Rounding the Sparsest-Cut SDP on Low Threshold-Rank Graphs

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(Joint work with Amit Deshpande, MSR and Prahladh Harsha, TIFR)

SDP and Matrix Methods, NUS 2016

Outline

1 Introduction

2 Cheeger or Spectral Approach

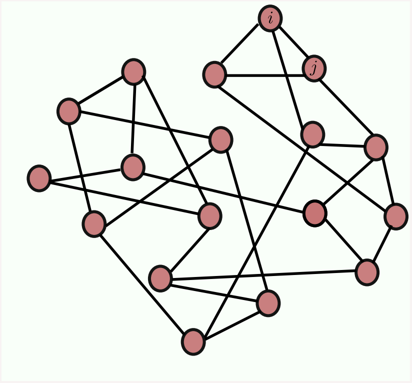
3 Rounding a stronger SDP
- Our Algorithm

4 Goemans' Theorem

5 Summary

The Sparsest Cut Problem

d-regular graph G



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$$\Phi(S) = \frac{|E(S,\bar{S})|}{|S||\bar{S}|}$$

- Sparsest Cut in G is

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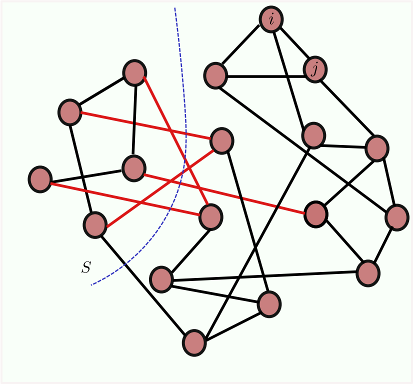

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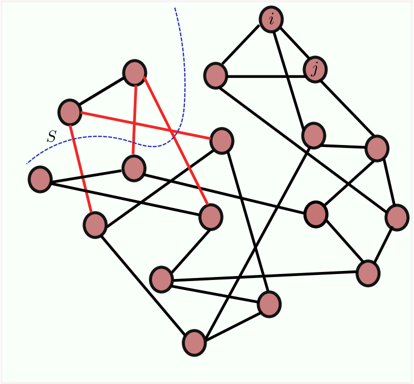

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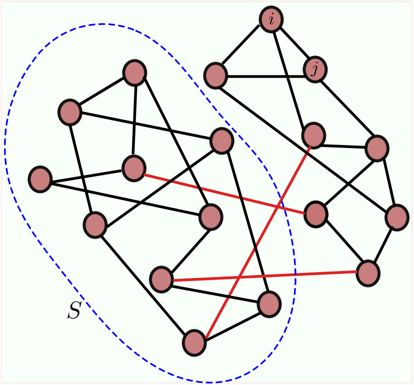

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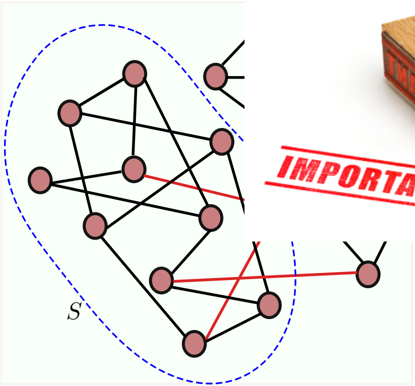

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Cut in G is

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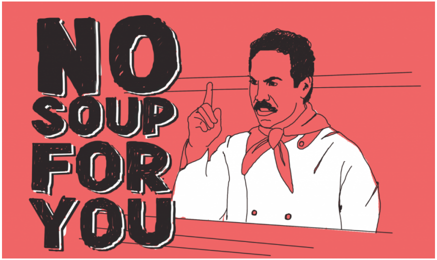
Complexity of Sparsest Cut: Lower bounds

- NP-hard to compute exactly
- Assuming the *Unique Games Conjecture*, it is NP-hard to *approximate* to any constant factor [CKKRS '05, Khot-Vishnoi '05]



