Separations in Query Complexity Based on Pointer Functions

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CWI

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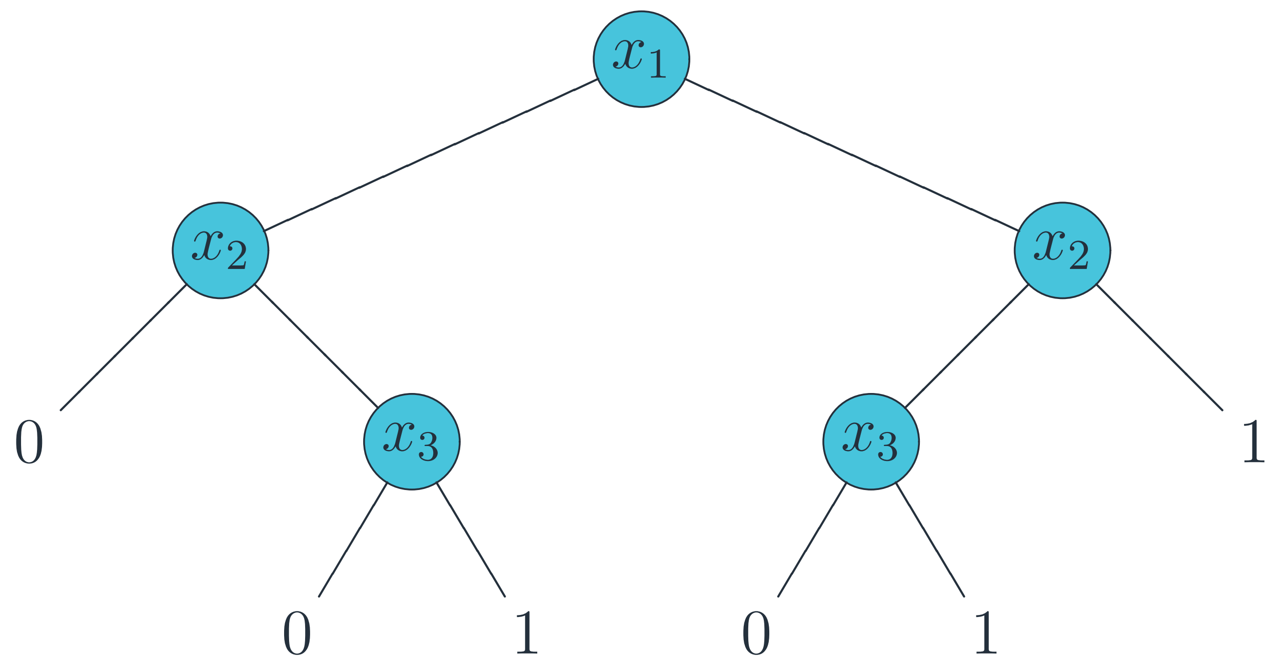
Introduction

D: Deterministic (Decision Tree)



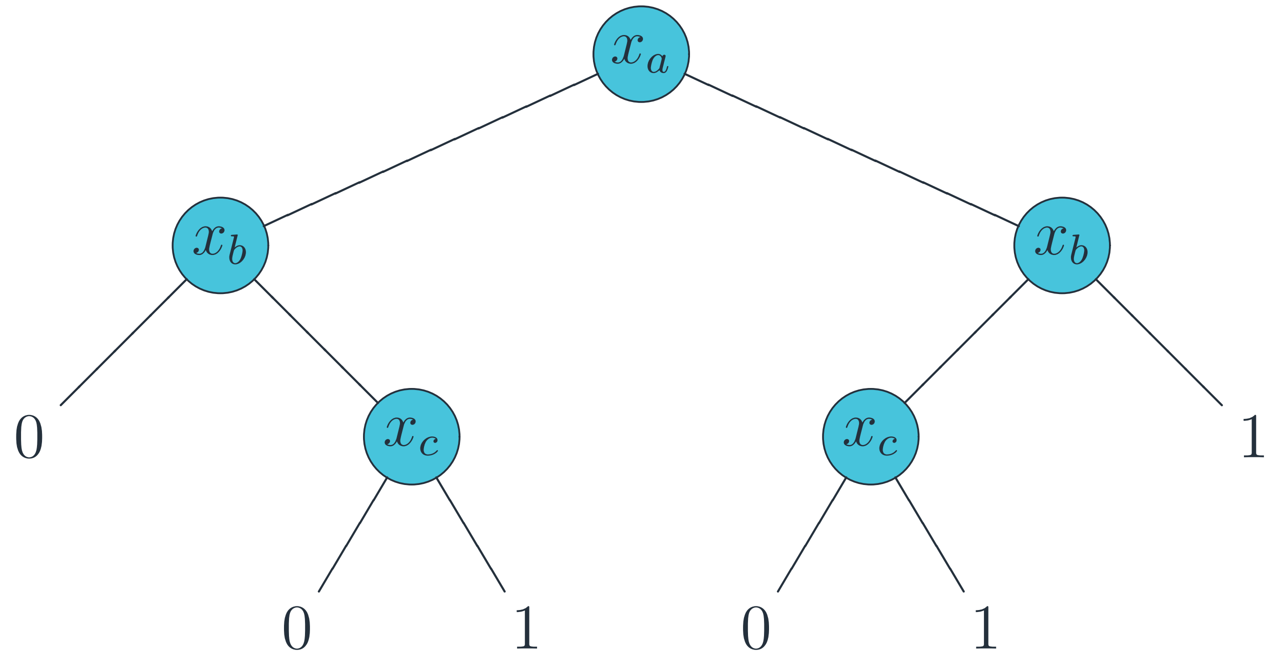
Complexity

• on input:	Number of queries	(length of the path)	2 or 3
• in total:	Worst input	(depth of the tree)	3

