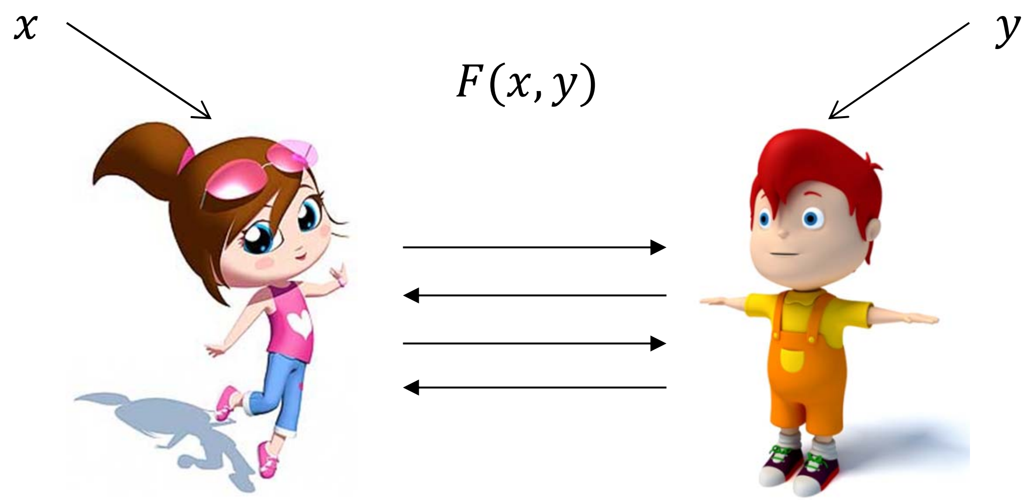
Logrank Conjecture for composed functions

Shengyu Zhang

The Chinese University of Hong Kong

Communication complexity



- Two parties, Alice and Bob, jointly compute a function f on input (x, y).
  - x known only to Alice and y only to Bob.
- Communication complexity*1: how many bits are needed to be exchanged?

*1. A. Yao. STOC, 1979.

Computation modes

- Deterministic: Players run determ. protocol. ---$D(F)$
- Randomized: Players have access to random bits; small error probability allowed. --- $R(F)$
- Quantum: Players send quantum messages. --- $Q(F)$
- Superscript: shared resource.
  - *: entanglement.
  - pub: public coins
- Subscript: error allowed.
- $Q_{\epsilon}^{*}(F)$: ϵ-bounded error, share entanglement.
- $Q_E(F)$: fixed length, zero-error, no shared entanglement.

Log-rank conjecture: quantum version

- Rank lower bound

Log-Rank Conjecture *1

$$\log_2 rank(M_F) \leq D(F) \leq \log^{O(1)} rank(M_F)$$

- Quantum: rank lower bounds *2
  - $\frac{1}{2}\log_2 rank(M_F) \leq Q^*(F) \leq \log^{O(1)} rank(M_F)$
  - $\Omega(\log_2 rank_\epsilon(M_F)) \leq Q_\epsilon(F) \leq \log^{O(1)} rank_\epsilon(M_F)$
    - $rank_\epsilon(M) = \min_{M' : |M(x,y) - M'(x,y)| \leq \epsilon} rank(M')$

Quantum Log-Rank Conjecture

*1. Lovász, Saks. *FOCS*, 1988.

*2. Buhrman, de Wolf. *CCC*, 2001.

Log-rank conjecture for XOR functions

- Log-rank conjecture appears too hard in its full generality.
- let's try some special class of functions.
- Composed functions: $f(g_1, \dots g_n)$
- When all g_i's are one-bit functions.
- XOR functions: $f(x \oplus y)$. --- $F = f \circ \oplus$
  - Examples: Equality, (Gapped) Hamming Distance.
- AND functions: $f(x \wedge y)$. --- $F = f \circ \wedge$
  - Examples: Disjointness, Inner Product, Nisan-Wigderson-Kushilevitz functions *[1].

*1. Nisan and Wigderson. *Combinatorica*, 1995.

Outline

- XOR functions: connection to Fourier, solved cases.
  - Deterministic protocols.
  - Quantum protocol.

- AND functions: connection to real polynomial.
  - Deterministic protocol.

XOR functions and Fourier

- Connections to Fourier analysis of functions on $\{0,1\}^n$.

1. $rank(M_{f\circ\oplus}) = \|\hat{f}\|_0$.
2. $rank_\epsilon(M_{f\circ\oplus}) \geq \|\hat{f}\|_{1,\epsilon}$ *1
   - $rank_\epsilon(M) = \min_{M' : |M(x,y)-M'(x,y)|\leq\epsilon} rank(M')$
   - $\|\hat{f}\|_{1,\epsilon} = \min_{g : \|f-g\|_\infty\leq\epsilon} \|\hat{g}\|_1$

*1. Lee and Shraibman. *Foundations and Trends in Theoretical Computer Science*, 2009.

