Classification of the Joint Numerical Range of Three Hermitian 3-by-3 Matrices

Workshop on Positive Semidefinite Rank
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Abstract

The joint numerical range W_n of $n = 2$ or $n = 3$ hermitian 3-by-3 matrices $F_1, \ldots, F_n$ is a convex and compact subset of $\mathbb{R}^n$ equal to the projection of the set of positive semi-definite 3-by-3 matrices of trace one onto the span of the F_i.

The classification of W_2 follows from the discussion of a planar algebraic curve whose convex hull is W_2. We provide a convex geometric classification of W_3 and remark on some problems with its counterpart of algebraic surfaces.

Outline

Introduction

- Convexity of the joint numerical range
- Some recent research
- Classification for $n = 2$ matrices

Classification for $n = 3$ matrices

- Algebraic classification of W_3 — what can go wrong
- Convex geometric classification of W_3

Convexity of the joint numercial range

$n = 2$ or $n = 3$ hermitian matrices $F_1, \dots, F_n \in H_3$
(M_3 = complex 3-by-3 matrices, $H_3 \subset M_3$ hermitian matrices)

joint numerical range
$W_n := \{\langle x, F_i(x)\rangle_{i=1}^n \mid x \in \mathbb{C}^3, \langle x, x\rangle = 1\} \subset \mathbb{R}^n$

if $A = F_1 + \mathrm{i}\, F_2$ then W_2 is the **numerical range** of A which is

$$\{\langle x, A(x)\rangle \mid x \in \mathbb{C}^3, \langle x, x\rangle = 1\} \subset \mathbb{C} \cong \mathbb{R}^2$$

W_n is convex (Hausdorff 1919 and Au-Yeung, Poon 1979) and
$\{\langle xx^*, F_i\rangle_i \mid \langle x, x\rangle = 1\} \subset \{\langle \rho, F_i\rangle_i \mid \rho \in M_3, \rho \succeq 0, \mathrm{tr}(\rho) = 1\}$ so

$$W_n = \{\langle \rho, F_i\rangle_i \mid \rho \in M_3, \rho \succeq 0, \mathrm{tr}(\rho) = 1\},$$ not for $n > 3$!

Some recent research

$\mathcal{M}_n = \{\rho \in M_n \mid \rho \succeq 0, \mathrm{tr}(\rho) = 1\}$ is the **state space** of a quantum mechanical system admitting n energy levels

generalizations of the numerical range spread in **quantum many-party physics**, e.g. marginal problems (Coleman, Erdahl, Ruskai, Klyachko, etc.) or **quantum error correction** (Choi, Kribs and Życzkowski ’06 or Li, Poon and Sze ’09, etc.). Still, the simplest example W_3 is not understood geometrically!

W_3 is studied in **operator theory** where the numerical range (of operators) is a central object for almost a century

e.g. Krupnik and Spitkovsky ’06, or Chien and Nakazato ’10 *Joint numerical range and its generating hypersurface*

Classification of W_2 — Kippenhahn's theorem

homogeneous polynomial

$p(u_0, u_1, \dots, u_n) := \det(u_0 \mathbb{1} + u_1 F_1 + \cdots + u_n F_n)$

determinantal variety

$S_n := \{(u_0 : u_1 : \cdots : u_n) \in \mathbb{PC}^{n+1} \mid p(u_0 : u_1 : \cdots : u_n) = 0\}$

dual variety $S_n^\wedge \subset \mathbb{PC}^{n+1}$ (closure of the set of tangent spaces to non-singular points of S_n)

$\alpha : \mathbb{PC}^{n+1} \setminus \{x_0 = 0\} \to \mathbb{C}^n$, $(x_0 : x_1 : \cdots : x_n) \mapsto (\frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0})$

boundary generating curve ($n = 2$**) / surface (**$n = 3$**)**

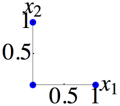
$S_n^\wedge(\mathbb{R}) := \mathbb{R}^n \cap \alpha(S_n^\wedge)$

Theorem 1. (Kippenhahn 1951) $W_2 = \mathrm{conv}(S_2^\wedge(\mathbb{R}))$

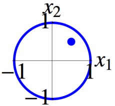
Classification of W_2 — boundary generating curves

the boundary generating curve $S_2^{\wedge}(\mathbb{R}) \subset \mathbb{R}^2$ of W_2 (blue) belongs to one of four classes (Kippenhahn 1951)