Computation Models

D: Deterministic (Decision Tree)

R: Randomized (Probability distribution on decision trees)



Complexity

- on input: **Expected** number of queries 2 or $\frac{8}{3}$
- in total: Worst input $\frac{8}{3}$

D: Deterministic (Decision Tree)

R: Randomized (Probability distribution on decision trees)

R_0: Zero-error (Las Vegas)

- always outputs a correct output

R_1: One-sided error

- always rejects a negative input
- accepts a positive input with probability $\geq \frac{1}{2}$ (or vice versa)

R_2: Bounded-error (Monte Carlo)

- rejects a negative input with probability $\geq \frac{2}{3}$
- accepts a positive input with probability $\geq \frac{2}{3}$

Computation Models

D: Deterministic (Decision Tree)

R: Randomized (Probability distribution on decision trees)

- R_0: Zero-error (Las Vegas)
- R_1: One-sided error
- R_2: Bounded-error (Monte Carlo)

Q: Quantum bounded-error

- Q_E: Quantum exact

Easy for **partial** functions

Separations

Easy for **partial** functions

Example: Deutsch-Jozsa problem (almost)

- Reject iff all input variables are zeroes

0	0	0	0	0	0	0	0

- Accept iff exactly half of the variables are ones

0	1	0	1	1	0	0	1

Separations

Easy for **partial** functions

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$$R_1 = 1$$

Separations

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$$R_1 = 1, \quad Q_E = 1, \qquad\qquad R_0 = n/2 + 1$$

0		0			0	0	

Easy for **partial** functions

Example: Deutsch-Jozsa problem (almost)

- Reject iff all input variables are zeroes

0	0	0	0	0	0	0	0

- Acc

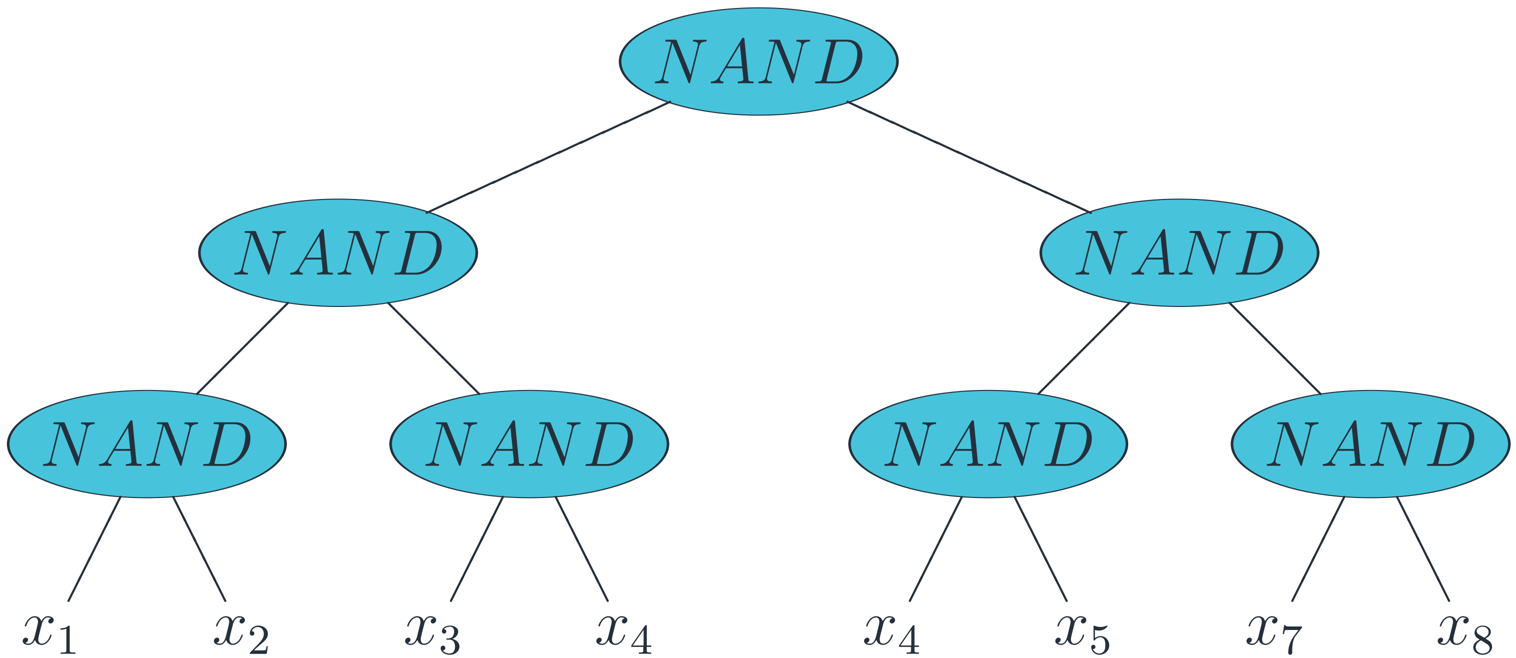
Total Functions — ???