Recall: Fourier analysis

- $\forall f: \{0,1\}^n \to \mathbb{R}$ can be written as

$$f = \sum_{\alpha \in \{0,1\}^n} \hat{f}(\alpha) \chi_\alpha$$

  - $\chi_\alpha(x) = (-1)^{\alpha \cdot x}$, and characters are orthogonal
  - $\{\hat{f}(\alpha): \alpha \in \{0,1\}^n\}$: Fourier coefficients of f
  - Parseval: If $Range(f) = \{\pm 1\}$, then $\sum_\alpha \hat{f}(\alpha)^2 = 1$.
- Two specific norms:
  - $\|\hat{f}\|_1 = \sum_\alpha |\hat{f}(\alpha)|$ --- *Spectral norm*.
  - $\|\hat{f}\|_0 = |\{\alpha: \hat{f}(\alpha) \neq 0\}|$ --- *Fourier sparsity*.

Log-rank Conj. For XOR functions

- Since $rank(M_{f\circ\oplus}) = \|\hat{f}\|_0$, Log-rank Conj. for XOR functions becomes

$$D(f \circ \oplus) \leq \log_2^{O(1)} \|\hat{f}\|_0 .$$

- One approach*1: $D(f \circ \oplus) \leq 2DT_\oplus(f)$.
  - $DT_\oplus(f)$: Parity decision tree complexity.
    - Decision tree with queries like "$x_1 \oplus x_3 \oplus x_4 = ?$"
  - $DT_\oplus(f) \leq DT(f)$
  - simulating 1 $\oplus$-query by 2 bits of communication.

*1. Zhang and Shi. *Theoretical Computer Science*, 2009.

XOR functions: 1

- Logrank Conj. holds for the following $f \circ \oplus$.
  - f: Symmetric *[1]
  - f: LTF *[2]
  - f: monotone *[2]
  - f: AC^0 *[3]

(Symmetric, LTF): $\log\|\hat{f}\|_0 = \Omega(n)$

(monotone, AC^0): $\deg(f) = \tilde{O}\left(\log\|\hat{f}\|_0\right)$

*1. Zhang and Shi. *Quantum Information & Computation*, 2009.
*2. Montanaro and Osborne. *arXiv*:0909.3392v2, 2010.
*3. Kulkarni and Santha. *CIAC*, 2013.

One easy case

- $\deg(f) = \max\{|\alpha|: \hat{f}(\alpha) \neq 0\}$
  - The degree of f viewed as a polynomial over $\mathbb{R}$.

- If $\deg(f) = \log^{O(1)} \|\hat{f}\|_0$, then even the standard decision tree complexity is small *[1,2].
  - $DT(f) = O(\deg^3(f)) = \log^{O(1)} \|\hat{f}\|_0$.

*1. Nisan and Smolensky. *Unpublished*.
*2. Midrijanis. *arXiv/quant-ph/0403168*, 2004.

Degree suffices?

- Question: Are all nonzero Fourier coefficients always located in low levels?
- Answer*1: Not even after change of basis.
  - $\exists f$ with $D_{\oplus}(f) \leq \log n$ but $\min_L DT(Lf) \geq n/4$.
- $\deg_2(f)$: degree of f as a polynomial over $\mathbb{F}_2$.
- Fact. $\deg_2(f) \leq \deg(f)$.
- Low $\mathbb{F}_2$-degree already admits big family of functions with elusive structures *2.
- Fact*3. $\deg_2(f) \leq \log\|\hat{f}\|_0$.

*1. Zhang and Shi. *Theoretical Computer Science*, 2009.

*2. Haramaty and Shpilka. *STOC*, 2010.

*3. Bernasconi and Codenotti. *IEEE Transactions on Computers,* 1999.

XOR functions: 2

- Logrank Conj. holds for the following $f \circ \bigoplus$.
  - f: Symmetric
  - f: LTF
  - f: monotone
  - f: AC^0
  - f: low $\mathbb{F}_2$-degree *[1]
  - f: small spectral norm *[1]

*1. Tsang, Wong, Xie, Zhang. *FOCS*, 2013.