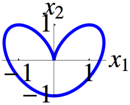
three points, e.g. $F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



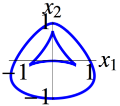
ellipse and point, e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & .5 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & .5 \end{pmatrix}$



degree-4 curve, e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



degree-6 curve, e.g. $F_1 = \begin{pmatrix} 0 & 0 & .5 \\ 0 & 0 & 1 \\ .5 & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$



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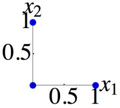
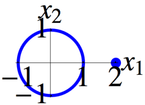


ellipse and point, e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



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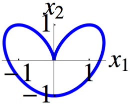

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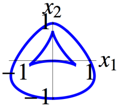
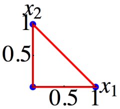


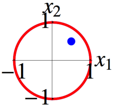
Classification of W_2 — boundaries

boundary of W_2 (red), observe: **one-dimensional faces of W_2 intersect mutually**

three points, e.g. $F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



ellipse and point, e.g. $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & .5 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & .5 \end{pmatrix}$



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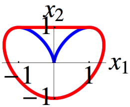

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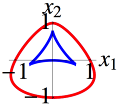

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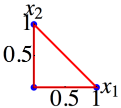

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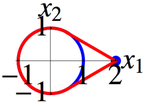

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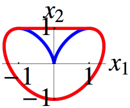

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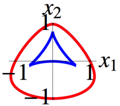


Classification of W_2 — further results

the classification of W_2 is worked out in terms of matrix entries and invariants (trace, determinant, eigenvalues)

see Keeler, Rodman and Spitkovsky 1997, Rodman and Spitkovsky '05, Rault, Sendova and Spitkovsky '13

closures of subsets of M_3 with the same shape of W_2 have been computed (sort of perturbation of numerical ranges)

see Spitkovsky and W '15 (arXiv:1509.05676 [math.FA])

classification of W_2 by convex duality to spectrahedra

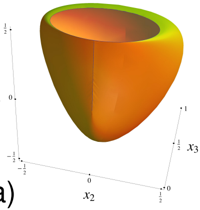
see Henrion '10, Helton, Spitkovsky '12

Algebraic classification of W_3 — what can go wrong

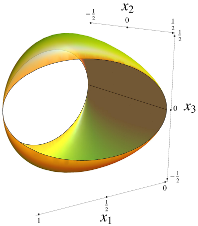
an example with $S_3^\wedge(\mathbb{R}) \not\subset W_3$ was found by Chien and Nakazato '10 so $W_n = \mathrm{conv}(S_n^\wedge(\mathbb{R}))$ holds for $n = 2$ but not for $n = 3$!

the discrepancy $S_3^\wedge(\mathbb{R}) \setminus W_3 \neq \emptyset$ lies in a Zariski-closed subset of $S_3^\wedge(\mathbb{R})$ of dimension one while $S_3^\wedge(\mathbb{R})$ has dimension two

Examples: a) $F_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



a)



b)

$S_3^\wedge(\mathbb{R}) \cap$ (boundary of W_3) is depicted — the x_1- and x_2-axes lie in $S_3^\wedge(\mathbb{R})$!

equation of $S_3^\wedge(\mathbb{R})$
$-4x_1^2x_3^2 - 4x_2^2x_3^2 + 4x_3^3 - 4x_3^4 + 4x_1x_2^2x_3 - x_2^4 = 0$

Algebraic classification of W_3 — what can go wrong

an example with $S_3^\wedge(\mathbb{R}) \not\subset W_3$ was found by Chien and Nakazato '10 so $W_n = \mathrm{conv}(S_n^\wedge(\mathbb{R}))$ holds for $n = 2$ but not for $n = 3$!