Sparsest Cut Objective

$$\Phi_{OPT} = \min_S \frac{|E(S,\bar{S})|}{|S||\bar{S}|}$$

- Original Objective

$$\Phi_{OPT} = \min_{x_i \in \{0,1\}} \frac{\sum_{ij \in E}(x_i - x_j)^2}{\sum_{ij \in V \times V}(x_i - x_j)^2}$$

- Where $x_i = 1$ if $i \in S$, $x_i = 0$ if $i \in \bar{S}$

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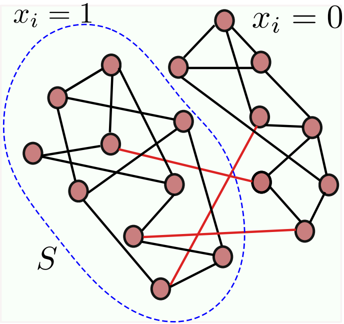

Relaxing the Objective

- Assign a vector $x_i \in \mathbb{R}^m$ for each 0/1 variable
- Ideally, the vectors should be just scalars 0 or 1, i.e. one-dimensional
- Sparsest Cut objective relaxation (a SDP):

$$\Phi_1 = \min_{x_i \in \mathbb{R}^m} \quad \frac{\sum_{ij \in E} \|x_i - x_j\|_2^2}{\sum_{ij \in V \times V} \|x_i - x_j\|_2^2}$$

- $$\boxed{\Phi_1 \le \Phi_{OPT}}$$
- Can add in more constraints on vectors that 0/1 variables satisfy, e.g.

$$x_i^{\text{old}} \in [0,1] \text{ becomes } \|x_i\|^2 \le 1$$

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Rounding the Relaxation

- Extract out a 1-dimensional 0/1 solution $\{y_i\}$ from the SDP solution $\{x_i\}$ in m dimensions
- We will lose in objective value since $\Phi_{\mathrm{ALG}} = \Phi(\{y_1, \ldots, y_n\}) \geq \Phi(\{x_1, \ldots, x_n\}) = \Phi_1$
- For Sparsest Cut, suffices to get an *embedding* into ℓ_1, rather than pure $\{0, 1\}$ solutions for y_i's.

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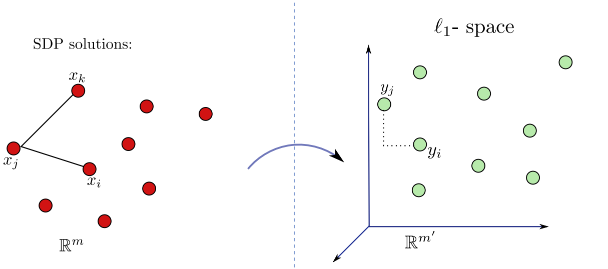
Connection to ℓ_1 embeddings [LLR '94, AR '94]

- Given a mapping of the points: $Y : V \to \mathbb{R}^{m'}$, we can produce a cut T of cost:

$$\Phi(T) \leq \frac{\sum_{ij \in V} \|y_i - y_j\|_1}{\sum_{kl \in V \times V} \|y_k - y_l\|_1}$$

- Sufficient to produce an *embedding* of the SDP solutions into ℓ_1-space

ℓ_1 embeddings of SDP solutions



$$OBJ = \frac{\sum_e \|x_i - x_j\|_2^2}{\sum_{i,j \in V} \|x_i - x_j\|_2^2} \approx OBJ(\ell_1) = \frac{\sum_e \|y_i - y_j\|_1}{\sum_{i,j \in V} \|y_i - y_j\|_1}$$

- Will compare $\|y_i - y_j\|_1$ to $\|x_i - x_j\|^2$

Average distortion and guarantee

If

- (Contraction):

$$\|y_i - y_j\|_1 \leq \|x_i - x_j\|^2, \text{ for every } i, j$$

- (Average Dilation)

$$\sum_{ij} \|y_i - y_j\|_1 \geq \frac{1}{D} \cdot \sum_{ij} \|x_i - x_j\|^2$$

then we have a $O(D)$ - approximation.

