Separations in Query Complexity using Cheat Sheets

Scott Aaronson, Shalev Ben-David, Robin Kothari

Query Complexity

- Fix a Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$
- How many queries to an unknown binary string x do we need in order to compute f(x)?
  - D(f) = deterministic queries
  - R(f) = randomized queries (with bounded error)
  - Q(f) = quantum queries (with bounded error)

Gap Between D and R

- f(x) = 1 if x has 2/3 or more 1s
- f(x) = 0 if x has 1/3 or less 1s
- We assume that x satisfies one of the above conditions

- R(f)=1,   D(f) ≈ n

Gap Between R and Q

- Simon's problem [Simon '94]:
  - Suppose x consists of 2^m blocks of m bits
  - The position of a block is an m-bit index
  - Assume there's a hidden m-bit string s such that two blocks are equal iff the xor of their positions is s
  - Goal: find the first bit of s

- Quantum query complexity: $O(\log^2 n)$
- Randomized query complexity: $\tilde{\Omega}(\sqrt{n})$

Gap Between R and Q

- Forrelation [Aaronson, Ambainis 2014]
  - Split x into two parts, each containing 2^m blocks of m bits
  - Interpret this as two functions from $\{0,1\}^m$ to $\{0,1\}^m$
  - Assume that the first function is either highly correlated with the Fourier transform of the second, or else has near-zero correlation
  - Goal: determine which is the case

- Quantum Query Complexity: 1
- Randomized Query Complexity: $\Omega(\sqrt{n}/\log n)$

Gap between R and Q

- k-fold Forrelation:
  - Quantum Query Complexity O(k)
  - Conjectured to have randomized query complexity $\Omega(n^{1-1/k})$
- If so, this is the optimal separation
- k = log n gives O(log n) vs. Ω(n)

What about total functions?

- [BBCMdW '98]:

$$D(f) = O(R(f)^3)$$
$$D(f) = O(Q(f)^6)$$

- So none of these constructions can work for total functions!

What about total functions?

- Saks, Wigderson '86:
  - And-Or tree of depth log n
  - Deterministic query complexity $\Omega(n)$
  - Randomized query complexity $\Theta(n^{0.753})$

- Grover '96:
  - OR
  - Randomized query complexity $\Omega(n)$
  - Quantum query complexity $\Theta(\sqrt{n})$

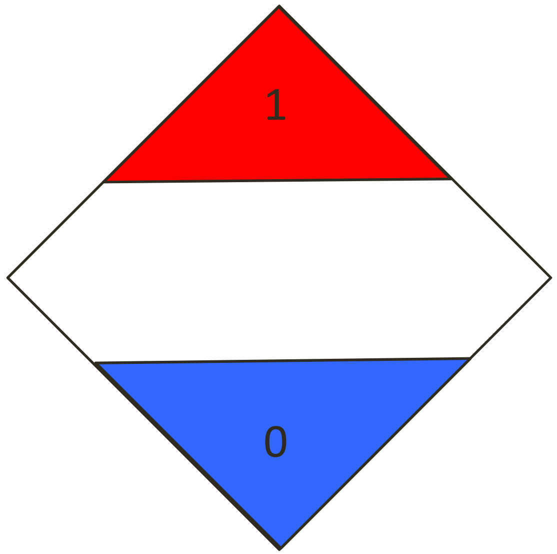
Separations in 2015

- April 4 (Göös, Pitassi, Watson):
  - Introduced the idea of pointer functions
  - Quadratic separation between D(f) and deg(f)
- June 16 (Ambainis, Balodis, Belovs, Lee, Santha, Smotrovs):
  - Quadratic separation between D(f) and R(f)
  - Power 4 separation between D(f) and Q(f)
  - Many other separations, involving $R_0(f)$ and $Q_E(f)$
- June 26 (B.):
  - Power 2.5 separation between R(f) and Q(f)
  - Introduced cheat sheets
- Nov 5 (Aaronson, B., Kothari):
  - Used cheat sheets to reprove many of the other separations
  - Power 4-o(1) separation between Q(f) and approximate degree

Cheat Sheets

Turning partial functions total

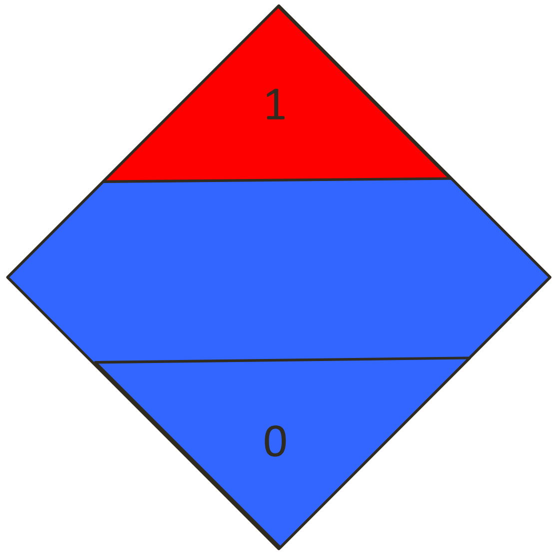
- Given a partial function f that has a good separation, how can we turn it total?
- For concreteness, set f to be f(x) = 1 if x is 2/3 ones, 0 if x is 2/3 0s (“two-thirds”)



Turning partial functions total

- **Attempt**:

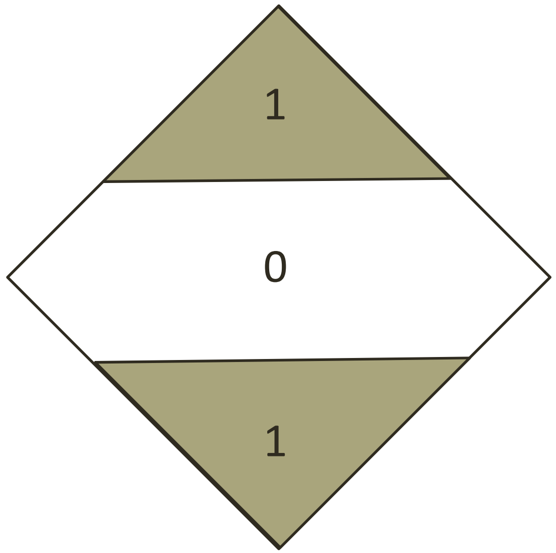
$$f'(x) = \begin{cases} f(x) & if\ x \in Dom(f) \\ 0 & otherwise \end{cases}$$



Turning partial functions total

- The problem is that the *promise* is difficult for a randomized algorithm to calculate

$$p_f(x) = \begin{cases} 1 & if\ x \in Dom(f) \\ 0 & otherwise \end{cases}$$



Cheat sheet step 1: Make things easy to certify

- Change the function so that it is easy to certify if an input satisfies the promise
- Also, make it easy to certify whether f(x) is 0 or 1
- But make sure not to decrease D(f)!

