Factor-11/8 NP-inapproximability for 2-Variable Linear Equations

Sangxia Huang

EPFL
Switzerland

Joint work with Johan Håstad
Rajsekar Manokaran
Ryan O'Donnell
John Wright

2-Variable Linear Equations

Definition ($\text{E2L}\textsc{in}_q$)

System of $\mathbb{F}_q$ linear equations, each containing exactly two variables.

Example of an $\text{E2L}\textsc{in}_2$ instance

$$\begin{aligned} x_1 + x_4 &= 1 \\ x_3 + x_5 &= 0 \\ x_2 + x_5 &= 0 \\ &\dots \end{aligned}$$

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Equivalent to Unique-Games

[Khot, Kindler, Mossel, O'Donnell '07], [Mossel, O'Donnell, Oleszkiewicz '10]

Approximation for 2-Variable Linear Equations

Definition ($(\varepsilon, \varepsilon')$-approximation algorithm)

Given an instance in which the best solution falsifies at most an ε-fraction of the equations, the algorithm finds a solution falsifying at most an ε'-fraction of the equations.

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- Simple observation: there is a $(0, 0)$-approximation algorithm for E2LIN$_q$.

Approximation for 2-Variable Linear Equations

- $(\varepsilon, \frac{2}{\pi}\sqrt{\varepsilon} + o(\varepsilon))$-approximation algorithm for E2LIN$_2$
  [Goemans, Williamson '94]

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  [Charikar, Makarychev, Makarychev, '06]

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- UGC implies that improving on the above algorithms is NP-hard
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- UGC implies that improving on the above algorithms is NP-hard
  [Khot, Kindler, Mossel, O'Donnell '07], [Mossel, O'Donnell, Oleszkiewicz '10]
- If there exists $q = q(\varepsilon)$ such that $(\varepsilon, \omega(\sqrt{\varepsilon}))$-approximating $\textsc{E2Lin}_q$ is NP-hard, then UGC holds
  [Rao '11]

NP-hardness for Approximating E2LIN$_2$

Theorem (Håstad '97)

For any $C < \frac{5}{4}$ *and* $0 < \varepsilon < \varepsilon_0 = \frac{1}{4}$, $(\varepsilon, C\varepsilon)$*-approximating* E2LIN$_2$ *is NP-hard.*

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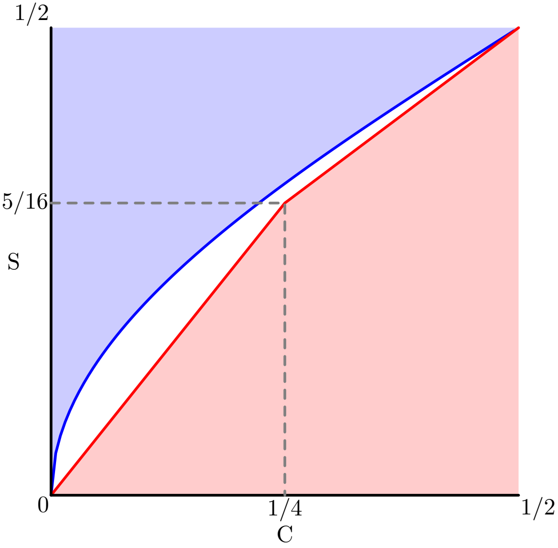
- The same hardness holds for any q, and $\varepsilon_0 \to \frac{1}{2}$ as $q \to \infty$ [O'Donnell, Wright '12]

NP-hardness for Approximating E2LIN$_2$



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For any $C < \frac{11}{8}$, $0 < \varepsilon < \varepsilon_0 = \frac{1}{8}$, *it is NP-hard to* $(\varepsilon, C\varepsilon)$*-approximate* E2LIN$_2$.