$$R_1 = 1, \quad Q_E = 1, \qquad R_0 = n/2 + 1$$

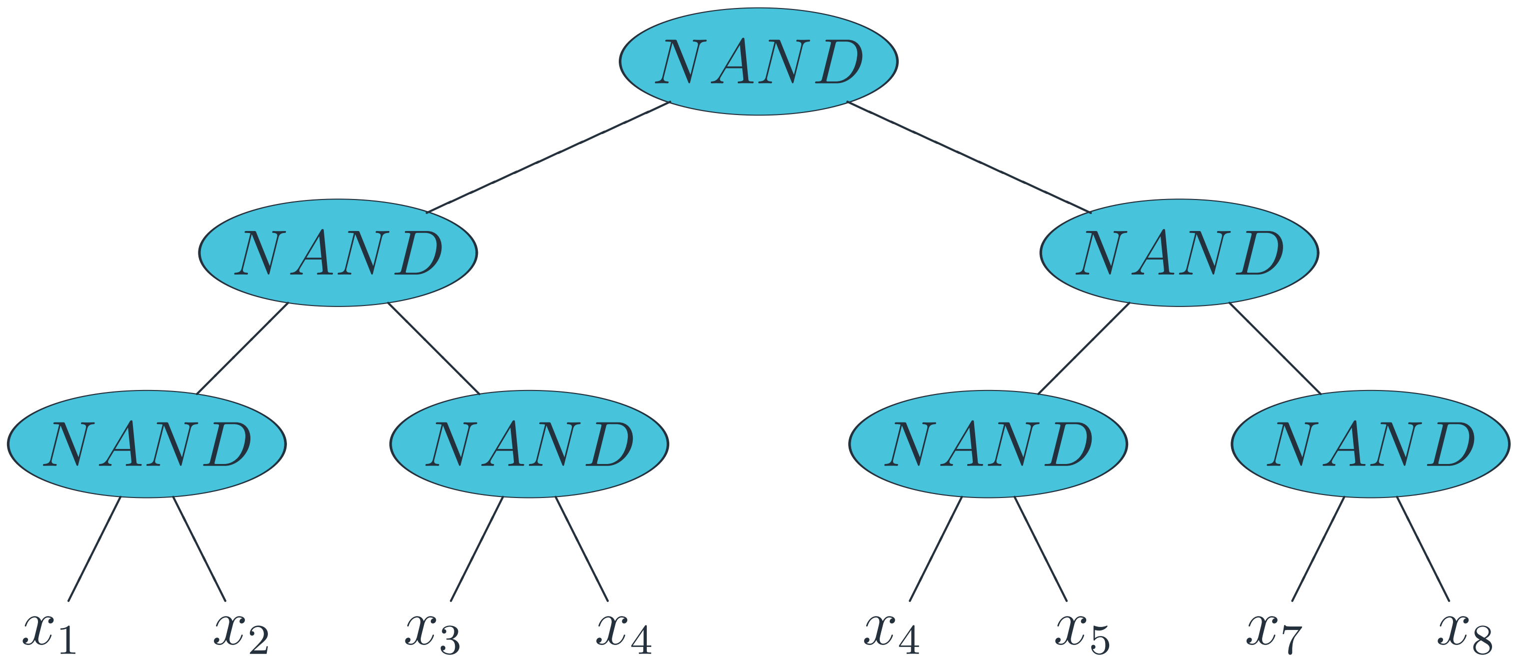
0		0			0	0	

A Previous Result

Iterated NAND: record-holder for R_0, R_1, R_2 versus D



Iterated NAND: record-holder for R_0, R_1, R_2 versus D



We have [Snir'85, Saks & Wigderson'86]:

$$R_0 = R_1 = R_2 = O(n^{0.7537\ldots}), \qquad D = n$$

It is known [Nisan'89]

$$D = O(R_1^2)$$

We get functions with:

$$D = \widetilde{\Theta}(R_0^2)$$

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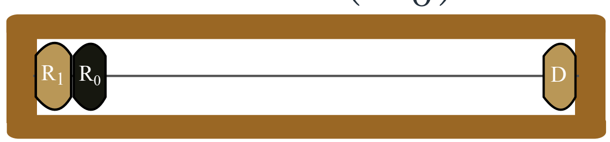
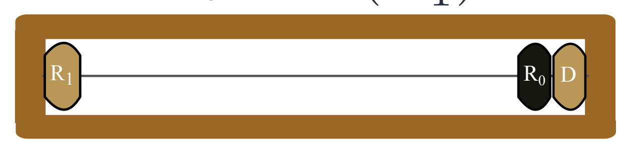


$$R_0 = \widetilde{\Theta}(R_1^2)$$



The last one also saturates [Kulkarni & Tal'13, Midrijānis'05]

$$R_0 = \widetilde{O}(R_2^2)$$

Göös-Pitassi-Watson

Paper

Electronic Colloquium on Computational Complexity, Report No. 50 (2015)

ECCC

Deterministic Communication vs. Partition Number

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Department of Computer Science, University of Toronto

April 1, 2015

Abstract

We show that deterministic communication complexity can be superlogarithmic in the partition number of the associated communication matrix. We also obtain near-optimal deterministic lower bounds for the Clique vs. Independent Set problem, which in particular yields new lower bounds for the log-rank conjecture. All these results follow from a simple adaptation of a communication-to-query simulation theorem of Raz and McKenzie (Combinatorica 1999) together with lower bounds for the analogous query complexity questions.

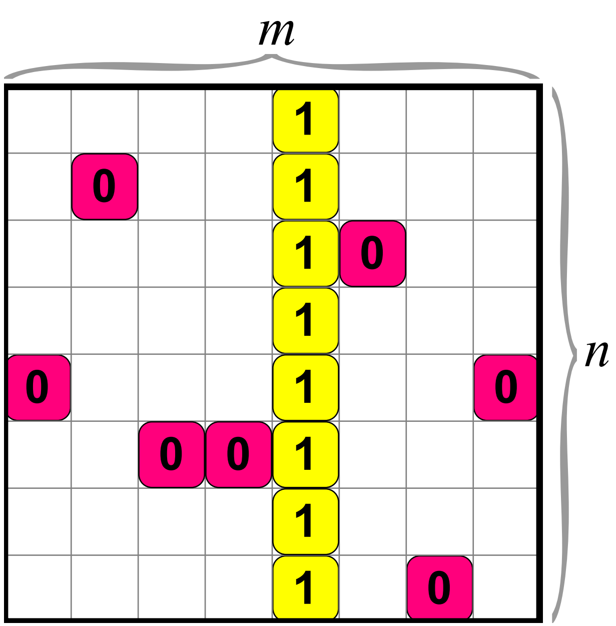
- Clique vs. Independent Set in communication complexity
- Reduce to a problem in query complexity: Find a function that
  - has large deterministic complexity
  - has small unambiguous 1-certificates

  There exists a number of 1-certificates such that each positive input satisfies exactly one of them.