One good ⇒ all good

- Upper bound for $DT_{\oplus}(f)$: longest path from root to leaf is short.
- Suffices to show the shortest path is short.
- $C_{\oplus,min}(f)$: the minimum co-dim of an affine subspace on which f is constant.
- Roughly: $DT_{\oplus}(f) \leq C_{\oplus,min}(f) \cdot \deg_2(f)$

Low degree result

- Theorem. $DT_{\oplus}(f) \leq 2^{d^2/2} \log^{d-2} \|\hat{f}\|_1$
- Thus Log-rank Conj. holds for f with $\deg_2(f) = O(1)$.
- For such f, *Fourier sparse* $\overset{poly}{\Longleftrightarrow}$ *short* $\oplus$-*DT*

$$\log \|\hat{f}\|_0 \leq DT_{\oplus}(f) \leq \log^{O(1)} \|\hat{f}\|_0$$

Small spectral norm

- Theorem. For any Boolean f,
  - $C_{\oplus,min}(f) = O\left(\|\hat{f}\|_1\right)$.
  - $D_{\oplus}(f) = O\left(\|\hat{f}\|_1 \cdot \deg_2(f)\right)$.
- Independent work*1:
  - $C_{\oplus,min}(f) = O\left(\|\hat{f}\|_1^2\right)$,
  - $D_{\oplus}(f) = O\left(\|\hat{f}\|_1^2 \cdot \log\|\hat{f}\|_0\right)$

*1. Shpilka and Volk. *ECCC*, 2013.

Linear rank and the main protocol

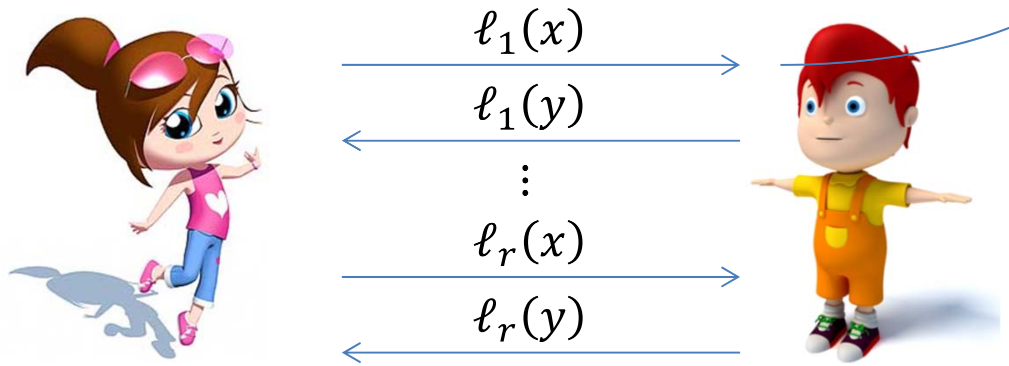
- *Linear rank (lin-rank)*: min r s.t.

$$f(z) = \ell_1(z)f_1(z) + \cdots + \ell_r(z)f_r(z) + f_0(z)$$

where $\forall \ell_i$ is linear and $\forall f_i$ has $\deg_2(f_i) \leq d - 1$.

- Main protocol: $d = \deg_2(f)$ rounds; each round reduces $\mathbb{F}_2$-degree by at least 1
  - regardless of values of $\ell_i(x)$ and $\ell_i(y)$

$z = x + y$
$\ell_i(x + y) = \ell_i(x) + \ell_i(y)$



Linear rank conjecture

- Communication cost depends on $r = \mathit{lin\text{-}rank}(f)$.
- Linear Rank Conjecture. $\forall$ Boolean f,

$$\mathit{lin\text{-}rank}(f) = \log^{O(1)} \|\hat{f}\|_0$$

- Linear Rank Conj. $\Rightarrow$ Log-rank Conj. for all XOR fn's.
- Fact. $\mathit{lin\text{-}rank}(f) \leq C_{\oplus,min}(f)$,
  - "Decreasing $\deg_2(f)$ by 1" is easier than "decreasing $\deg_2(f)$ to 0".
- Most our results are obtained by bounding $C_{\oplus,min}(f)$.

Low degree result

- How to bound $C_{\oplus,min}(f)$?
- Degree reduction again.
- Induction on $d \stackrel{\text{def}}{=} \deg_2(f)$. Apply IH on (discrete) derivative.
- Derivative: $\Delta_t f(x) = f(x+t) + f(x)$.
  - All plus: over $\mathbb{F}_2$
  - Fact. $\deg_2(\Delta_t f) \le \deg_2(f) - 1$.
  - Fact. $\|\widehat{\Delta_t f}\|_1 \le \|\hat{f}\|_1^2, \|\widehat{\Delta_t f}\|_0 \le \|\hat{f}\|_0^2$.