- **Idea**: Compose with AND-OR

0	0	1	0	1	0	1	0
1	1	1	1	1	1	0	0
1	0	1	0	0	0	1	0
1	0	1	1	1	1	1	1
0	0	1	0	1	1	1	0
1	1	1	1	1	1	0	1
0	1	1	0	0	1	0	1
1	0	1	1	0	1	1	1

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0	0	**1**	0	1	0	1	0
1	1	**1**	1	1	1	0	0
1	0	**1**	0	0	0	1	0
1	0	**1**	1	1	1	1	1
0	0	**1**	0	1	1	1	0
1	1	**1**	1	1	1	0	1
0	1	**1**	0	0	1	0	1
1	0	**1**	1	0	1	1	1

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- **Idea**: Compose with AND-OR

0	0	1	0	1	0	1	0
1	1	1	1	1	1	**0**	**0**
1	0	**0**	**0**	**0**	**0**	1	0
1	**0**	1	1	1	1	1	1
0	0	1	0	1	1	1	0
1	1	1	1	1	1	0	1
0	1	1	0	0	1	0	1
1	0	1	1	0	1	1	1

Cheat sheet step 1: Make things easy to certify

- Change the function so that it is easy to certify if an input satisfies the promise
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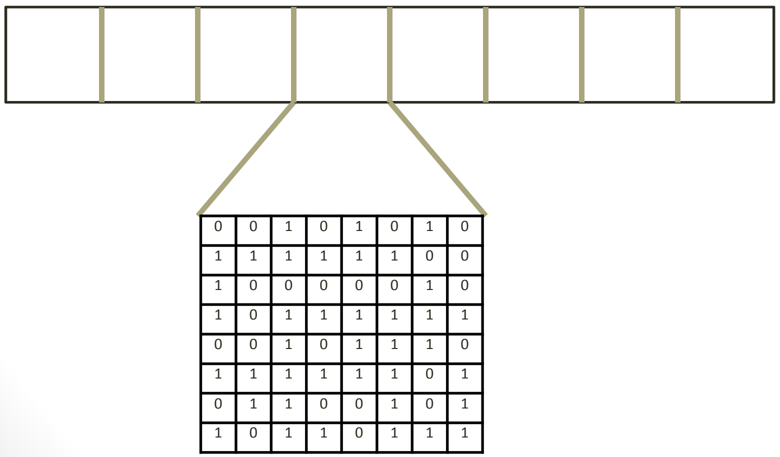
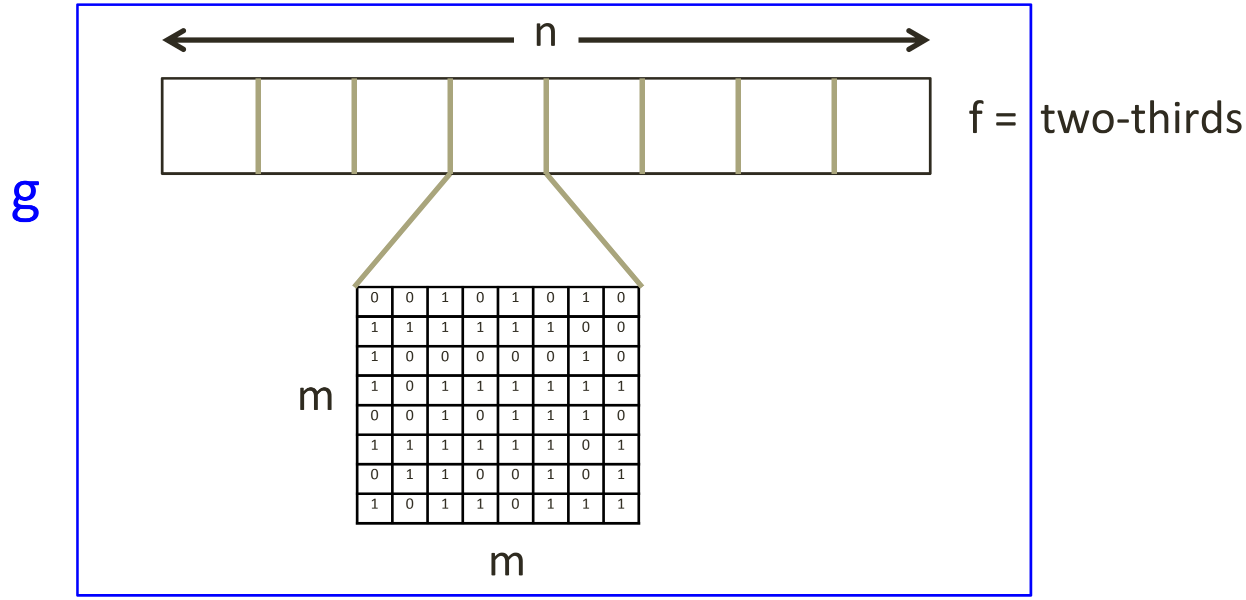


0	0	1	0	1	0	1	0
1	1	1	1	1	1	0	0
1	0	0	0	0	0	1	0
1	0	1	1	1	1	1	1
0	0	1	0	1	1	1	0
1	1	1	1	1	1	0	1
0	1	1	0	0	1	0	1
1	0	1	1	0	1	1	1

Properties of the composition



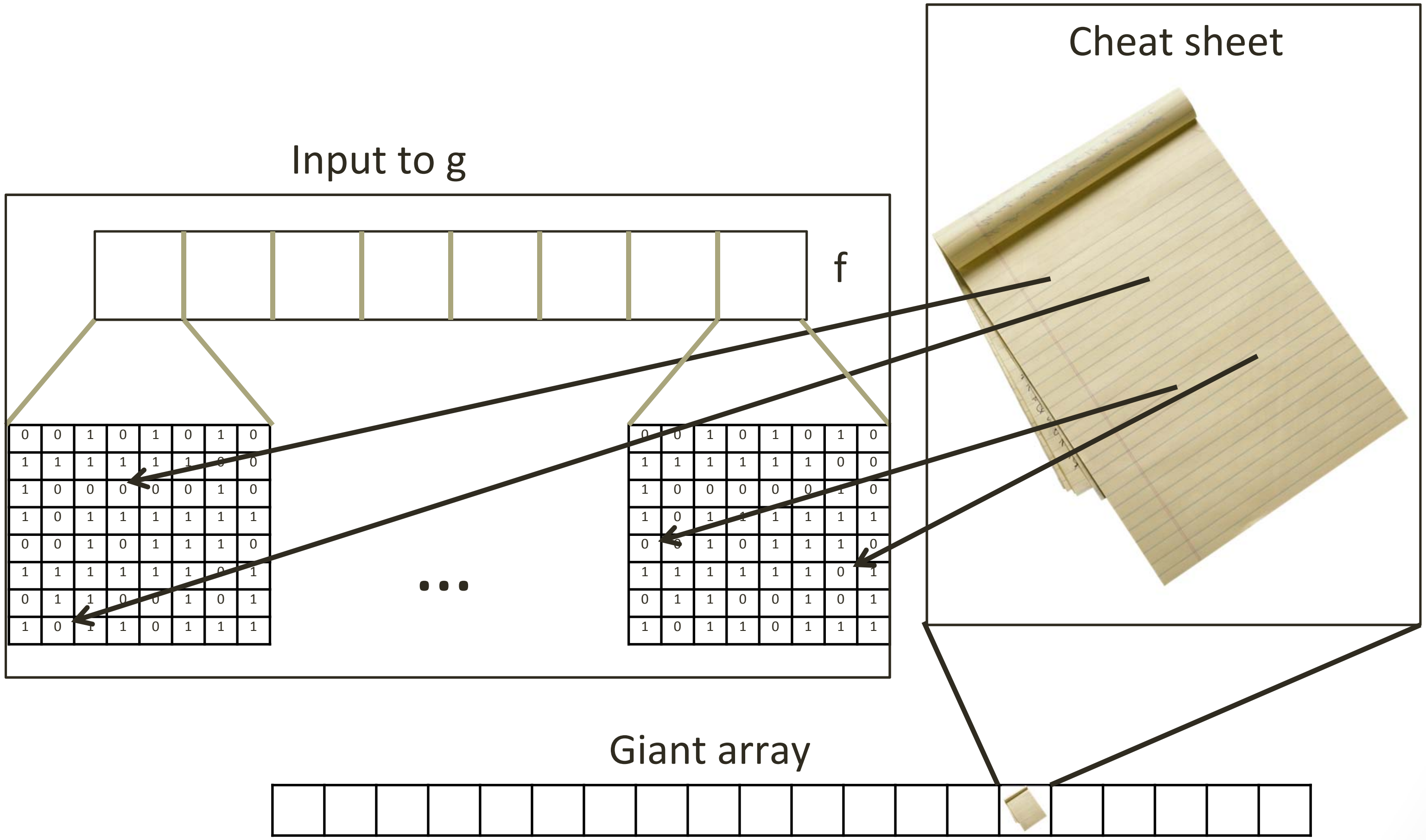
- D(AND-OR) = m^2,   R(AND-OR) = $\Omega(m^2)$
- D(g) = nm^2,   R(g) = $O(m^2)$
- An input x can be certified to be in the promise of g by certifying all the AND-OR copies (using nm bits)
- This also certifies whether g(x) is 0 or 1
- **But g is still not a total function!**