D versus 1-certificates

Function on nm Boolean variables

- **Accept** iff there exists a unique all-1 column



- $D = nm$
- short 1-certificates ($n + m - 1$), **BUT not** unambiguous.

D versus 1-certificates

Function on nm Boolean variables

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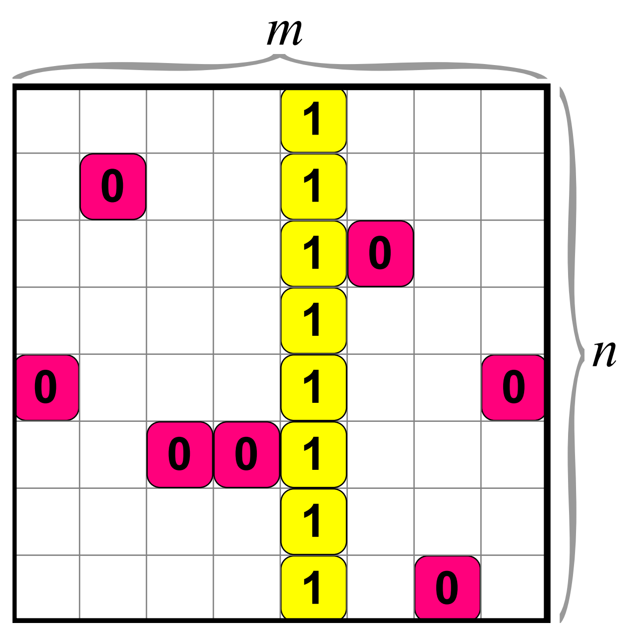
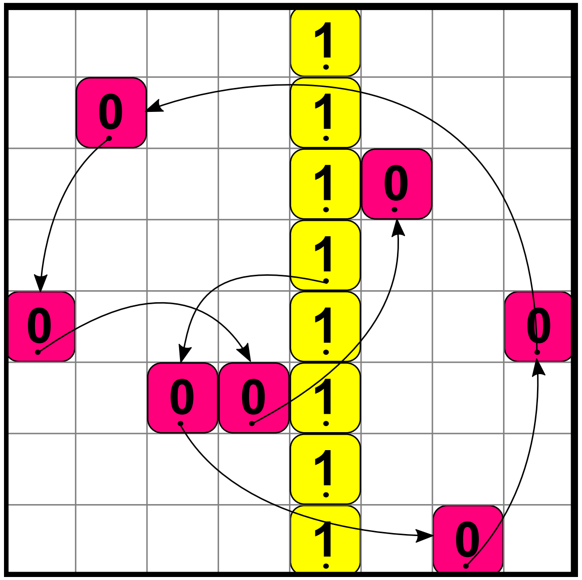
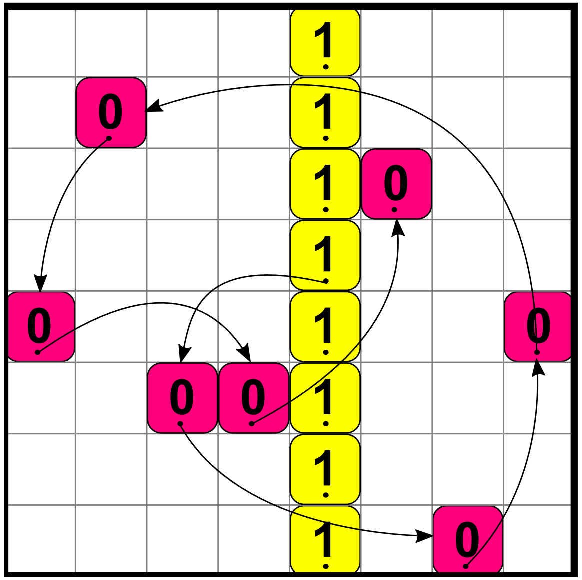


- $D = nm$
- short 1-certificates $(n + m - 1)$, **BUT not** unambiguous.
  Should specify which zero to take in each column

Pointers



- **Alphabet:** $\{0,1\} \times ([n] \times [m] \cup \{\perp\})$
  Not Boolean, but we can encode using $O(\log(n+m))$ bits.
- **Accept** iff
  - There is a (unique) all-1 column b;
  - in b, there is a unique element r with non-zero pointer;
  - following the pointers from r, we traverse through exactly one zero in each column but b.



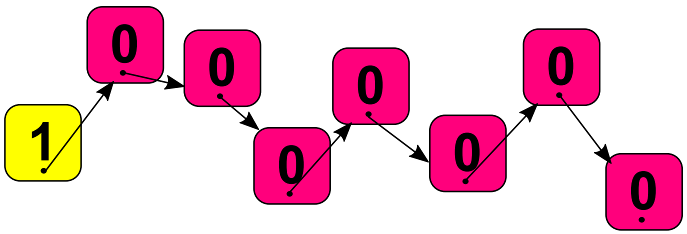
- Still have $D = nm$
- short unambigous 1-certificates ($n + m - 1$)

Highly elusive
(flexible)

