Lieb's concavity theorem, matrix geometric means, and semidefinite optimization

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Feb. 3, 2016

Joint work with Hamza Fawzi (MIT)

"How to optimize with quantum relative entropy?" (O. Fawzi)

Quantum relative entropy

$$S(A||B) = \mathrm{tr}[A \log(A) - A \log(B)] \quad \text{for } A, B \succ 0$$

Why?

- compute classical-quantum channel capacities (Sutter, Sutter, Esfahani, Renner 2014)
- test conjectures about properties of $S(\cdot||\cdot)$

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- test conjectures about properties of $S(\cdot||\cdot)$

Unsatisfactory answer: use (sub)gradient methods

"Can we use semidefinite optimization?" (H. Fawzi)

Does *epigraph*

$$\mathrm{epi}(S) = \{(A, B, t) : \mathrm{tr}[A \log(A) - A \log(B)] \leq t\}$$

have PSD lift?

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Obvious answer: No!

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Obvious answer: No!

Motivating questions:

- Does (compact base of) epi(S) have an approximate PSD lift?
- Is there a lift with size depending on $\log(1/\epsilon)$?

One approach: Lieb $\to$ Tsallis $\to$ $S(\cdot||\cdot)$

Theorem (Lieb 1973)

If $K \in \mathbb{C}^{m\times n}$ is fixed and $t \in [0,1]$ then

$$(A,B) \mapsto \mathrm{tr}[KA^{1-t}K^*B^t]$$

is jointly concave for positive definite A, B.

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Convexity of $S(A||B) = \mathrm{tr}[A\log(A) - A\log(B)]$ follows from

$$S(A||B) = \lim_{t\to 0} \mathrm{tr}\left[A^{1-t}\left(\frac{A^t - I}{t} - \frac{B^t - I}{t}\right)\right]$$
$$= \lim_{t\to 0} \frac{1}{t}\mathrm{tr}\left[A - A^{1-t}B^t\right]$$

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Central question:

Does epigraph of Lieb's function have small PSD lift (for $t \in \mathbb{Q}$)?

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Outline

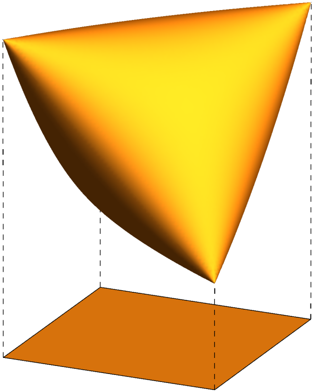
- Preliminaries
- Weighted matrix geometric mean
- Lieb-Ando function (and beyond)
- Back to approximating quantum relative entropy

PSD lifts

A convex set $C \subseteq \mathbb{R}^n$ has a PSD lift of size d if

$$C = \left\{ x \in \mathbb{R}^n : \quad \exists y \in \mathbb{R}^m \text{ s.t. } A_0 + \sum_{i=1}^n A_i x_i + \sum_{j=1}^m B_j y_j \succeq 0 \right\}$$

for symmetric $d \times d$ matrices $A_0, \ldots, A_n, B_1, \ldots, B_m$



Matrix concave/convex functions

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a convex set.

Definition: A function $f : \mathcal{C} \to \mathcal{S}^k$ is *matrix convex* if for all $\lambda \in [0, 1]$ and all $A, B \in \mathcal{C}$

$$f(\lambda A + (1 - \lambda)B) \preceq \lambda f(A) + (1 - \lambda)f(B)$$

- For matrix convex functions $f : \mathcal{C} \to \mathcal{S}^k$ want PSD lift of $\mathcal{S}^k_+$-*epigraph*:

$$\text{epi}(f) = \{(x, T) \in \mathcal{C} \times \mathcal{S}^k : f(x) \preceq T\}$$

- For matrix concave functions $f : \mathcal{C} \to \mathcal{S}^k$ want PSD lift of $\mathcal{S}^k_+$-*hypograph*:

$$\text{hyp}(f) = \{(x, T) \in \mathcal{C} \times \mathcal{S}^k : f(x) \succeq T\}$$

Scalar weighted geometric mean

Weighted geometric mean:

$$(a, b) \mapsto a^{1-t} b^t$$

for $t \in \mathbb{R}$ and $a, b > 0$

Concavity/convexity properties:

- If $t \in [0, 1]$ jointly concave in $a, b > 0$
- If $t \in [-1, 0] \cup [1, 2]$ jointly convex in $a, b > 0$

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PSD lift [Ben-Tal and Nemirovski (2001)]
If $p/q \in [0, 1]$ is rational,

$$\text{hyp}_t := \{(a, b, \tau) : a \geq 0,\ b \geq 0,\ a^{1-p/q} b^{p/q} \geq \tau\}$$

has PSD (actually SOCP) lift of size $O(\log(q))$

(Similar result for $t \in [-1, 0] \cup [1, 2]$ and epigraph)

Scalar weighted geometric mean

Weighted geometric mean:

$$(a, b) \mapsto a^{1-t} b^t = a(b/a)^t = a^{1-t} \otimes b^t$$

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If A and B positive definite and $t \in \mathbb{R}$ define

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}$$

If A and B commute then $A \#_t B = A^{1-t} B^t$

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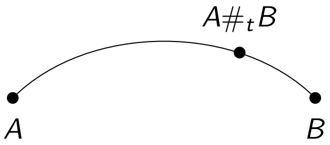
Geometric interpretation



If $s, t \in \mathbb{R}$

- $t \mapsto A \#_t B$ parameterizes geodesic through A and B
- $A \#_t B = B \#_{1-t} A$
- $A \#_s (A \#_t B) = A \#_{st} B$

Concavity properties

If $t \in [0,1]$ then $(A,B) \mapsto A\#_t B$ is matrix concave

Proof idea (Bhatia) For $t \in (0,1)$

$$A\#_t B = \int_0^\infty (\lambda A^{-1} + B^{-1})^{-1} \lambda^{t-1} \frac{\sin(\pi t)}{\pi}\, d\lambda$$

If $t \in [-1,0] \cup [1,2]$ then $(A,B) \mapsto A\#_t B$ is matrix convex

Semidefinite descriptions of $A\#_t B$

Theorem

- If $t = p/q \in [0,1]$ is rational then

$$\mathrm{hyp}_{p/q} = \{(A, B, T) : A \succeq 0,\ B \succeq 0,\ A\#_{p/q}B \succeq T\}$$

has a PSD lift of size at most $(4\lfloor \log_2(q) \rfloor + 3)n$.

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- Concave case due to Sagnol (2013), not very explicit
- We give explicit lifts—CVX code is available

Flavor of construction

Dyadic rational case ($0 \leq p \leq q = 2^{\ell}$): Construction comes from

- PSD lift for $p/q = 1/2$ (base case)
- Digits of binary expansion of $p/2^{\ell}$

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Example: $5/8$ in binary is $(0.101)_2$

$$A\#_{5/8}B = A\#_{(0.101)_2}B = B\#_{1/2}(A\#_{1/2}(B\#_{1/2}A))$$

Leads to (free) PSD lift:

$$\text{hyp}_{5/8} = \Bigg\{(A, B, Z_3) : \exists Z_1, Z_2 \text{ s.t.} \\ \begin{bmatrix} B & Z_3 \\ Z_3 & Z_2 \end{bmatrix} \succeq 0,\ \begin{bmatrix} A & Z_2 \\ Z_2 & Z_1 \end{bmatrix} \succeq 0,\ \begin{bmatrix} A & Z_1 \\ Z_1 & B \end{bmatrix} \succeq 0 \Bigg\}$$

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- Proof requires monotonicity of matrix geometric mean
- Other cases obtained by transformations that reduce to dyadic rational case

Lieb-Ando theorem

Theorem (Lieb 1973, Ando 1979)

Fix $K \in \mathbb{C}^{m \times n}$.

$$\mathcal{S}_+^n \times \mathcal{S}_+^m \ni (A, B) \mapsto \mathrm{tr}[KA^{1-t}K^*B^t]$$

- *jointly concave for* $t \in [0, 1]$
- *jointly convex for* $t \in [-1, 0] \cup [1, 2]$

Equivalently:

$$\mathcal{S}_+^n \otimes \mathcal{S}_+^m \ni (A, B) \mapsto A^{1-t} \otimes B^t$$

- jointly matrix concave for $t \in [0, 1]$
- jointly matrix convex for $t \in [-1, 0] \cup [1, 2]$