Step 2: hide a cheat sheet

- So far:
  - D(g) large
  - R(g) small
  - For every input x in the promise of g, there is a small cheat sheet that tells us everything about it

- Next step:
- change g so it contains the cheat sheet inside it
- Make sure only a randomized algorithm will be able to find the cheat sheet

Step 2: hide a cheat sheet

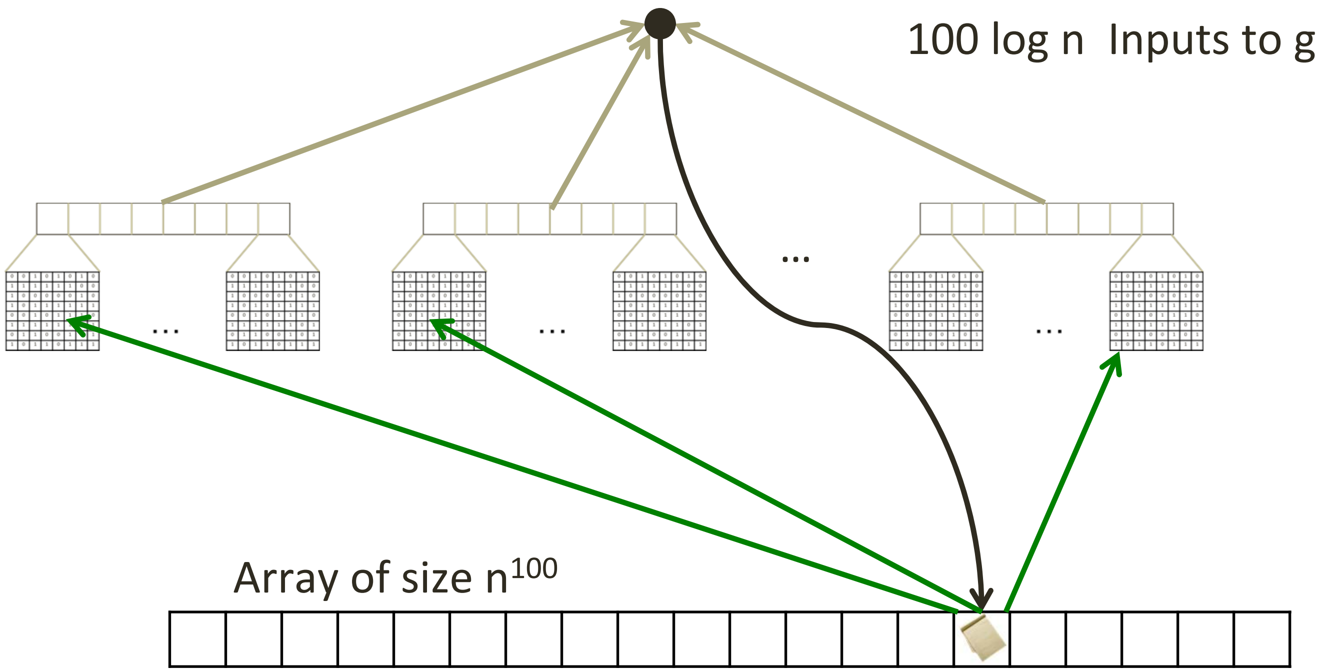


Step 3: find the cheat sheet

- We want to let a randomized algorithm find the cheat sheet
- We want a deterministic algorithm to NOT find it
- What can a randomized algorithm do that a deterministic one can't?
- Solve g!

- **Idea**: let g(x) describe where the cheat sheet is
- **Problem**: g(x) is only one bit
- **Solution**: use 100 log n copies of g to describe a position in an array of size n^{100}

Step 3: Find the cheat sheet



The final function

- Let g_{CS} be defined by
  - $g_{CS}(x) = 1$ if x has a valid cheat sheet in the right spot of the array
  - $g_{CS}(x) = 0$ otherwise
- Then g_{CS} is a total function!

- $R(g_{CS}) = ?$
- Need to compute g 100 log n times
  - Each takes $O(m^2)$
- Need to check that the cheat sheet is valid
  - There are nm pointers per copy of g, times 100 log n copies
  - Each takes O(log nm) queries to read, so Õ(nm)
- Total is $nm + m^2$ (times log factors)

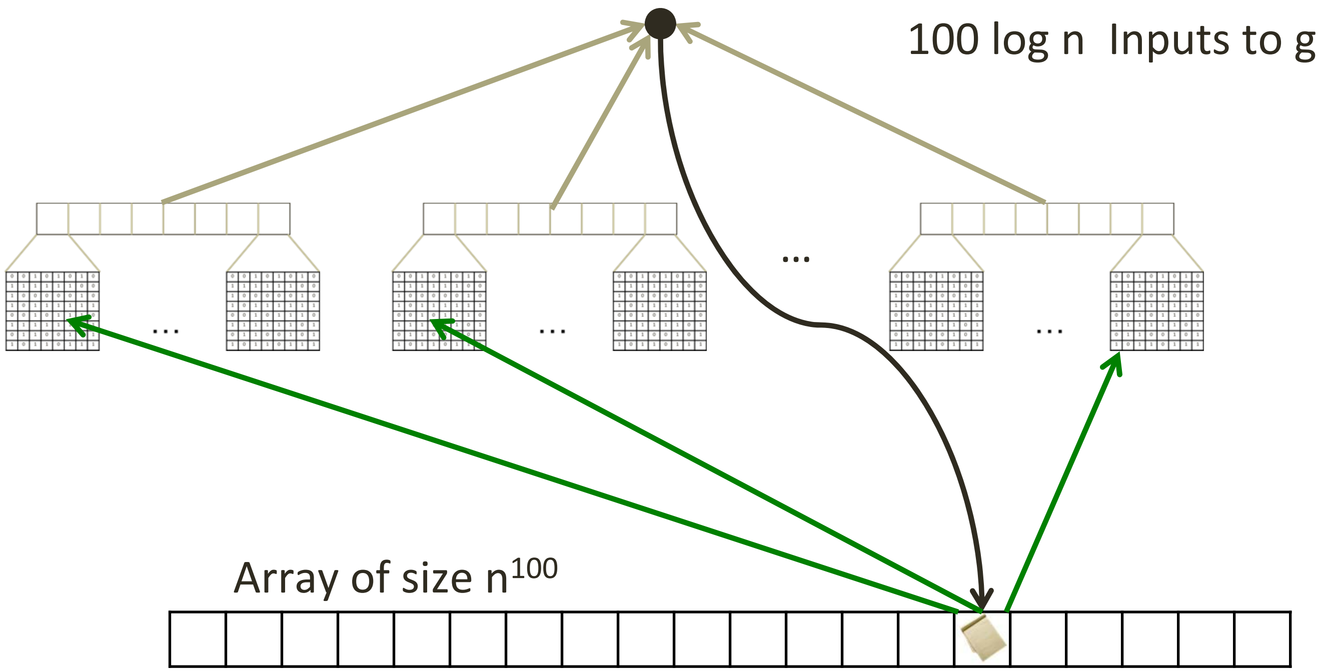
The final function

- Let g_{CS} be defined by
  - $g_{CS}(x) = 1$ if x has a valid cheat sheet in the right spot of the array
  - $g_{CS}(x) = 0$ otherwise
- Then g_{CS} is a total function!

