \delta > 0$ positive integers t, q, Δ, we have for any $0 < \zeta < 1$ and n large enough

$$\mathrm{fc}_{\mathrm{LP}}(\textsc{UniqueGames}_{\Delta}(n, q), 1 - \zeta, \delta) - n\Delta^t q^{t+1} \leq \mathrm{fc}_{\mathrm{LP}}(Q\text{-}{\neq}\text{-CSP}, (1 - \varepsilon)(1 - 1/q)(1 - \zeta t), \eta)$$

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– Proof idea is similar to 1F-CSP.

Matching over 3-regular graphs has no small LP

Theorem

For any n and $0 \leq \varepsilon < 1$, there exists a 3-regular graph D_{2n} with $2n(2n-1)$ vertices, so that any LP approximating $\textsc{Matching}(D_{2n})$ *within $1 - \varepsilon / |V(D_{2n})|$ has $2^{\Omega(\sqrt{|V(D_{2n})|})}$ inequalities.*

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Proof Idea.

- Reduce from $\textsc{Matching}(K_{2n})$ by replacing every vertex by $(2n-1)$-cycles
- Connect corresponding vertices to each other
- Lift perfect matchings in the "obvious" way

□

3-regular matchings continued

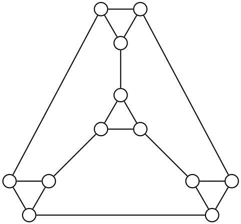

Figure : The graph D_{2n} for $n = 2$ in the reduction to 3-regular Matching.

– The base hard problem is the perfect matching problem MATCHING(K_{2n}), Rothvoß (2014).

SDP hardness of MaxCut

Theorem

For any $\delta, \varepsilon > 0$ there are infinitely many n such that there is a graph G with n vertices and

$$\mathrm{fc}_{\mathrm{SDP}}\left(\textsc{MaxCut}(G), \frac{4}{5} - \varepsilon, \frac{3}{4} + \delta\right) = n^{\Omega(\log n / \log\log n)}. \tag{1}$$

SDP hardness of MAXCUT

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Proof Idea.

- Reduce from MAX-3-XOR$/0$, every predicate is of the form $x_{i1} + x_{i2} + x_{i3} \pmod 2 = 0$
- Use the existing reduction by Trevisan et al. (2000).
- Hardness is due to Lee et al. (2014) combined with Schoenebeck (2008)'s Lasserre inapproximability result.

□

SDP hardness of MAXCUT continued

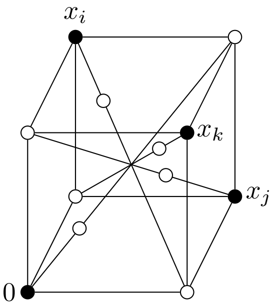

Figure : The gadget H_C for the clause $C = (x_i + x_j + x_k = 0)$ in the reduction from MAX-3-XOR/0 to MAXCUT. Solid vertices are shared by gadgets, the empty ones are local to the gadget.

Hardness of SPARSESTCUT

Theorem

For any $\varepsilon \in (0,1)$ *there are* $\eta_{LP} > 0$ *and* $\eta_{SDP} > 0$ *such that for every large enough n the following hold*

$$\mathrm{fc}_{\mathrm{LP}}\left(\textsc{SparsestCut}, \eta_{LP}(1+\varepsilon), \eta_{LP}(2-\varepsilon)\right) \geq n^{\Omega\left(\frac{\log n}{\log\log n}\right)},$$

$$\mathrm{fc}_{\mathrm{SDP}}\left(\textsc{SparsestCut}, \eta_{SDP}\left(1+\frac{4\varepsilon}{5}\right), \eta_{SDP}\left(\frac{16}{15}-\varepsilon\right)\right) \geq n^{\Omega\left(\frac{\log n}{\log\log n}\right)}.$$

even if the supply graph has treewidth 2.

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Proof Idea.

– Use the reduction from MAXCUT due to Gupta et al. (2013) using the fractional reduction framework.

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Proof Idea.

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– Base LP hardness of MAXCUT is due to Chan et al. (2013).

□

Lasserre is suboptimal for INDEPENDENTSET

Theorem

For any small enough $\gamma > 0$ there are infinitely many n, such that there is a graph G with n vertices with the largest independent set of G having size $\alpha(G) = O(n^\gamma)$ but there is a $\Omega(n^\gamma)$-round Lasserre solution of size $\Theta(n)$, i.e., the integrality gap is $n^{1-\gamma}$. However $\mathrm{fc}_{\mathrm{LP}}(\textsc{IndependentSet}(G), 2\sqrt{n}) \leq 3n + 1$.

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Proof Idea.

– Use the reduction mechanism within a Lasserre/SoS framework by reducing Max-k-CSP to IndependentSet.

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Proof Idea.

– Use the reduction mechanism within a Lasserre/SoS framework by reducing MAX-k-CSP to INDEPENDENTSET.

– The reduction is a simple conflict graph over the partial assignments.

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Proof Idea.

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- The reduction is a simple conflict graph over the partial assignments.
- The SoS/Lasserre integrality gap for Max-k-CSP is due to Bhaskara et al. (2012).

□

BALANCEDSEPARATOR cannot be approximated to any constant factor by a LP

Definition

The BALANCEDSEPARATOR is similar to the SPARSESTCUT cut problem. There are two functions $c : E(K_n) \to \mathbb{R}_{\geq 0}$ (capacity) and $d : E(K_n) \to \mathbb{R}_{\geq 0}$ (demand). Goal is to minimize the capacity of all cuts that are “balanced“, i.e. cut atleast $\frac{1}{4}$ of total demand.

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– Reduce from UNIQUEGAMES using the long code test of Khot and Vishnoi (2015).

Bounded treewidth graphs are LP-easy

Recall the definition of treewidth:

Definition

A *tree decomposition* of a graph G is a tree T together with a vertex set of G called *bag* $B_t \subseteq V(G)$ for every node t of T, satisfying the following conditions:

- $V(G) = \bigcup_{t \in V(T)} B_t$
- For every adjacent vertices u, v of G there is a bag B_t containing both u and v
- For all nodes t_1, t_2, t of T with t lying between t_1 and t_2 (i.e., t is on the unique path connecting t_1 and t_2) we have $B_{t_1} \cap B_{t_2} \subseteq B_t$

The *width* of the tree decomposition is $\max_{t \in V(T)} |B_t| - 1$. The *treewidth* $\text{tw}(G)$ of G is the minimum width of its tree decompositions.

Small uniform LPs for bounded treewidth problems

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- MATCHING, VERTEXCOVER, INDEPENDENTSET and CSPs such as MAXCUT and UNIQUEGAMES are admissible problems.

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Thank you for listening!