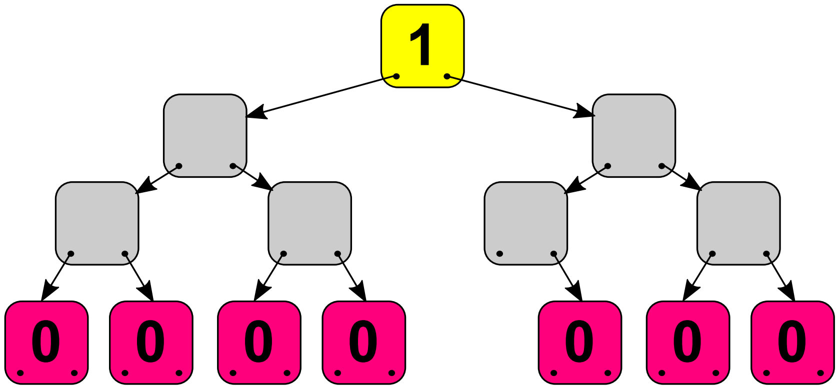
Still traversable
(if know where to start).

Our Modifications

Instead of a list

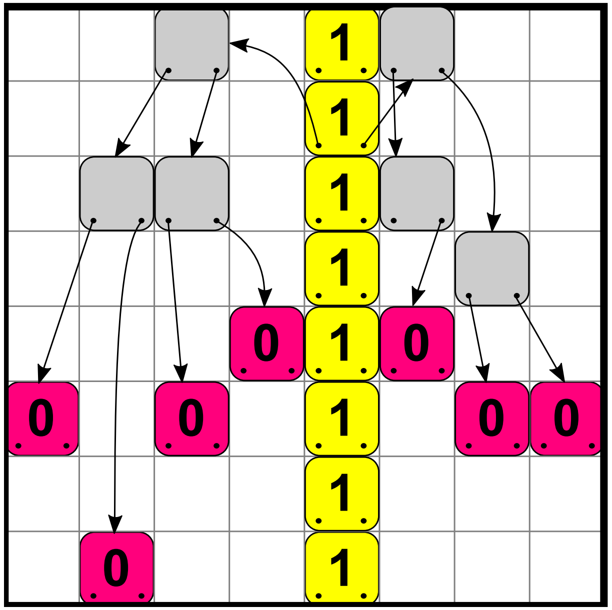


we use a balanced binary tree



- More elusive
- Random access

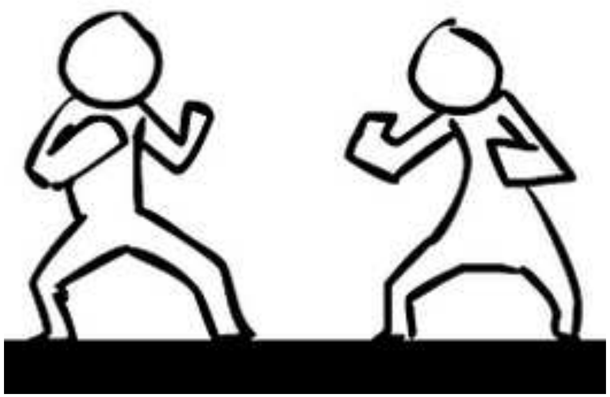
Definition (base)



Accept iff

- There is a (unique) all-1 column b;
- in b, there is a unique element r with non-zero pointers;
- for each $j \neq b$, following a path $T(j)$ from r gives a zero in the jth column.
- Some additional information is contained in the leaves (to be defined).

R_1 versus R_0



- NO separation was known even between R_2 and R_0.

(Iterated functions are not of much help here.)

Partial Separation

Recall the separation for a partial function

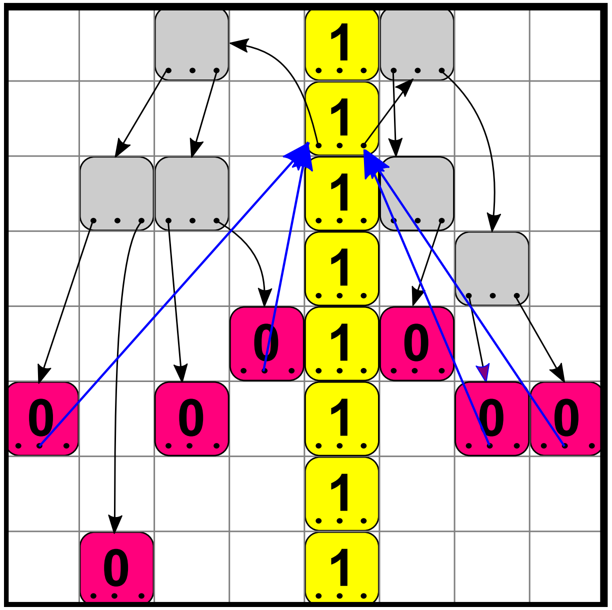
- Reject iff all input variables are zeroes

0	0	0	0	0	0	0	0

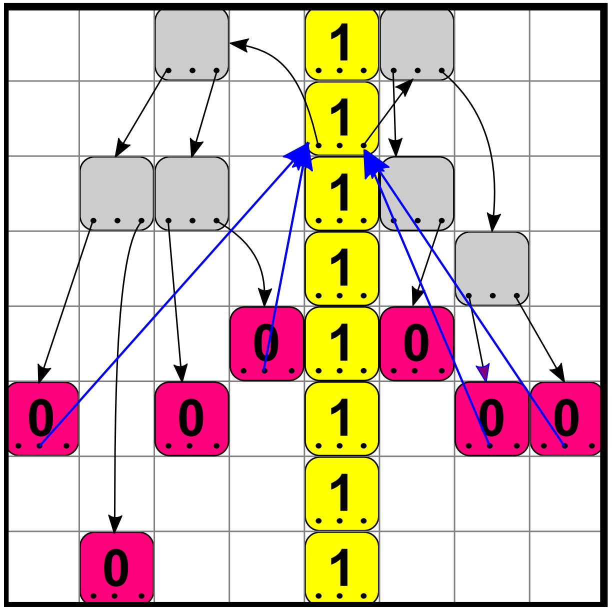
- Accept iff exactly half of the variables are ones

0	1	0	1	1	0	0	1

Definition



- Add a back pointer to each variable.
- **Accept** iff
  - ...
  - exactly $m/2$ of the leaves back point to the root r.