- $D(g_{CS}) = ?$
- Blindly searching the array is hopeless
- Must compute g at least once
- $D(g_{CS}) = \Omega(nm^2)$

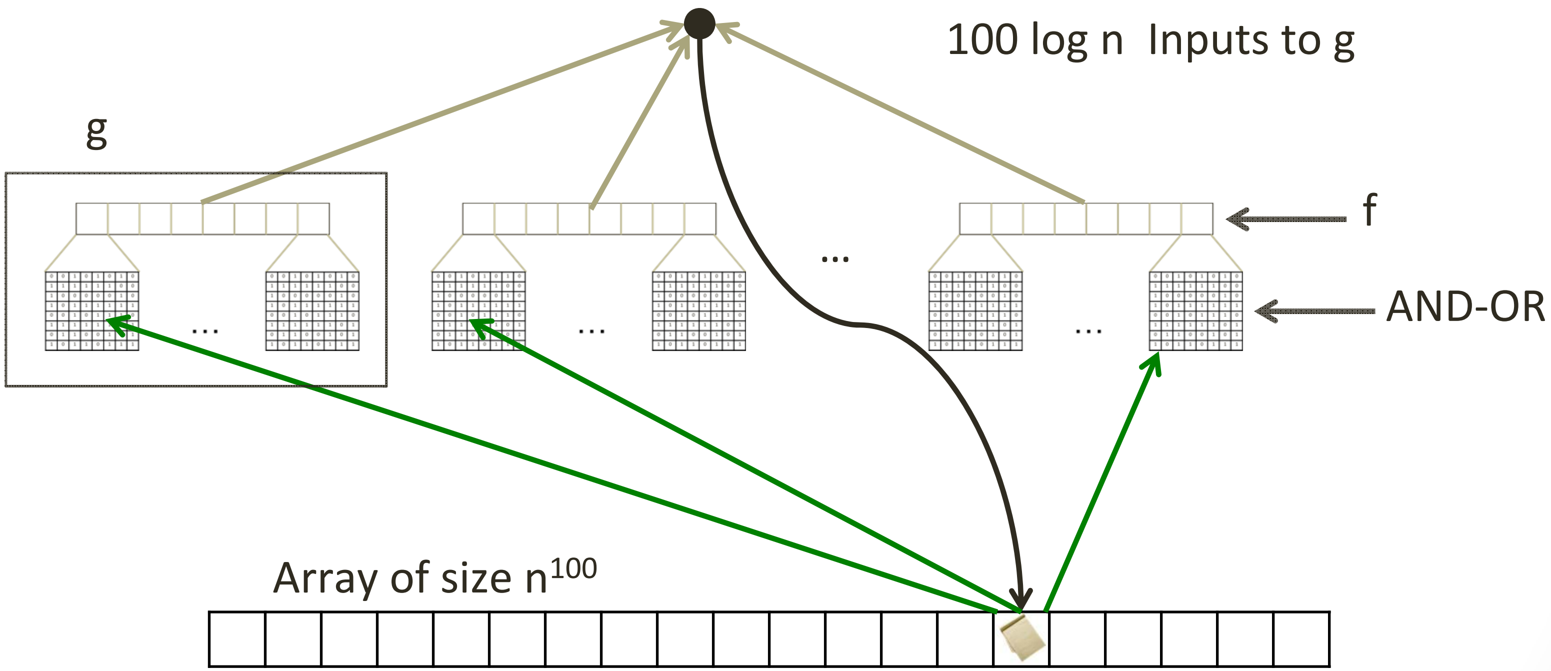
- Setting m=n gives $D(g_{CS}) \approx n^3$, $R(g_{CS}) \approx n^2$