Two interesting functions: $g_{0/1}$

- $F = (-1)^f$
- By IH, $\exists$ affine H_b with small co-dimension and $(\Delta_t f)|_{H_b} = b$.
- Define two new functions.

$$g_0(x) = \frac{F(x) +_{\mathbb{R}} F(x+t)}{2}, \quad g_1(x) = \frac{F(x) -_{\mathbb{R}} F(x+t)}{2}.$$

- g_0 and g_1 are non-Boolean. Range: $\{-1,0,+1\}$.
- $g_b|_{H_b} = F|_{H_b}, g_b|_{H_{\bar{b}}} = 0$.
  - On H_0, $F(x) = F(x+t)$, so $g_0 = F$ and $g_1 = 0$.
  - On H_1, $F(x) = -F(x+t)$, so $g_0 = 0$ and $g_1 = F$.

$g_{0/1}$ in Fourier domain

- $\widehat{g_0}(\alpha) = \begin{cases} \hat{F}(\alpha) & \alpha \in t^\perp \\ 0 & \alpha \in \overline{t^\perp} \end{cases}$, $\widehat{g_1}(\alpha) = \begin{cases} 0 & \alpha \in t^\perp \\ \hat{F}(\alpha) & \alpha \in \overline{t^\perp} \end{cases}$.
  - $\widehat{g_b}(\alpha) = \frac{1}{2}\left(\hat{F}(\alpha) + (-1)^b \hat{F}(\alpha)\chi_t(\alpha)\right)$
- Recall: $g_b|_{H_b} = F|_{H_b}$, $g_b|_{H_{\bar{b}}} = 0$.
- On H_0, $F = g_0$, which keeps all $\hat{F}(\alpha)$ for α in half-space $t^\perp$, and kills all $\hat{F}(\alpha)$ in the other half-space.
  - Similarly for H_1.
- So on H_b, half-space of the Fourier coefficients disappear.
  - During the linear restrictions, those $\hat{F}(\alpha)$ collide a lot and finally all annihilate.

Killing the Fourier coefficients

- Formally: $\|\hat{F}\|_1 = \|\widehat{g_0}\|_1 + \|\widehat{g_1}\|_1$.
- Thus either $\|\widehat{g_0}\|_1$ or $\|\widehat{g_1}\|_1$ is $\leq \frac{1}{2}\|\hat{F}\|_1$.
- Say it's g_b.
- $\|\widehat{F|_{H_b}}\|_1 = \|\widehat{g_b|_{H_b}}\|_1 \qquad (g_b|_{H_b} = F|_{H_b})$

  $\leq \|\widehat{g_b}\|_1 \qquad$ (subfn: smaller norm)

  $\leq \frac{1}{2}\|\hat{F}\|_1 \qquad$ (picked b for this)

Finishing induction

- Repeating this $\log\|\hat{F}\|_1$ times reduces $\|\hat{F}\|_1$ to ≤ 1, reaching a linear function. One more restriction makes it constant.
- So $C_{\oplus,min}(f) \leq C_{\oplus,min}(\Delta_t f) \log\|\hat{F}\|_1$.
- Use the following to finish the induction.

$$\deg_2(\Delta_t f) \leq \deg_2(f) - 1 \quad \text{and} \quad \|\widehat{\Delta_t f}\|_1 \leq \|\hat{f}\|_1^2$$

- And get $DT_{\oplus}(f) \leq 2^{d^2/2} \log^{d-2}\|\hat{f}\|_1$.

Sketch for $C_{\oplus,min}(f) = O\left(\|\hat{f}\|_1\right)$

- $D_{\oplus}(f) = O(L_1 \cdot \deg_2(f))$ follows.
  - $L_1 = \|\hat{f}\|_1$
- Greedy folding: boost $\|\hat{f}\|_\infty$ as quickly as possible.
- keep folding over the line $\beta = \alpha_1 + \alpha_2$,
  - $|\hat{f}(\alpha_1)| \geq |\hat{f}(\alpha_2)| \geq \cdots \geq |\hat{f}(\alpha_s)| > 0$.
- Two stages.
  - Before $|\hat{f}(\alpha_1)| < 1/2$: $\|\hat{f}\|_\infty$ increases by $\geq \frac{3}{4L_1}$.
  - Afterwards: L_1 drops by ≥ 1.
    - Don't always analyze $\|\hat{f}\|_\infty$ though the alg aims to boost it.