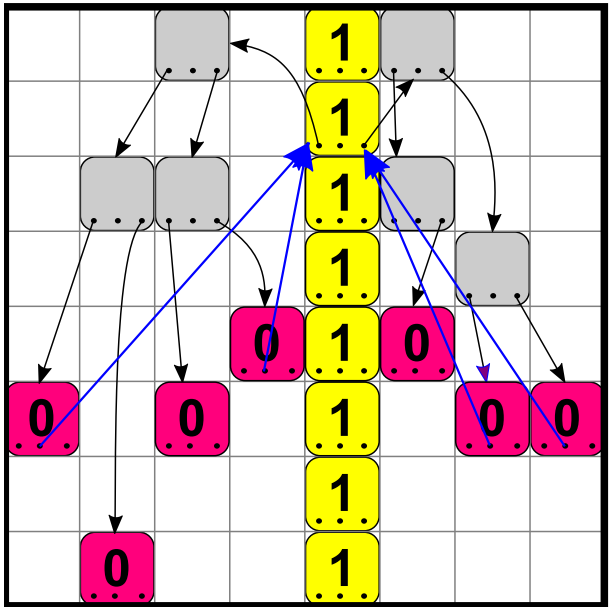
A column is good if it contains a leaf back pointing to the root of a legitimate tree.

- A positive input contains exactly $m/2$ good columns.
- A negative input contains no good columns.

A total function looks like a partial function.

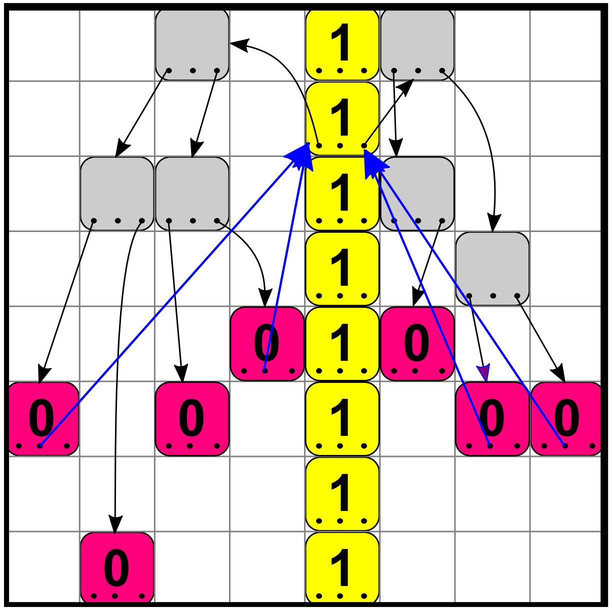
Deterministic subroutine

Given a column $c \in [m]$, accept iff it is good.



On each step we either

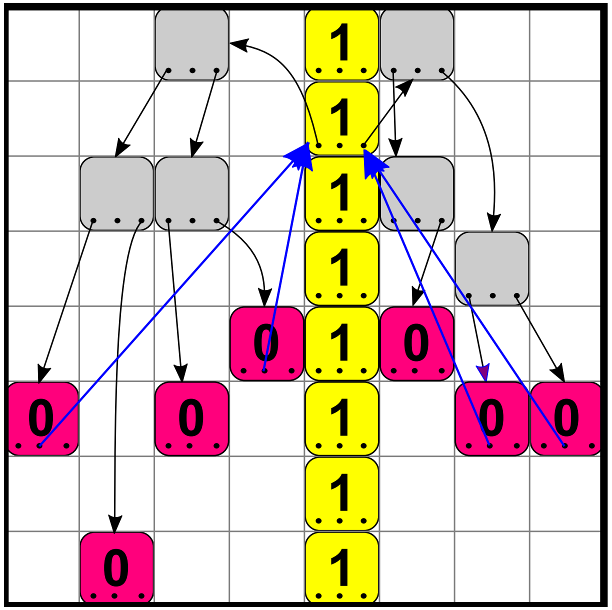
- eliminate a column: it is not the all-1 column; or
- eliminate an element in column c: it is not a leaf of the tree.



Deterministic subroutine

Given a column $c \in [m]$, accept iff it is good.

- **While** there is ≥ 2 non-eliminated columns:
  - Let a be a non-eliminated element in c. If none, **reject**.
  - Let r be the back pointer of a, and b be the column of r.
  - Let j be a non-eliminated column $\neq b$.
  - **If** the path $T(j)$ from r ends in a zero in column j, eliminate column j.
    Otherwise, eliminate element a.
- Verify the only non-eliminated column.

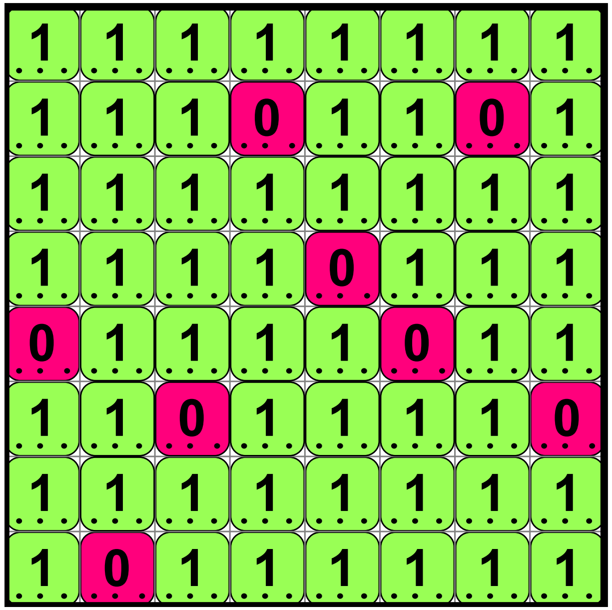


- On each iteration of the loop, either an element or a column gets eliminated. At most $n + m$ iterations.
  Complexity: $\widetilde{O}(n + m)$.