The final function

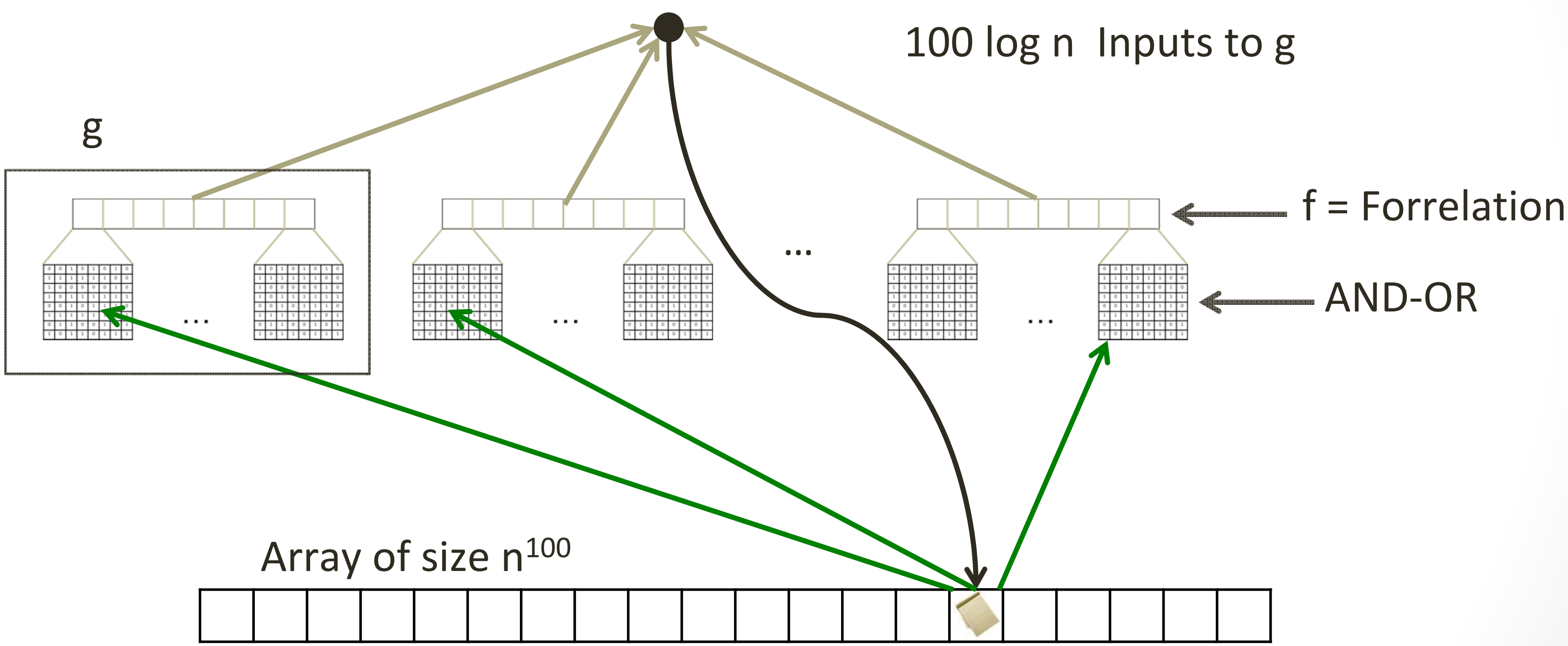


A Super-Grover Speedup

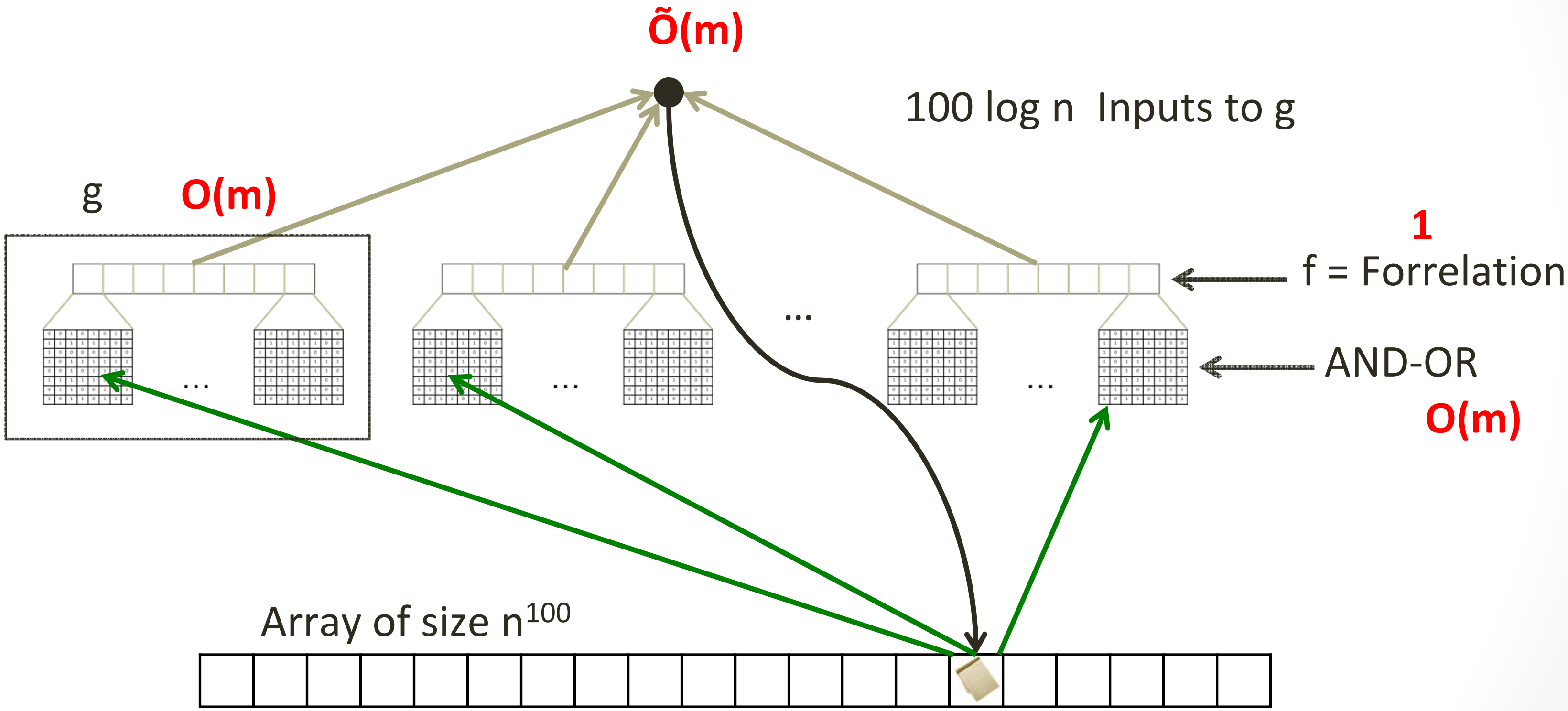
Cheat sheet framework



Cheat sheet framework



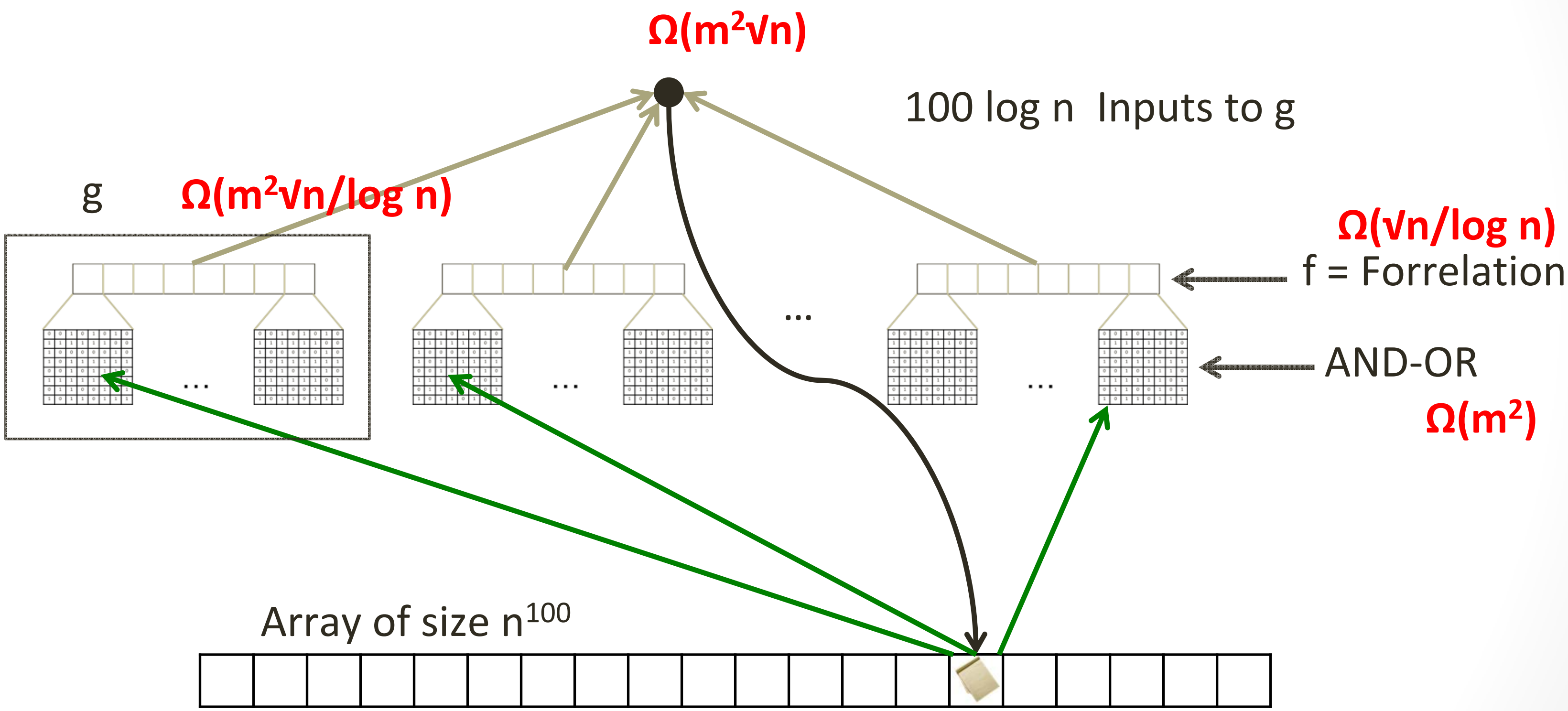
How many quantum queries?



Verifying certificate for g: **n** queries to read input to f, plus **√m√n** to Grover search over n certificates of size m looking for an error