Before $\left|\hat{f}(\alpha_1)\right| < 1/2$

- $\left\|\hat{f}\right\|_\infty$ increases by a_2.
  - $a_i = \left|\hat{f}(\alpha_i)\right|$.
- Parseval: $a_1^2 + a_2^2 + \cdots + a_s^2 = 1$.
- $1 - a_1^2 \le a_2(a_2 + \cdots + a_s) = a_2(L_1 - a_1)$.
- $a_2 \ge \frac{1-a_1^2}{L_1-a_1} \ge \frac{3}{4} \cdot \frac{1}{L_1}$
- So $\left\|\hat{f}\right\|_\infty$ increases by at least $\frac{3}{4L_1}$

After $\left|\hat{f}(\alpha_1)\right| \geq 1/2$

- $\left|\hat{f}(\alpha_1)\right|$ still increases, so it's always $\geq 1/2$.
- Fact. $\sum_{i,j:\alpha_i+\alpha_j=\beta} \hat{f}(\alpha_i)\hat{f}(\alpha_j) = 0$.
  - $\beta = \alpha_1 + \alpha_2 \neq 0$.
- e.g. $a_1 a_2 + a_5 a_7 = a_3 a_4 + a_6 a_8 + a_9 a_{10}$
  - Recall: $a_i = \left|\hat{f}(\alpha_i)\right|$.
- Fact. L_1 drops by $2(a_4 + a_8 + a_{10})$.
- $a_1 a_2 \leq a_3 a_4 + a_6 a_8 + a_9 a_{10} \leq a_3(a_4 + a_8 + a_{10})$
- L_1 drops $\geq \frac{2a_1 a_2}{a_3} \geq 2a_1 \geq 1$.

XOR functions: 3

- Logrank Conj. holds for the following $f \circ \bigoplus$.
  - f: Symmetric
  - f: LTF
  - f: monotone
  - f: AC^0
  - f: low $\mathbb{F}_2$-degree
  - f: small spectral norm
- Quantum Logrank Conj. holds for the following $f \circ \bigoplus$.
  - f: low $\mathbb{F}_2$-degree *[1]

*1. Zhang. *SODA*, 2014.

XOR functions: Quantum

- Theorem*1. $Q_E^*(f \circ \oplus) \geq \Omega\left(\log\|\hat{f}\|_0\right)$.
- Theorem*2. $Q_E(f \circ \oplus) = O\left(2^d \log\|\hat{f}\|_0\right)$.
  - where $d = \deg_2(f)$.
  - Confirms quantum Log-rank Conj. (Q_E) for $f \circ \oplus$ with $\deg_2(f) = O\left(\log\log\|\hat{f}\|_0\right)$.
  - Comparison: $D(f \circ \oplus) \leq 2^{d^2/2} \log^{d-2}\|\hat{f}\|_1$

*1. Buhrman and de Wolf. *CCC*, 2001.

*2. Zhang, *SODA*, 2014.

XOR functions: Quantum

- Theorem.
  $Q_\epsilon(f \circ \bigoplus) \geq \Omega(\log rank_\epsilon(M_{f\circ\bigoplus})) \geq \Omega\left(\log\|\hat{f}\|_{1,\epsilon}\right)$ *1
- Theorem. $Q_\epsilon(f \circ \bigoplus) \leq \tilde{O}\left(2^d \log\|\hat{f}\|_{1,\epsilon}\right)$,
  - Confirms quantum Log-rank Conj. (Q_ϵ) for $f \circ \bigoplus$ with $\deg_2(f) = O\left(\log\log\|\hat{f}\|_{1,\epsilon}\right)$.

*1. Lee and Shraibman. *Foundations and Trends in Theoretical Computer Science*, 2009.
*2. Zhang, *SODA*, 2014.

About quantum protocol

- Much simpler.
- $\log \|\hat{f}\|_0$ comes very naturally.
- Inherently quantum.
  - Not from quantizing any classical protocol.
- Computational cost is also very low.

Goal: compute $f(x+y)$
where $f:\{0,1\}^n \to \{\pm 1\}$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |\psi\rangle = \sum_{\alpha \in A} \hat{f}(\alpha)\chi_\alpha(x)|E(\alpha)\rangle \quad \xrightarrow{|\psi\rangle}$$

Add phase $\chi_\alpha(y)$

$$|\psi'\rangle = \sum_{\alpha \in A} \hat{f}(\alpha)\chi_\alpha(x)\chi_\alpha(y)|E(\alpha)\rangle$$

$$\xleftarrow{|\psi'\rangle}$$

$|E(\alpha)\rangle \to |\alpha\rangle$

$$\sum_{\alpha \in A} \hat{f}(\alpha)\chi_\alpha(x+y)|\alpha\rangle$$

Fourier: $|\alpha\rangle \to \sum_t \chi_\alpha(t)|t\rangle$

$$\sum_t \sum_{\alpha \in A} \hat{f}(\alpha)\chi_\alpha(x+y)\chi_\alpha(t)|t\rangle$$

$$= \sum_t f(x+y+t)|t\rangle$$

$$\sum_t \frac{|0\rangle + f(x+y+t)|1\rangle}{\sqrt{2}}|t\rangle$$

Goal: compute $f(x+y)$
where $f:\{0,1\}^n \to \{\pm 1\}$

$$|\psi\rangle = |+\rangle \sum_{\alpha \in A} \hat{f}(\alpha)\chi_\alpha(x)|E(\alpha)\rangle$$