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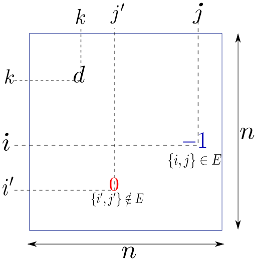
Cheeger Rounding (Alon-Milman '85)

$$\Phi_1 = \min_{x_i \in \mathbb{R}^m} \quad \frac{\sum_{ij \in E} \|x_i - x_j\|^2}{\sum_{ij \in V \times V} \|x_i - x_j\|^2}$$

- Given a SDP solution $\{x_i\}_i$ with objective value $\Phi_1 = \dfrac{\epsilon d}{n}$, can get a rounded solution (a cut) with value $O(\dfrac{\sqrt{\epsilon} d}{n})$.
- The ℓ_1 mapping is a simple one-dimensional embedding: find a specific co-ordinate t and set $y_i = x_i[t]$
- Works well for *expander* graphs

Spectral Graph Theory- Preliminaries

- Laplacian of a graph: $L = d\mathbb{I} - A$



- Eigenvalues of the Laplacian

$$0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2d$$

- Eigenvalues predict connectivity properties
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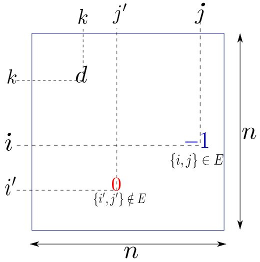

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Cheeger Rounding Guarantee

- Can show the following:
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  - x_i's are in fact, one-dimensional. Furthermore, $x_i = u_2(i)$, where

$$Lu_2 = \lambda_2 u_2$$

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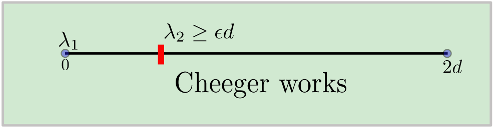
- $\Phi_{\text{ALG}} \leq O(\sqrt{\dfrac{d}{\lambda_2}})\ \Phi_{\text{OPT}}$
- This works when $\lambda_2 \geq \epsilon d$ (an *expander*)



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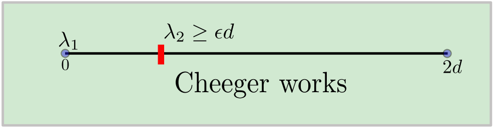

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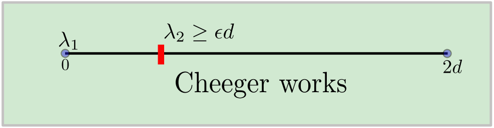
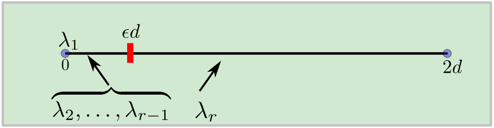


Improved Cheeger [KKLGT '13]

- An improved analysis by Kwok et al. gives the guarantee:

$$\Phi_{\mathsf{ALG}} \leq O(r)\sqrt{\frac{d}{\lambda_r}}\Phi_{\mathsf{OPT}}$$

- This is a $O(r)$ guarantee on graphs where $\lambda_r \geq \epsilon d$.
- Such graphs are said to have threshold rank r



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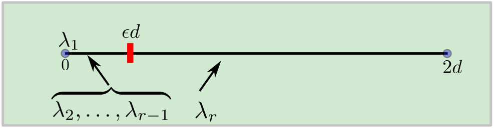

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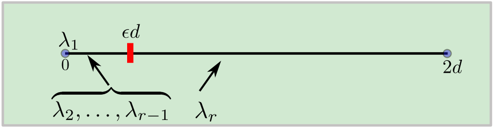

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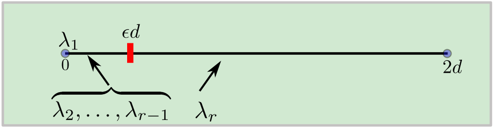