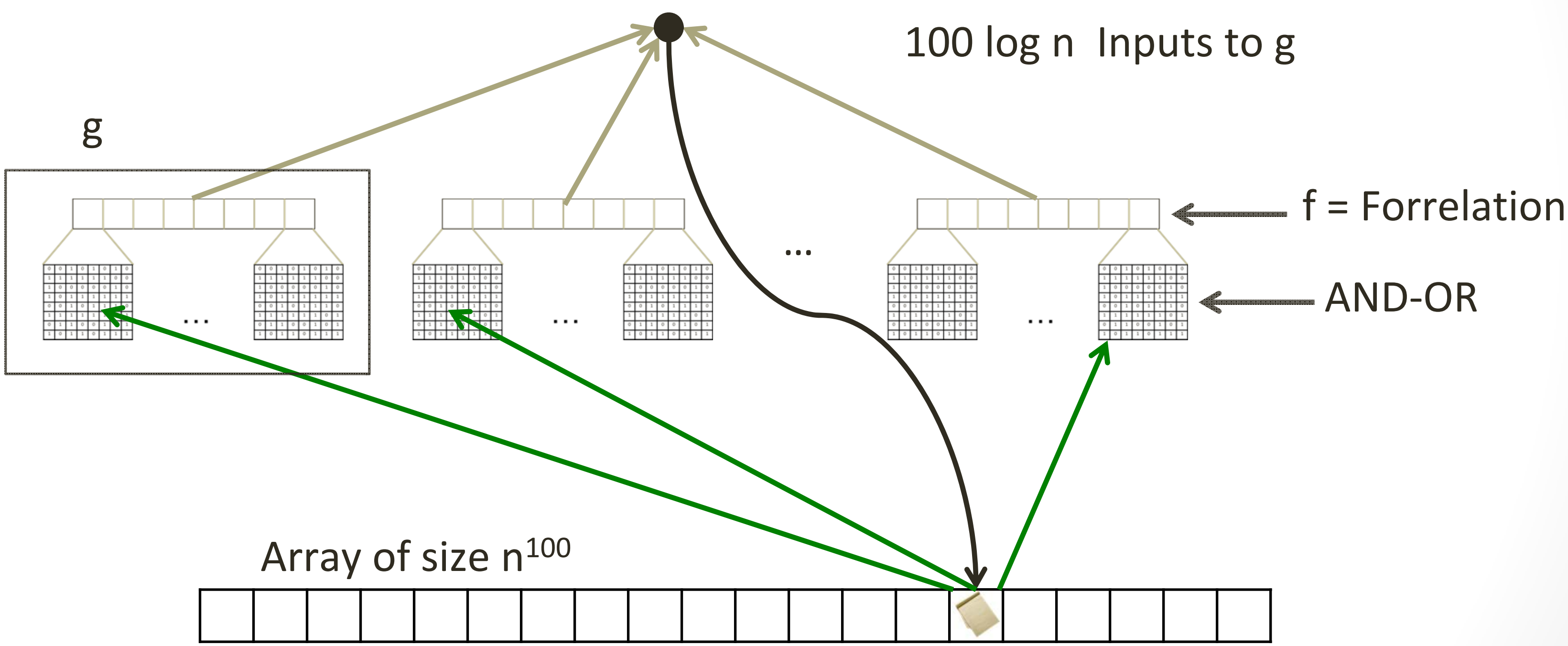
Total: $\tilde{O}(n + m + \sqrt{m}\sqrt{n}) = \tilde{O}(n)$ if $m = n$

How many random queries?



Total: $\Omega(n^{2.5})$ if $m = n$

Conclusion: power 2.5 speedup



Summary

- Power 2.5 separation between randomized and quantum query complexity
- Becomes power 3 separation if we can show a log n vs. n separation in the promise setting
- Best known upper bound is 6

Communication Complexity?

- We want to lift this to communication complexity
- We could use a measure that
  - Lower-bounds R(f)
  - Give a good lower bound for forrelation (or Simon’s)
  - Composes (with AND-OR)
  - Is preserved under addition of cheat sheets
  - Lifts to communication lower bound

- Alternatively, lift to communication complexity before adding cheat sheets
- Prove a lower bound on R for cheat sheet functions in communication complexity

More Complexity Measures

	D	R_0	R	C	RC	bs	Q_E	deg	Q	$\widetilde{\deg}$
D		2, 2 [ABB+15]	2*, 3 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GPW15]	4*, 6 [ABB+15]	4*, 6 [ABB+15]
R_0	1, 1 $\oplus$		2, 2 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GJPW15]	3, 6 [ABB+15]	4*, 6 [ABB+15]
R	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1.5, 3 [ABB+15]	2, 3 [GJPW15]	2.5, 6 Th. 1	4*, 6 [ABB+15]
C	1, 1 $\oplus$	1, 1 $\oplus$	1, 2 $\oplus$		2, 2 [GSS13]	2, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 4 $\wedge$	2, 4 $\wedge$
RC	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.5, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
bs	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
Q_E	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$		2, 3 Th. 4	2, 6 $\wedge$	4*, 6 Th. 2
deg	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1, 1 $\oplus$		2, 6 $\wedge$	2, 6 $\wedge$
Q	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 Th. 3	2*, 3 Th. 3	2*, 3 Th. 3	1, 1 $\oplus$	2, 3 Th. 4		4*, 6 Th. 2
$\widetilde{\deg}$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	7/6, 2 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	

New separations

Separations we reprove

More Complexity Measures

	D	R_0	R	C	RC	bs	Q_E	deg	Q	$\widetilde{\deg}$
D		2, 2 [ABB+15]	2*, 3 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GPW15]	4*, 6 [ABB+15]	4*, 6 [ABB+15]
R_0	1, 1 $\oplus$		2, 2 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GJPW15]	3, 6 [ABB+15]	4*, 6 [ABB+15]
R	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1.5, 3 [ABB+15]	2, 3 [GJPW15]	2.5, 6 Th. 1	4*, 6 [ABB+15]
C	1, 1 $\oplus$	1, 1 $\oplus$	1, 2 $\oplus$		2, 2 [GSS13]	2, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 4 $\wedge$	2, 4 $\wedge$
RC	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.5, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
bs	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
Q_E	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$		2, 3 Th. 4	2, 6 $\wedge$	4*, 6 Th. 2
deg	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1, 1 $\oplus$		2, 6 $\wedge$	2, 6 $\wedge$
Q	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 Th. 3	2*, 3 Th. 3	2*, 3 Th. 3	1, 1 $\oplus$	2, 3 Th. 4		4*, 6 Th. 2
$\widetilde{\deg}$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	7/6, 2 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	

New separations

Separations we reprove

Approximate Degree

- Lower bound for Q
- Previous separation: 1.3 (Ambainis 2003)
- This work: 4 - o(1)
- Most complicated function used in query complexity
  - At the time, at least…

Unambiguous Certificates

- A set of unambiguous 1-certificates is a set of 1-certificates for f such that
  - Any two of them contradict each other
  - Any 1-input to f contains one of them

- Example: $f = OR_4$
  - `1___`   `01__`   `001_`   `0001`

- Let $UC^{(1)}(f)$ be the size of the largest certificate in the best choice of unambiguous 1-certificates

Polynomials from $UC^{(1)}$

- Let S be a set of unambiguous 1-certificates for f
- For any certificate c in S, there is a low-degree polynomial p_c for checking if the input contains the certificate
  - $p_c(x)=1$ iff x contains c
  - $\deg(p_c) = |c|$

- Add up p_c for all c in S to get a polynomial p
- Each 1-input contains exactly one certificate in S
  - $p(x) = 1$ if $f(x)=1$
  - $p(x) = 0$ if $f(x)=0$

- Conclusion: $\deg(f) \leq UC^{(1)}(f)$

Approximate degree from $UC^{(1)}$

- Let S be a set of unambiguous 1-certificates for f
- Suppose that for any certificate c in S, there is a low-degree polynomial p_c for checking if the input contains the certificate
  - $p_c(x) \geq 2/3$ if x contains c
  - $p_c(x) = 0$ if x does not contain c

- Add up p_c for all c in S to get a polynomial p
- Each 1-input contains exactly one certificate in S
  - $p(x) \geq 2/3$ if $f(x)=1$
  - $p(x) = 0$ if $f(x)=0$

- Conclusion: $\text{adeg}(f) \leq$ Quantum complexity of checking $UC^{(1)}$ certificates

Cheat Sheets and $UC^{(1)}$

- **Observation**: $UC^{(1)}(f_{CS}) \approx C(f)$
- The unambiguous 1-certificates will be the correct cheat sheet cell and all the certificates it points to

- **Implication**: $deg(f_{CS}) \leq C(f)$
- So adding a cheat sheet to AND-OR gives a quadratic gap between deg and R
- Since certificates for AND-OR can be checked in $\sqrt{m}$ quantum queries, this gives a power 4 separation between adeg and R

- What about Q?