$|\psi\rangle$ →

↓ Add phase $\chi_\alpha(y)$

← $|\psi'\rangle$

↓ Decoding + Fourier

$$\sum_t \frac{|0\rangle + f(x+y+t)|1\rangle}{\sqrt{2}} |t\rangle$$

Measure$\{|+\rangle, |-\rangle\}$ ↓ ↓ Measure t

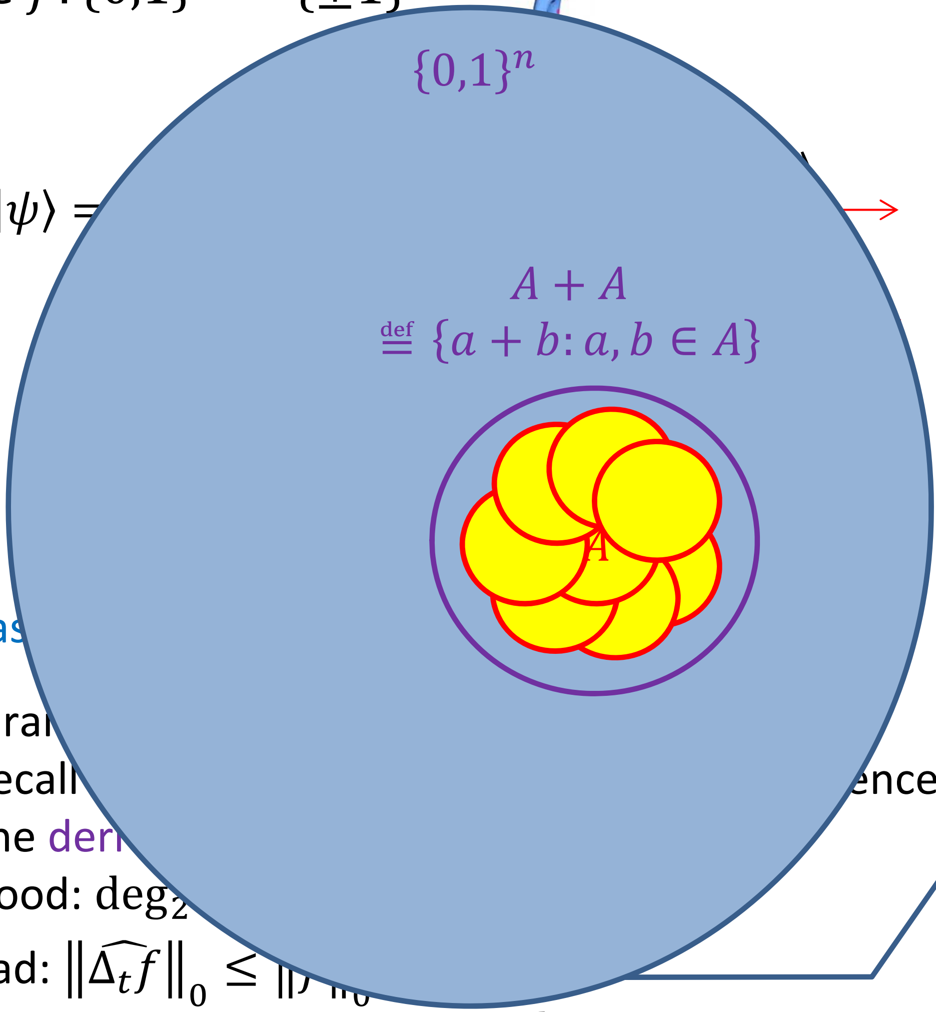
A random $t \in \{0,1\}^n$ and $f(x+y+t)$.
Recall our target: $f(x+y)$. What's the difference?
The derivative: $\Delta_t f(z) = f(z+t)f(z)$.
Good: $\deg_2(\Delta_t f) \le \deg_2(f) - 1$.
Bad: $\|\widehat{\Delta_t f}\|_0 \le \|\hat{f}\|_0^2$.
(That's where the factor of 2^d comes from.)