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- Our gadget is optimal among all gadget reductions from ϕ

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- Proof by gadget reduction from CSP with some predicate ϕ
- Our gadget is optimal among all gadget reductions from ϕ
- For any predicate ψ whose set of satisfying assignments supports a pairwise independent distribution, no gadget reduction from CSP with predicate ψ can establish NP-hardness factor better than $\frac{1}{1-e^{-1/2}} \approx 2.54$

Gadget Reduction for $(\varepsilon, \frac{5}{4}\varepsilon)$-hardness of E2LIN$_2$

[Håstad '97][Trevisan, Sorkin, Sudan, Williamson '00]

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Theorem (Håstad '97)

For any k ≥ 3 *and* $\varepsilon > 0$*,* $(\varepsilon, \frac{1}{2} - \varepsilon)$*-approximating* E*k*$\text{L}_{\text{IN}_2}$ *is NP-hard.*

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- Take $k = 3$
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$$x_1 + x_2 + x_3 = 0.$$

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$$x_1 + x_2 + x_3 = 0.$$

- Introduce variables y_1, y_2, y_3, y_4, and constraints

$$\begin{aligned}
z + y_1 &= 1, & z + y_2 &= 1, & z + y_3 &= 1, & z + y_4 &= 0,\\
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- Let σ be an assignment to x_1, x_2 and x_3
  - If σ satisfies the equation, then there is an assignment to the auxiliary variables that falsifies 4 equations
  - Otherwise any assignment to the auxiliary variables falsify at least 6 equations

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$\Rightarrow (\frac{4}{16} + \varepsilon, \frac{5}{16} - \varepsilon)$-hardness for E2LIN$_2$

Constructing Gadgets

[Trevisan, Sorkin, Sudan, Williamson '00]

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x_1	x_2	x_3	y_1	y_2	y_3	y_4	...
0	0	0	1	1	1	1	...
0	1	1	1	1	0	0	...
1	0	1	1	0	1	0	...
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- For each satisfying assignment to x_1,x_2,x_3, setting the auxiliary variables according to the matrix above achieves *unsat* at most C

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- Key Observation: If two columns y_i and y_j are identical, then merging them gives a new gadget that is as good

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- Key Observation: If two columns y_i and y_j are identical, then merging them gives a new gadget that is as good $\Rightarrow$ Need at most $2^4 = 16$ variables in the gadget
- Search of optimal gadget as LP
  - $\sim \binom{16}{2}$ variables and $\sim 2^{16}$ constraints
  - Certificate of optimality via dual LP

Stronger Hardness Guarantees

Definition (Had_r)

The Had_r predicate has $k = 2^r$ input variables, one for each subset $S \subseteq [r]$. The input $\{x_S\}_{S \subseteq [r]}$ satisfies Had_r if for each S, $|S| \geq 2$, $x_\emptyset + x_S = \sum_{i \in S}(x_\emptyset + x_{\{i\}})$.

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Theorem (Chan '13)

For every $r \geq 2$ and $\varepsilon > 0$, $(\varepsilon, 1 - 2^{r+1}/2^{2^r} - \varepsilon)$-approximating Had_r-CSP is NP-hard.

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- Distribution of ϕ-constraints $\mathcal{C} \sim \mathcal{I}$, where

$$\mathcal{C} = ((x_1, b_1), \ldots, (x_k, b_k)), \quad x_i \in V, b_i \in \{0, 1\}$$

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- Assignment $A : V \to \{0,1\}$ satisfies constraint $\mathcal{C}$ if

$$\phi(b_1 + A(x_1), \ldots, b_k + A(x_k)) = 1.$$

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- The value of A on $\mathcal{I}$: $unsat(A; \mathcal{I}) = \Pr_{\mathcal{C} \sim \mathcal{I}}[A \text{ falsifies } \mathcal{C}]$.