k-Sum

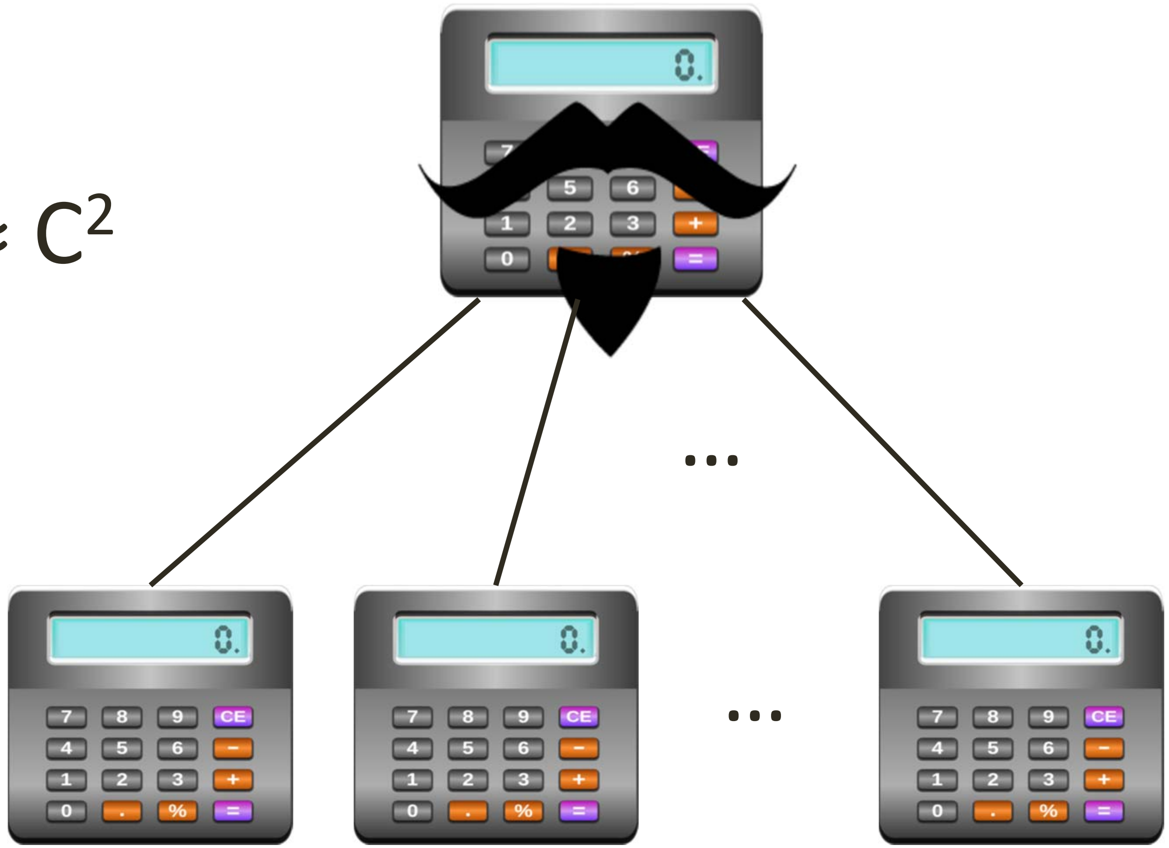
- Are there k elements summing to 0 mod M?
- Set k = log n
- $Q \approx n$, $C^{(1)} \approx \text{polylog } n$ (Belovs and Špalek 2013)

Block k-Sum

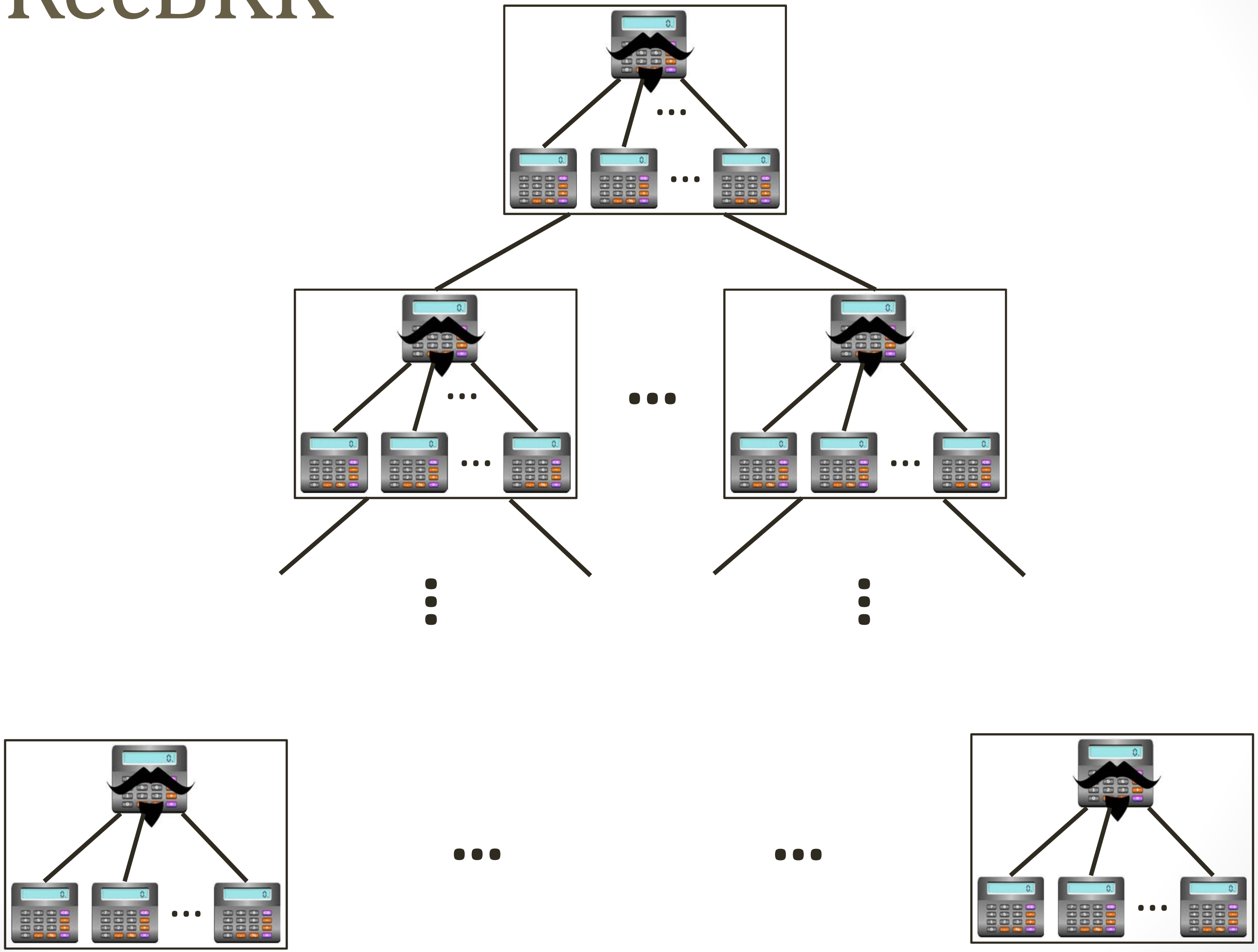
- Split input into blocks; a block is balanced if it has the same number of 0s and 1s
- Balanced blocks represent numbers
- If there are log n balanced blocks summing to 0 mod M and all other blocks have at least as many 1s as 0s, f(x) = 1
- Q is large, all certificates use almost only 1s