- One more issue: Only Alice knows t! Bob doesn't.
- It's unaffordable to send t.
- Obs: $supp(\widehat{\Delta_t f}) \subseteq A + A, \forall t$.
  - $A = supp(\hat{f})$

Goal: compute $f(x+y)$
where $f: \{0,1\}^n \to \{\pm 1\}$

$|\psi\rangle =$

Add phase $\chi_\alpha(y)$



Meas

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Recall ... ence?

The deri

Good: $\deg_2$

Bad: $\|\widehat{\Delta_t f}\|_0 \leq \|$

(That's where the factor of 2^d comes from.)

- One more issue: Only Alice knows t! Bob doesn't.
- It's unaffordable to send t.
- Obs: $supp(\widehat{\Delta_t f}) \subseteq A + A, \forall t$.
  - $A = supp(\hat{f})$

Goal: compute $f(x+y)$
where $f:\{0,1\}^n \to \{\pm 1\}$

$$|\psi\rangle = |+\rangle \sum_{\alpha \in A} \hat{f}(\alpha)\chi_\alpha(x)|E(\alpha)\rangle$$

$|\psi\rangle$ →

↓ Add phase $\chi_\alpha(y)$

← $|\psi'\rangle$

↓ Decoding + Fourier

$$\sum_t \frac{|0\rangle + f(x+y+t)|1\rangle}{\sqrt{2}}|t\rangle$$

Measure$\{|+\rangle, |-\rangle\}$ ↓ ↓ Measure t

A random $t \in \{0,1\}^n$ and $f(x+y+t)$.
Recall our target: $f(x+y)$. What's the difference?
The derivative: $\Delta_t f(z) = f(z+t)f(z)$.
Good: $\deg_2(\Delta_t f) \le \deg_2(f) - 1$.
Bad: $\|\widehat{\Delta_t f}\|_0 \le \|\hat{f}\|_0^2$.
(That's where the factor of 2^d comes from.)

- One more issue: Only Alice knows t! Bob doesn't.
- It's unaffordable to send t.
- Obs: $supp(\widehat{\Delta_t f}) \subseteq A + A, \forall t$.
  - $A = supp(\hat{f})$
- Thus in round 2, Alice and Bob can just encode the entire $A + A$.

Goal: compute $f(x+y)$
where $f:\{0,1\}^n \to \{\pm 1\}$

$$|\psi\rangle = |+\rangle \sum_{\alpha \in A} \hat{f}(\alpha)\chi_\alpha(x)|E(\alpha)\rangle$$

$|\psi\rangle$ →

↓ Add phase $\chi_\alpha(y)$

← $|\psi'\rangle$

↓ Decoding + Fourier

$$\sum_t \frac{|0\rangle + f(x+y+t)|1\rangle}{\sqrt{2}} |t\rangle$$

Measure$\{|+\rangle, |-\rangle\}$ ↓ ↓ Measure t

A random $t \in \{0,1\}^n$ and $f(x+y+t)$.
Compute $f_2 \stackrel{\text{def}}{=} \Delta_t f(z) = f(z+t)f(z)$.

At last, $\deg_2(f_d) = 0$, a constant function.

Cost: $\log|A| + \log|2A| + \log|4A| + \cdots + \log\left|2^{d-1}A\right| \leq 2^d \log|A|$.

Used trivial bound: $|kA| \leq |A|^k$

AND functions

- Contains Disjointness ($f = OR$), Inner Product ($f = Parity$) and the Nisan-Wigderson-Kushilevitz functions.
- Each $f: \{0,1\}^n \to \{0,1\}$ can be uniquely written as a real-polynomial
$$p(x) = \sum_i c(S_i) x_{S_i}$$
  - where $x_S \stackrel{\text{def}}{=} \prod_{i \in S} x_i$.
- Fact*[1,2]. $rank\left(M_{f \circ \wedge}\right) = \|c\|_0$, the sparsity of c.
- Log-rank Conj. for AND functions:
$$D(f \circ \wedge) = \log^{O(1)} \|c\|_0$$

*1. Nisan and Wigderson. *Combinatorica*, 1995.

*2. Buhrman and de Wolf. *CCC*, 2001.

AND functions: solved cases

- Logrank Conj. holds for the following $f \circ \bigoplus$.
  - f: Symmetric, LTF
  - f: monotone, AC^0
  - f: low $\mathbb{F}_2$-degree, small spectral norm
- Quantum Logrank Conj. Holds for the following $f \circ \bigoplus$.
  - f: low $\mathbb{F}_2$-degree
- Logrank Conj. holds for the following $f \circ \wedge$.
  - f: symmetric *[1]
  - f: monotone *[1]
  - f: close to monotone

*1. Buhrman and de Wolf. *CCC*, 2001.

monotone

- Recall Theorem *1. If f is monotone, then
$$D(f\circ\oplus) = O\left(\log^2 rank\left(M_{f\circ\oplus}\right)\right)$$
- Theorem *2. If f is monotone, then
$$D(f\circ\wedge) = O\left(\log^2 rank\left(M_{f\circ\wedge}\right)\right)$$
- Goal: extend to non-monotone functions.
- Need to be careful on distance measure.
  - Hamming distance: changing f at one point can change the $\log rank$ to close to n, making Logrank Conj. trivially hold for the new function.