Sticking into Deutsch-Jozsa, get R_1 and Q_E upper bound of

$$\widetilde{O}(n + m).$$

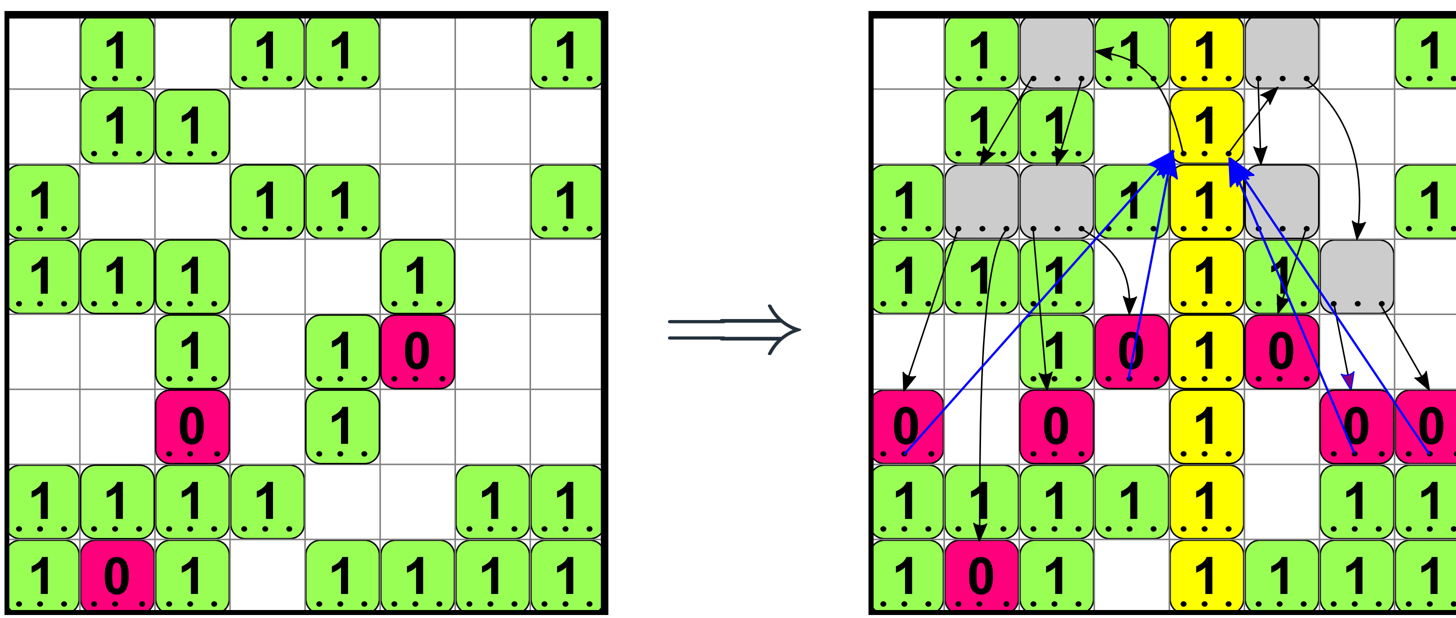
Lower Bound



(Negative) input with exactly one zero in each column.

- An R_0 algorithm can reject only if it has found $m/2$ zeroes.

Requires $\Omega(nm)$ queries.



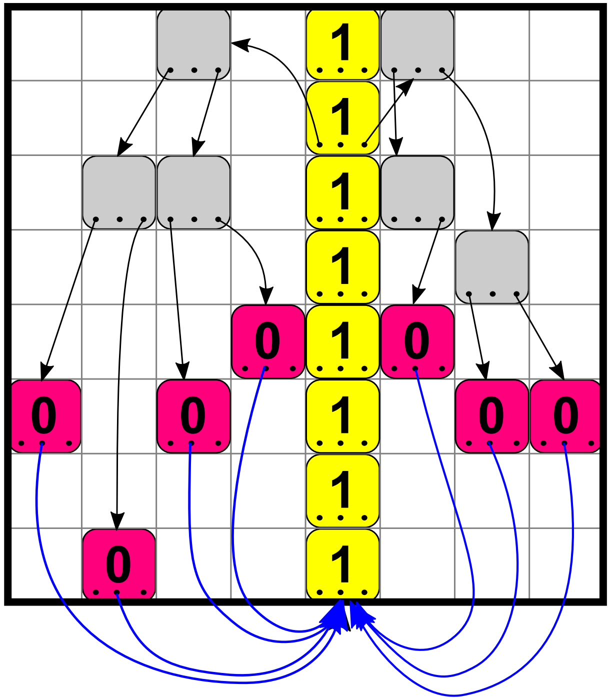
- Upper bound for R_1 and Q_E is $\widetilde{O}(n+m)$.
- Lower bound for a R_0 algorithm is $\Omega(nm)$.