BKK

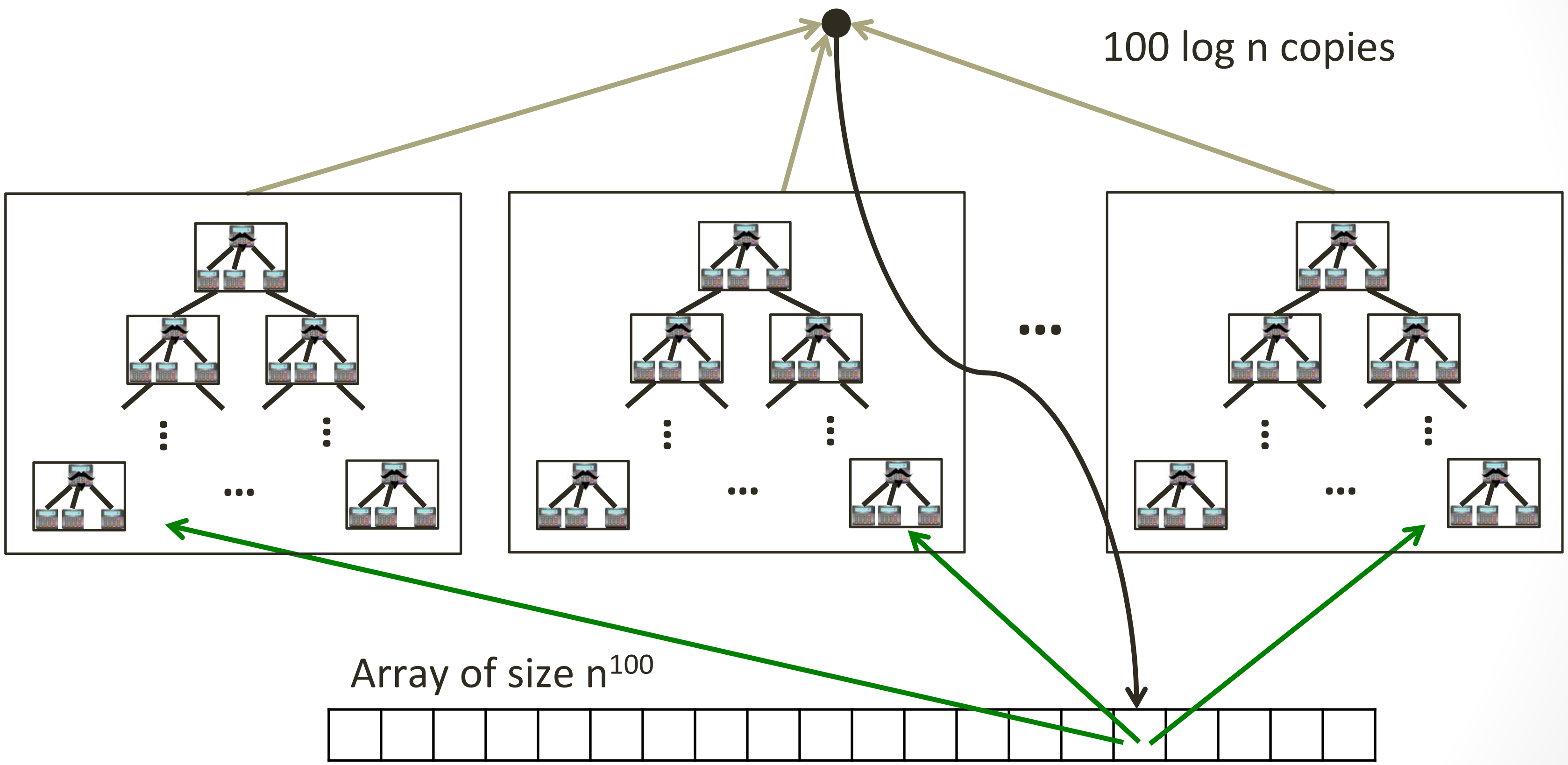
$Q \approx C^2$



RecBKK



$RecBKK_{CS}$



Q ≈ approxdeg$^{4-o(1)}$

Open Problems

	D	R_0	R	C	RC	bs	Q_E	deg	Q	$\widetilde{\deg}$
D		2, 2 [ABB+15]	2*, 3 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GPW15]	4*, 6 [ABB+15]	4*, 6 [ABB+15]
R_0	1, 1 $\oplus$		2, 2 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GJPW15]	3, 6 [ABB+15]	4*, 6 [ABB+15]
R	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1.5, 3 [ABB+15]	2, 3 [GJPW15]	2.5, 6 Th. 1	4*, 6 [ABB+15]
C	1, 1 $\oplus$	1, 1 $\oplus$	1, 2 $\oplus$		2, 2 [GSS13]	2, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 4 $\wedge$	2, 4 $\wedge$
RC	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.5, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
bs	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
Q_E	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$		2, 3 Th. 4	2, 6 $\wedge$	4*, 6 Th. 2
deg	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1, 1 $\oplus$		2, 6 $\wedge$	2, 6 $\wedge$
Q	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 Th. 3	2*, 3 Th. 3	2*, 3 Th. 3	1, 1 $\oplus$	2, 3 Th. 4		4*, 6 Th. 2
$\widetilde{\deg}$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	7/6, 2 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	

Open Problems

	D	R_0	R	C	RC	bs	Q_E	deg	Q	$\widetilde{\deg}$
D		2, 2 [ABB+15]	2*, 3 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GPW15]	4*, 6 [ABB+15]	4*, 6 [ABB+15]
R_0	1, 1 $\oplus$		2, 2 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GJPW15]	3, 6 [ABB+15]	4*, 6 [ABB+15]
R	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1.5, 3 [ABB+15]	2, 3 [GJPW15]	2.5, 6 Th. 1	4*, 6 [ABB+15]
C	1, 1 $\oplus$	1, 1 $\oplus$	1, 2 $\oplus$		2, 2 [GSS13]	2, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 4 $\wedge$	2, 4 $\wedge$
RC	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.5, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
bs	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
Q_E	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$		2, 3 Th. 4	2, 6 $\wedge$	4*, 6 Th. 2
deg	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1, 1 $\oplus$		2, 6 $\wedge$	2, 6 $\wedge$
Q	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 Th. 3	2*, 3 Th. 3	2*, 3 Th. 3	1, 1 $\oplus$	2, 3 Th. 4		4*, 6 Th. 2
$\widetilde{\deg}$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	7/6, 2 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	

Open Problems

	D	R_0	R	C	RC	bs	Q_E	deg	Q	$\widetilde{\deg}$
D		2, 2 [ABB+15]	2*, 3 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GPW15]	4*, 6 [ABB+15]	4*, 6 [ABB+15]
R_0	1, 1 $\oplus$		2, 2 [ABB+15]	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2, 3 [ABB+15]	2, 3 [GJPW15]	3, 6 [ABB+15]	4*, 6 [ABB+15]
R	1, 1 $\oplus$	1, 1 $\oplus$		2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1.5, 3 [ABB+15]	2, 3 [GJPW15]	2.5, 6 Th. 1	4*, 6 [ABB+15]
C	1, 1 $\oplus$	1, 1 $\oplus$	1, 2 $\oplus$		2, 2 [GSS13]	2, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 4 $\wedge$	2, 4 $\wedge$
RC	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.5, 2 [GSS13]	1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
bs	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$		1.1527, 3 [Amb13]	$\log_3 6$, 3 [NW95]	2, 2 $\wedge$	2, 2 $\wedge$
Q_E	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$		2, 3 Th. 4	2, 6 $\wedge$	4*, 6 Th. 2
deg	1, 1 $\oplus$	1.3267, 2 $\bar{\wedge}$-tree	1.3267, 3 $\bar{\wedge}$-tree	2, 2 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	2*, 3 $\wedge \circ \vee$	1, 1 $\oplus$		2, 6 $\wedge$	2, 6 $\wedge$
Q	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	2, 2 Th. 3	2*, 3 Th. 3	2*, 3 Th. 3	1, 1 $\oplus$	2, 3 Th. 4		4*, 6 Th. 2
$\widetilde{\deg}$	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	7/6, 2 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	7/6, 3 $\wedge \circ$ ED	1, 1 $\oplus$	1, 1 $\oplus$	1, 1 $\oplus$	

Thanks