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Example:

$$\phi(x_1, x_2, \overline{x_3})$$
$$\phi(x_3, x_1, x_4)$$
$$\phi(\overline{x_2}, x_3, x_4)$$

Assignment $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$.

$$\Pr_{(b_1,b_2,b_3)\sim\mathcal{D}}[(b_1, b_2, b_3) = (0, 1, 1)] = 1/3$$
$$\Pr_{(b_1,b_2,b_3)\sim\mathcal{D}}[(b_1, b_2, b_3) = (0, 0, 1)] = 2/3.$$

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Theorem (Håstad '97, restated)

For any $\varepsilon > 0$*, given a* $\text{E}k\text{L}\textsc{in}_2$ *instance, it is NP-hard to distinguish between the following two cases*

Completeness *unsat*$(\mathcal{I}) \leq \varepsilon$

Soundness *For every assignment A,* $d_{TV}(\mathcal{D}(A,\mathcal{I}),\mathcal{U}_k) \leq \varepsilon$

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Theorem (Chan '13, restated)

For every $r \geq 2$ and $\varepsilon > 0$, given an Had_r-CSP instance $\mathcal{I}$, it is NP-hard to distinguish between the following two cases:

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- If $x_1 + x_2 + x_3 + x_4 = 0$, we can set y such that only 1 of the equations are violated

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  $\Rightarrow$ If $unsat(\mathcal{I}) \leq \varepsilon$, then $unsat(\mathcal{J}) \leq \frac{1}{4} + \frac{3}{4}\varepsilon$

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- If $x_1 + x_2 + x_3 + x_4 = 0$, we can set y such that only 1 of the equations are violated
  $\Rightarrow$ If $\mathit{unsat}(\mathcal{I}) \leq \varepsilon$, then $\mathit{unsat}(\mathcal{J}) \leq \frac{1}{4} + \frac{3}{4}\varepsilon$
- If $x_1 = x_2 = x_3 = 1 + x_4$ then $y = x_1$ satisfies all equations

Stronger Hardness Guarantees

A simple $(\varepsilon, \frac{5}{4}\varepsilon)$-hardness gadget reduction from a E4LIN$_2$ instance $\mathcal{I}$ to a E2LIN$_2$ instance $\mathcal{J}$:

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$$x_1 + y = 0, x_2 + y = 0, x_3 + y = 0, x_4 + y = 1$$

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- In all other cases, the best assignment to y violates 2 equations
- If for every assignment A, $d_{TV}(\mathcal{D}(A, \mathcal{I}), \mathcal{U}_k) \leq \varepsilon$, then $unsat(\mathcal{J}) \geq \frac{2}{16} \cdot \frac{0}{4} + \frac{8}{16} \cdot \frac{1}{4} + \frac{6}{16} \cdot \frac{2}{4} - \varepsilon = \frac{5}{16} - \varepsilon$

Constructing Gadgets for $Had_r \to$ E2LIN$_2$

Definition (Had_r)

The Had_r predicate has $k = 2^r$ input variables, one for each subset $S \subseteq [r]$. The input $\{x_S\}_{S \subseteq [r]}$ satisfies Had_r if for each S, $|S| \geq 2$, $x_S + x_\emptyset = \sum_{i \in S}(x_i + x_\emptyset)$.

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- For $i \in \{0,1\}^r$, define the *i*-th dictator assignment $D_i(f) := f(i)$
  - The value of the primary variables under D_i and $1 + D_i$ satisfy Had_r

Constructing Gadgets for $Had_r \to$ E2LIN$_2$

Definition

A (c, s)-gadget reducing Had_r to E2LIN$_2$ is a gadget $\mathcal{G}$ satisfying the following properties:

Completeness For every dictator assignment D_i, $unsat(D_i, \mathcal{G}) \leq c$

Soundness The expected $unsat(\cdot; \mathcal{G})$ over uniformly random assignments to the primary variables is at least s.