*1. Montanaro and Osborne. *arXiv*:0909.3392v2, 2010.

*2. Buhrman and de Wolf. *CCC*, 2001.

Good measure

- Alternating number: --- $alt(f)$
  - Walk along any monotone path from 0^n to 1^n on $\{0,1\}^n$.
  - Count the number of alternations of f value.
  - Take the maximum over all monotone paths.
- Inversion complexity: --- $inv(f)$
  - The minimum number of negation gates needed in any Boolean circuit computing f.
- *Theorem* *1. $inv(f) \approx \log_2 alt(f)$.

*1. Markov, *Doklady Akademii Nauk SSSR*, 1957. (English translation: *JACM, 1958*.)

- Recall: If f is monotone, then

$$D(f \circ \bigoplus) = O\left(\log^2 rank\left(M_{f \circ \bigoplus}\right)\right)$$

- *Theorem* *[1].

$$D(f \circ \bigoplus) = O\left(alt(f) \cdot \log^2 rank\left(M_{f \circ \bigoplus}\right)\right)$$

- Recall: If f is monotone, then

$$D(f \circ \wedge) = O\left(\log^2 rank\left(M_{f \circ \wedge}\right)\right)$$

- *Theorem* *[1].

$$D(f \circ \wedge) = \tilde{O}\left(\log^{(alt(f)+3)/2} rank\left(M_{f \circ \wedge}\right)\right)$$

*1. Zhang, *work in progress*.

Summary

- XOR and AND functions are important yet challenging special cases.
  - Good targets.
- Log-rank Conjecture is confirmed on some special classes of XOR/AND functions.
- Most protocols are *ad hoc*, using the specific structures of those classes.

- One exception: the (classical) $\mathbb{F}_2$-degree reduction protocol.
  - We believe its cost is already $\log^{O(1)}\|\hat{f}\|_0$. We "just" need to tighten our analysis.
- Call for efforts: Prove linear rank conjecture

$$lin\text{–}rank(f) = \log^{O(1)}\|\hat{f}\|_0,$$

  which then solves all XOR functions.
- More open questions next.

Open questions

- *Question 1*. Log-rank for XOR and AND functions

  $\Rightarrow$ Log-rank for all functions?

- *Question 2*. Log-rank for $f \circ \bigoplus$

  $\Rightarrow C_{\oplus,min} = \log^{O(1)} \|\hat{f}\|_0$?

- *Theorem* *1: $DT_{\oplus}(f) = poly\left(D^{(4)}(f \circ \bigoplus)\right)$, where $D^{(4)}$ is the 4-partite CC.

*1. Lovett, unpublished. Yao, arXiv:1506.02936.

Open questions

- *Theorem**1. $R^{pub}(F) = \log^{c} rank(M_F)$
  $\Rightarrow D(F) = \log^{c+2} rank(M_F)$

- *Question 3*. $Q^{*}(F) = \log^{O(1)} rank(M_F)$
  $\Rightarrow D(F) = \log^{O(1)} rank(M_F)$?
- *Question 3.1*: For XOR functions? AND functions?

*1. Gavinsky and Lovett, *ICALP*, 2013.

Open questions

- If $D(F) = c$, then the protocol partition M_F into 2^c monochromatic rectangles R. Thus the largest one has size $|R| \geq |M_F|/2^c$.
- *Theorem*[*1]. $\forall F$, max mono. rectangle R of M_F is large ($\log \frac{|R|}{|M_F|} = \log^{O(1)} rank(M_F)$) $\Rightarrow$ Logrank Conjecture holds.
- *Question 4*. $\forall f$, max mono. rectangle of $M_{f \circ \bigoplus}$ is large $\Rightarrow$ Logrank Conj. holds for $\forall f \circ \bigoplus$? AND functions?

*1. Nisan and Wigderson, *Combinatorica* , 1995.

Open questions

- *Question 5*. Let $A = supp(\hat{f})$. What can we say about its additive properties? Could it be true that $|tA| \ll |A|^t$?
- *Question 5.1*. $\left\|\widehat{\Delta_{t_1 \dots t_k} f}\right\|_1 \ll \left\|\hat{f}\right\|_1^{2^k}$?

Thanks