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A stronger SDP relaxation

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$$\text{s.t.} \quad \|x_i - x_j\|^2 + \|x_j - x_k\|^2 \geq \|x_i - x_k\|^2 \quad \forall i, j, k \in [n]$$

$(\ell_2^2$ inequality constraints$)$

- Constraints are triangle inequalities on *squares* of distances
  - Satisfied by $0, 1$ integral solutions

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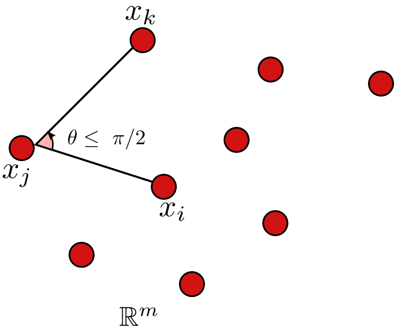
$$(\ell_2^2 \text{ inequality constraints})$$

- Here: A simple rounding algorithm for the above SDP with an $O(r)$ approximation when $\lambda_r \geq \epsilon d$.

ℓ_2^2 constraints

- $\|x_i - x_j\|^2 + \|x_j - x_k\|^2 \geq \|x_i - x_k\|^2$



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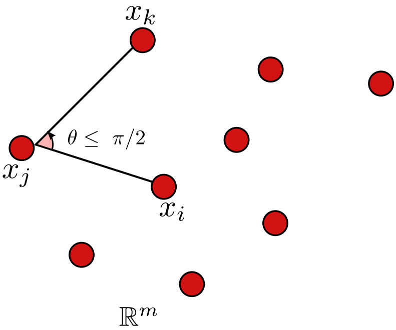

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Do ℓ_2^2 inequalities help?

- Best known unconditional guarantee for Sparsest Cut by Arora-Rao-Vazirani (ARV) rounds the above SDP to give

$$\boxed{\Phi_{\mathsf{ARV}} \leq O(\sqrt{\log n})\ \Phi_\Delta}$$

- Can we leverage them to do better on low threshold-rank graphs?
- Note: Can assume that $\Phi_\Delta \leq \dfrac{\epsilon d}{100n} \leq \dfrac{\lambda_r}{100n}$
  - Else, use Cheeger rounding to get a cut of sparsity $O(\frac{\sqrt{\epsilon} d}{n}) \leq \frac{1}{\sqrt{\epsilon}} \Phi_\Delta$

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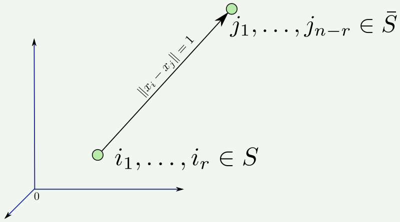
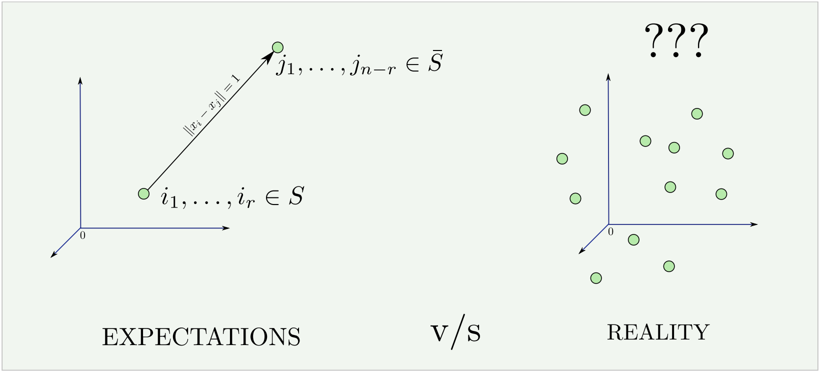
SDP Solutions



EXPECTATIONS

SDP Solutions

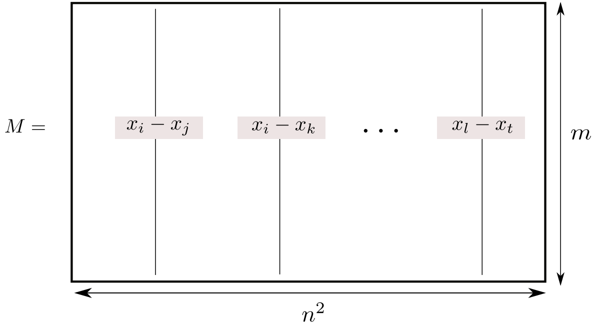
Can't beat NP-Hardness:



Life 101...