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Proposition

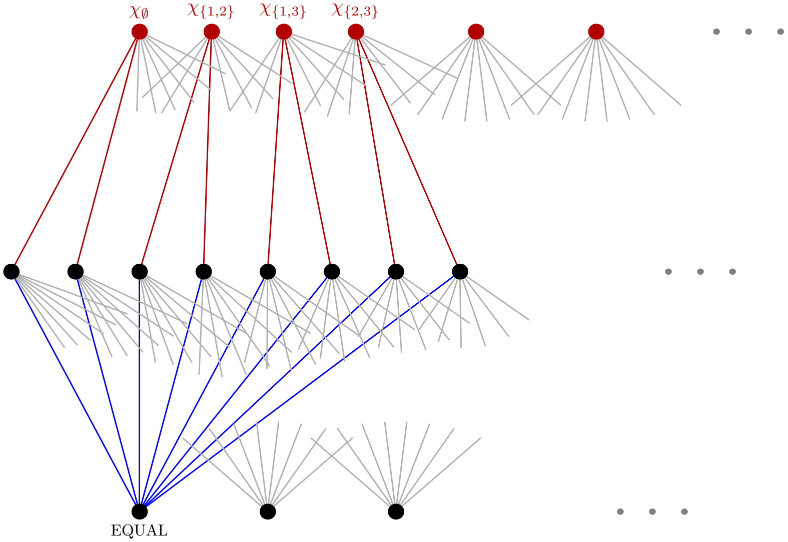
Suppose there exists a (c, s)-gadget reducing Had_r to E2LIN$_2$*, then given an* E2LIN$_2$ *instance $\mathcal{I}$, it is NP-hard to distinguish*

Completeness $unsat(\mathcal{I}) \leq c + \varepsilon$

Soundness $unsat(\mathcal{I}) \geq s - \varepsilon$

Constructing Gadgets for $Had_r \rightarrow$ E2LIN$_2$

Our $\left(\frac{1}{8}, \frac{11}{64}\right)$-Gadget



Constructing Gadgets for $Had_r \rightarrow$ E2LIN$_2$

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  ...

- Had_∞: ?
  $s/c \geq 3/2$ if a certain Game Show Conjecture is true

Limitation of Gadget Construction

Theorem

Let $\mathcal{G}$ be a (c, s)-gadget reducing Had_r to $\text{E2L}\textsc{in}_2$. *Then*

$$\frac{s}{c} \leq \frac{1}{1 - e^{-1/2}} \approx 2.54$$

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Remark

We can assume that all equations in the gadget has the form $x + y = 0$, where x and y are at Hamming distance 1, as long as we only consider assignments A such that $A(1 + x) = 1 + A(x)$.

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For $x, y \in \{0, 1\}^K$, define their similarity

$$sim(x, y) := Pr_{i \sim [K]}[x_i = y_i],$$

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$$sim(x, P^{\pm}) = \max_{y \in P^{\pm}} sim(x, y).$$

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Observation

For any $x \in V$, if there exists $y \in P^{\pm}$, such that $sim(x, y) > \frac{3}{4}$, then for all other $y' \in P^{\pm}$, $sim(x, y') < \frac{3}{4}$.

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1. Pick a number $t \sim \mathcal{D}$
2. Pick a uniformly random index $i \in [K]$, and let D_i be the corresponding dictator
3. For variable $x \in V$, if $sim(x, P^{\pm}) > t$, assign x the value of its closest primary variable, otherwise assign x according to D_i.

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If $t \in [3/4, s)$, then $x + y = 0$ always satisfied.

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If $t \in [s, s + 1/K)$, expected *unsat* is $1/2$.

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If $t \in [3/4, s)$, then $x + y = 0$ always satisfied.

If $t \in [s, s + 1/K)$, expected *unsat* is $1/2$.

If $t \in [s + 1/K, 1]$, expected *unsat* is $1/K$.

Summary

- Optimal gadget reduction for E2LIN$_2$ from stronger hardness guarantees for Had_3-CSP
- Limitations of gadget construction for showing hardness of E2LIN$_2$

Thank you!