Taking $n = m$, we get a quadratic separation between R_1 and R_0, as well as between Q_E and R_0

NB. The previous separation was [Ambainis'12]:

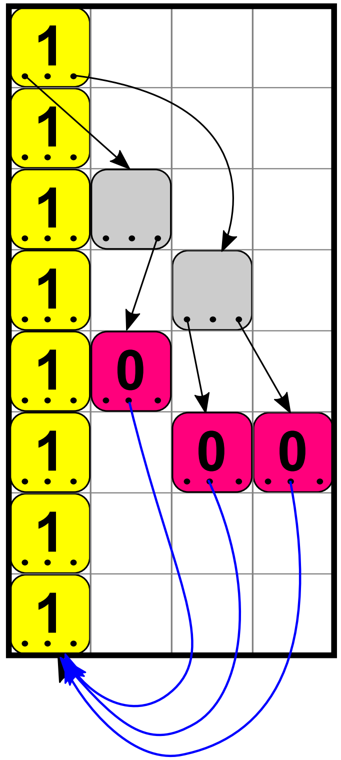
$$Q_E = O(R_0^{0.8675\ldots})$$

R_0 versus D



- Back pointers are to columns.
- **Accept** iff
  - ...
  - all the leaves back point to the all-1 column b.

Lower Bound



Adversary Method.

Let $n = 2m$.

If the kth element is queried in a column:

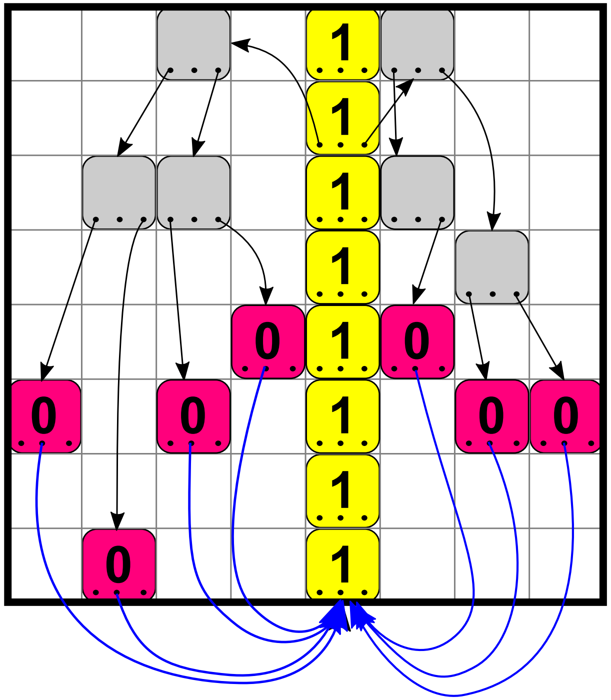
- If $k \le m$, return $\boxed{1}$.
- Otherwise, return $\boxed{0}$ with back pointer to column $k - m$.

At the end, the column contains m $\boxed{1}$ and m $\boxed{0}$ with back pointers to all columns $1, 2, \ldots, m$.

- The algorithm does not know the value of the function until it has queried $> m$ elements in each of m columns.

Lower bound: $\Omega(m^2)$.

Upper Bound: Informal



- Each column contains a back pointer to the all-1 column.
  BUT which one is the right one—?

We try each back pointer by quering few elements in the column, and proceed to a one where no zeroes were found.

- Even if this is not the all-1 column,
  we can arrange that it contains fewer zeroes whp.

Upper Bound: Formal

Algorithm

- Let c be the first column, and $k \leftarrow n$.
- **While** $k > 1$,
  - Let $c \leftarrow$ProcessColumn(c, k), and $k \leftarrow k/2$.

ProcessColumn(column c, integer k)

- Query all elements in column c.
- **If** there are no zeroes, verify column c.
- **If** there are more then k zeroes, query all nm variables, and output the value of the function.
- **For** each zero a:
  - Let j be the back pointer of a.
  - Query $\widetilde{O}(n/k)$ elements in column j. (Probability $< \frac{1}{(nm)^2}$ that no zero found if there are $> k/2$ of them).
  - **If** no zero was found, **return** j.
- **Reject**

Summary

Take $n = 2m$.

- Lower bound for a D algorithm is $\Omega(m^2)$.
- Upper bound for a R_0 algorithm is $\widetilde{O}(n + m)$.

We get a quadratic separation between R_0 and D.

Take $n = 2m$.

- Lower bound for a D algorithm is $\Omega(m^2)$.
- Upper bound for a R_0 algorithm is $\widetilde{O}(n+m)$.

We get a quadratic separation between R_0 and D.

- Also, upper bound for a Q algorithm is $\widetilde{O}(\sqrt{n+m})$.

We get a quartic separation between Q and D.

NB. Previous separation was quadratic: Grover's search.

Conclusion

$$R_1 = \widetilde{O}(R_0^{1/2})$$
$$Q_E = \widetilde{O}(R_0^{1/2})$$
$$R_0 = \widetilde{O}(D^{1/2})$$
$$Q = \widetilde{O}(D^{1/4})$$

$$Q = \widetilde{O}(R_0^{1/3})$$
$$Q_E = \widetilde{O}(R_2^{2/3})$$
$$\widetilde{\deg} = \widetilde{O}(R_2^{1/4})$$

We have resolved $R_2 \leftrightarrow R_0$ and $R_1 \leftrightarrow D$.

Can we resolve $R_2 \leftrightarrow D$ too?
Known: $R_2 = \Omega(D^{1/3})$ and $R_2 = \widetilde{O}(D^{1/2})$.

- Can we overcome the "certificate complexity barrier"?
  Obtain a function with $R_2 = o(C)$?

- The same about $Q \leftrightarrow D$
  Known: $Q = \Omega(D^{1/6})$ and $Q = \widetilde{O}(D^{1/4})$.
- and $Q_E \leftrightarrow D$?
  Known: $Q_E = \Omega(D^{1/3})$ and $Q_E = \widetilde{O}(D^{1/2})$.

Any questions?