There is structure, though

- The *Difference Matrix M* has some structure



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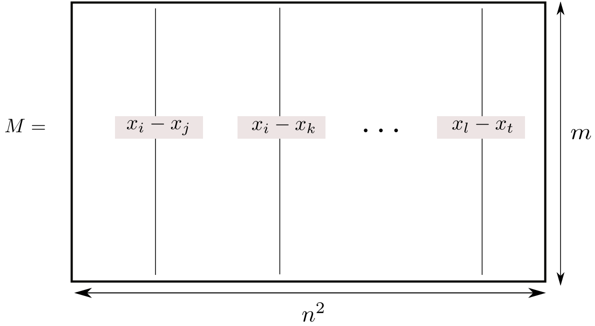

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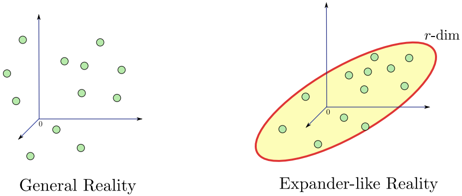
Tool 1 : Approximately Low-Dimensional solutions [GS'13]



$$\boxed{\frac{\lambda_r}{n} \geq 100\Phi_\Delta \implies \sum_{i=1}^{r} \sigma_i^2 \geq 0.99 \sum_{i=1}^{n} \sigma_i^2}$$

- Constant fraction of the squared mass of the vectors $\{x_i - x_j\}_{ij}$ lies in a r-dimensional subspace
- Shift vectors x_i to have centroid as origin, above works with x_i
- *Stable Rank*: $\mathrm{sr}(M) \triangleq \frac{\|M\|_F^2}{\sigma_1(M)^2} \leq r/0.99$

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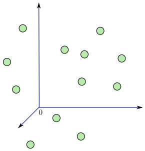
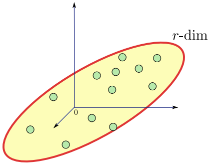


General Reality



Expander-like Reality

$$\frac{\lambda_r}{n} \geq 100\Phi_\Delta \implies \sum_{i=1}^{r} \sigma_i^2 \geq 0.99 \sum_{i=1}^{n} \sigma_i^2$$

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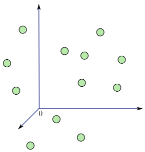

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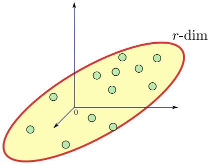

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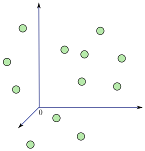

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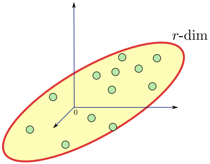

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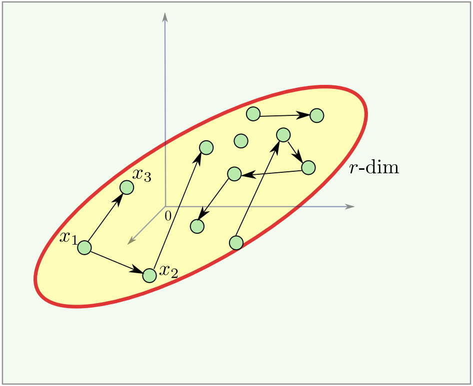

Tool 2: Stronger Cauchy-Schwarz using ℓ_2^2 inequalities

Proposition

If x_i satisfy ℓ_2^2-inequalities, then $\forall i, j, k, l$, we have:

$$| \langle x_i - x_j, x_k - x_l \rangle | \leq \min \left\{ \|x_i - x_j\|^2, \|x_k - x_l\|^2 \right\}$$

Proof.