Lieb-Ando function

If A and B are positive definite and $t \in \mathbb{R}$ consider

$$(A, B) \mapsto A^{1-t} \otimes B^t$$

Lieb-Ando vs weighted matrix geometric mean

Nikoufar, Ebadian, Gordji (2013): Since $A \otimes I$ and $I \otimes B$ commute

$$A^{1-t} \otimes B^t = (A^{1-t} \otimes I)(I \otimes B^t) = (A \otimes I)^{1-t}(I \otimes B)^t = (A \otimes I) \#_t (I \otimes B)$$

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Corollaries:

- For $t \in [0, 1]$, $A^{1-t} \otimes B^t$ jointly matrix concave (Lieb)
- For $p/q \in [0, 1]$ rational, matrix hypograph has PSD lift of size $O(mn \log(q))$
- For $t \in [-1, 0] \cup [1, 2]$ $A^{1-t} \otimes B^t$ jointly matrix convex (Ando)
- For $p/q \in [0, 1]$ rational, matrix epigraph has PSD lift of size $O(mn \log(q))$

Lieb-Ando function

$$(A, B) \mapsto A^{1-t} \otimes B^t$$

Concavity/convexity properties

- used in many proofs in quantum information
- extends to $(A_1, \ldots, A_k) \mapsto A^{t_1} \otimes \cdots \otimes A^{t_k}$ where $t_1, \ldots, t_k \geq 0$ and $\sum_i t_i = 1$
- also hold when restrict to symmetry classes of tensors, Hadamard products, etc.

Straightforward to obtain PSD lifts for

- multivariate case (but they get much bigger...)
- matrix hypo/epigraphs when restricted to symmetry classes of tensors, Hadamard products, etc.

Related functions

Function	Properties	PSD lift size ($t = p/q$)
Matrix geometric mean $(A, B) \mapsto A \#_t B$	$\mathcal{S}_+^n$-concave if $t \in [0, 1]$ $\mathcal{S}_+^n$-convex if $t \in [-1, 0] \cup [1, 2]$	$O(n \log(q))$
Lieb-Ando function $(A, B) \mapsto A^{1-t} \otimes B^t$	$\mathcal{S}_+^{mn}$-concave if $t \in [0, 1]$ $\mathcal{S}_+^{mn}$-convex if $t \in [-1, 0] \cup [1, 2]$	$O(mn \log(q))$
$A \mapsto t\,\mathrm{tr}[(K^* A^t K)^{1/t}]$ (fixed $n \times m$ matrix K)	concave if $t \in (0, 1]$ convex if $t \in [-1, 0) \cup [1, 2]$	$O(mn \log(q))$
Tsallis relative entropy $(A, B) \mapsto \frac{1}{t}\mathrm{tr}[A - A^{1-t} B^t]$	convex if $t \in [0, 1]$ $\to S(A\|\|B)$ as $t \to 0$	$O(n^2 \log(q))$

Motivation: towards quantum relative entropy

Quantum relative entropy

$$S(A||B) = \mathrm{tr}[A\log(A) - A\log(B)]$$

- Limit of Tsallis relative entropy

$$S(A||B) = \frac{1}{\epsilon}\mathrm{tr}[A - A^{1-\epsilon}B^{\epsilon}] + O(\epsilon)$$

- Get good approximation by choosing $\epsilon = 2^{-k}$
- PSD lift has size $O(n^2 k)$.
- But standard numerical algorithms become ill-conditioned as k grows

Conclusion

This talk

- Matrix hypograph of $A\#_{p/q}B$ has PSD lift of size $O(n\log(q))$
- Gives PSD lift of hypograph/epigraph of Lieb-Ando function (and others) for rational parameters
- Gives approximations of quantum relative entropy

Problems

- Improved approximate PSD lifts for quantum relative entropy
- PSD lifts for epigraph of sandwiched Renyi entropy

$$(A, B) \mapsto \operatorname{tr}\left[\left(A^{\frac{1-t}{2t}} B A^{\frac{1-t}{2t}}\right)^{t}\right] \qquad \text{for rational } t > 1$$

For more information

Preprint

- Fawzi, Saunderson, "Lieb's concavity theorem, matrix geometric means, and semidefinite optimization" arXiv:1512.03401

Code

- http://www.mit.edu/~hfawzi/lieb_cvx.html

Thank you!