Left as easy exercise (see board). $\square$

- Note: Simple Cauchy Schwarz would give: $| \langle x_i - x_j, x_k - x_l \rangle | \leq \|x_i - x_j\| \|x_k - x_l\|$

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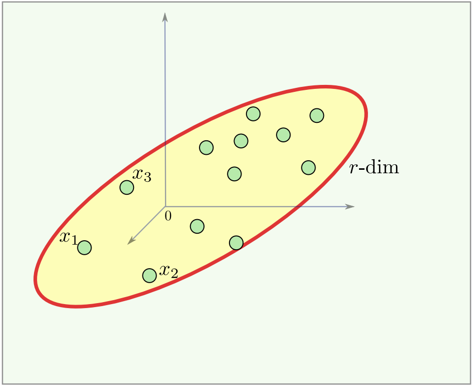

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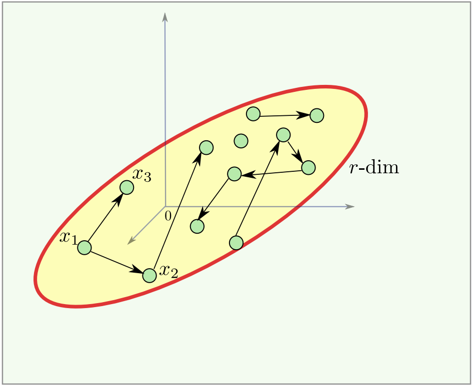

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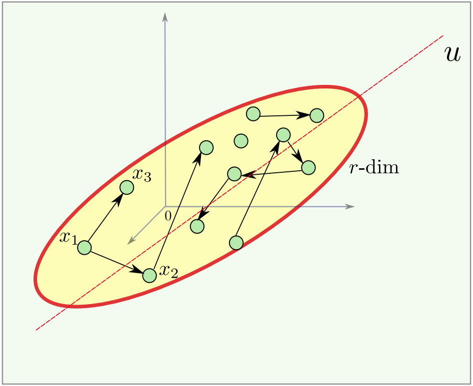

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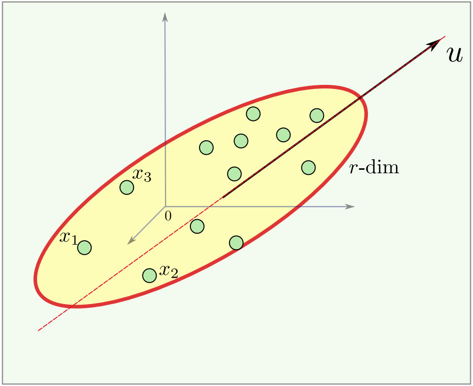

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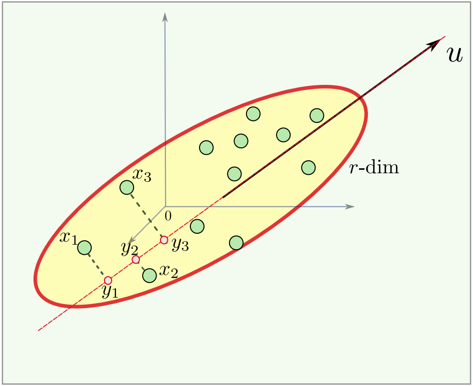

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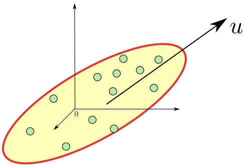

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1. Compute the top left-singular vector u, with singular value σ_1 of the matrix M
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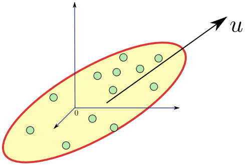

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Analysis

$$\boxed{y_i = \frac{\sigma_1}{\|v\|_1} \langle x_i, u \rangle}$$

Will show:

- Contraction:

$$|y_i - y_j| \leq \|x_i - x_j\|^2 \quad \forall i, j$$

- (Average) Dilation:

$$\mathbb{E}_{ij}\left[|y_i - y_j|\right] \geq \frac{0.99}{r}\, \mathbb{E}_{ij}\left[\|x_i - x_j\|^2\right]$$

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Analysis: Contraction

- $Mv = \sigma_1 u$, or equivalently, $\sigma_1 u = \sum_{kl} v_{kl}(x_k - x_l)$

Pick any (i,j). We have: $|y_i - y_j|_1 =$

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- Algorithm by Guruswami and Sinop based on SoS hierarchy at level $O(r)$ gives a better result: $O(1)$ approximation
- GS algorithm runs in time $2^{O(r)}\texttt{poly}(n)$, but needs a specific solver

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Outline

1 Introduction

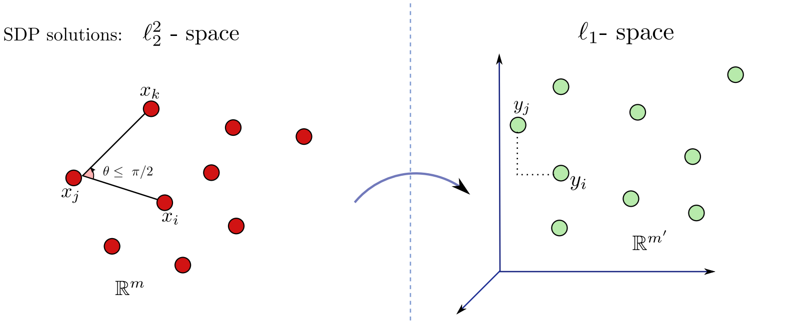
2 Cheeger or Spectral Approach

3 Rounding a stronger SDP
- Our Algorithm

4 Goemans' Theorem

5 Summary

Recap: Embedding



$$OBJ = \frac{\sum_e \|x_i - x_j\|_2^2}{\sum_{i,j \in V} \|x_i - x_j\|_2^2} \approx OBJ(\ell_1) = \frac{\sum_e \|y_i - y_j\|_1}{\sum_{i,j \in V} \|y_i - y_j\|_1}$$

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- Goemans' (unpublished) result:

Theorem (Goemans '00)

A set of points in $\mathbb{R}^m$ satisfying ℓ_2^2 triangle inequalities can be embedded into ℓ_1 with distortion $O(\sqrt{m})$

- Implies a $\sqrt{m}$ approximation to SPARSEST CUT on instances where solution has dimension m
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- Our rounding technique gives an embedding for ℓ_2^2 points with low *stable rank*: $\|M\|_F^2 \,/\, \|M\|^2$
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- Standard Johnson-Lindenstrauss dimension reduction preserves ℓ_2^2 triangle inequalities *approximately* (in $O(\log n/\epsilon^2)$ dimensions)

$$\|z_i - z_j\|^2 + \|z_k - z_j\|^2 \geq (1 - O(\epsilon)) \|z_i - z_k\|^2$$

- Goemans' theorem is true with approximate ℓ_2^2 inequalities, but requires ARV analysis [Trevisan]
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$$\|z_i - z_j\|^2 + \|z_k - z_j\|^2 \geq (1 - O(\epsilon)) \|z_i - z_k\|^2$$

- Goemans' theorem is true with approximate ℓ_2^2 inequalities, but requires ARV analysis [Trevisan]
- Can we modify our algorithm to work with approximate triangle inequalities?
  - Or 'fix' ℓ_2^2 inequalities without blowing up approximate dimension

Outline

1 Introduction

2 Cheeger or Spectral Approach

3 Rounding a stronger SDP
- Our Algorithm

4 Goemans' Theorem

5 Summary

Summary and Future directions

- A simple SDP algorithm that gives non-trivial guarantees, using ℓ_2^2 inequalities
- Unconditional guarantees?
- Dimension reduction techniques to get ARV-like guarantees?

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Thank you.