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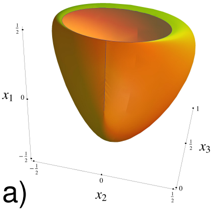
Examples: b) $F_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $F_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$



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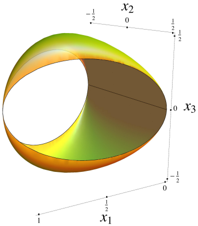

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$S_3^\wedge(\mathbb{R}) \cap$ (boundary of W_3) is depicted — the x_1- and x_2-axes lie in $S_3^\wedge(\mathbb{R})$!

equation of $S_3^\wedge(\mathbb{R})$

$$-x_1^2 x_2^2 + x_1 x_3^2 - x_1^2 x_3^2 - x_3^4 = 0$$

Classification of W_3 — normal cones

the set of maximizers of a linear functional on a convex set is called **exposed face**, $\emptyset$ is an exposed face by definition

exposed faces of W_3 are: $\emptyset$, singletons (exposed points), **segments, ellipses**, and W_3 (lifts are 3D-balls $\cong \mathcal{M}_2$)

the set of vectors which do not make an acute angle with the translation vector from a point x of W_3 to any point of W_3 is called **normal cone** $N(x)$ at x

lattice isomorphism $N : \{\text{exposed faces of } W_3\} \to \{\text{normal cones}\}$, $F \mapsto N(x)$, $x \in \mathrm{relint} F$ (for all convex sets $\neq$ singleton)

Theorem 2. (W '12) every non-empty face of every normal cone of W_3 is a normal cone of W_3

Classification of W_3 — corner points

a point of a convex subset of $\mathbb{R}^3$ which has a 3D normal cone is called **corner point**

studies by Binding and Li 1991 (conical point) show that if W_3 has a corner point p then W_3 is the convex hull of p and the joint numerical range of three hermitian 2-by-2 matrices

Lemma 1. if W_3 has a corner point p and $\dim(W_3) = 3$ then either

- W_3 is the convex hull of p and an ellipse whose affine hull does not contain p
- W_3 is the convex hull of p and an ellipsoid not containing p

Classification of W_3 — definition

using Theorem 2 and the isomorphism N we notice that the normal cone lattice of W_3 is atomistic and the exposed face lattice coatomistic: if $\dim(W_3) = 3$ then the coatoms are smooth exposed points or **large faces** (segments and ellipses)

Idea. for the classification of W_3 we use the sublattice $\mathcal{L}$ of the exposed faces generated by the large faces (up to isomorphisms of the set of segments and set of ellipses)

- it turns out that the numbers of **segments** s and **ellipses** e specify $\mathcal{L}$
- the bound $e \leq 4$ was proved by Chien and Nakazato '10 studying singularities of $\det(u_0 \mathbb{1} + u_1 F_1 + u_2 F_2 + u_3 F_3) = 0$

Classification of W_3 — lemmata

we assume in the following $\dim(W_3) = 3$, the case $\dim(W_3) \leq 2$ reduces to the known classification of W_2

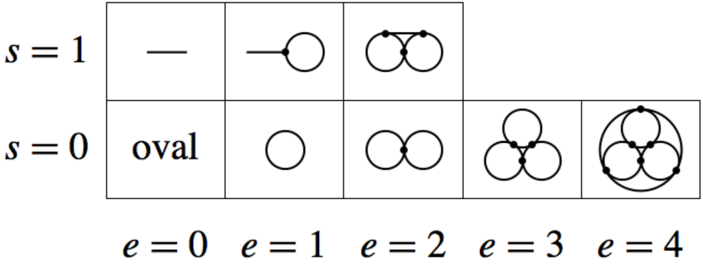
- each two large faces intersect in a singleton (proof: project W_3 onto the plane spanned by the normal vectors of two large faces and use the known classification of W_2)
- if three large faces intersect then W_3 has a corner point (proof: use the isomorphism N, and notice that a two-dimensional cone cannot have three extreme rays)
- if $\mathcal{L}$ contains two segments then W_3 has a corner point (proof: show that F_1, F_2, F_3 have block diagonal form)

Theorem 3. (SWZ '16) if W_3 has no corner point then a complete graph is embedded into the union of large faces having one vertex on each large face

Classification of W_3 — the classes

- since the boundary of W_3 is homeomorphic to the sphere S^2, the complete graph has at most four vertices (Ringel, Youngs 1968)
- if $\mathcal{L}$ contains a segment then the vertex degree is at most two, so the graph has at most three vertices

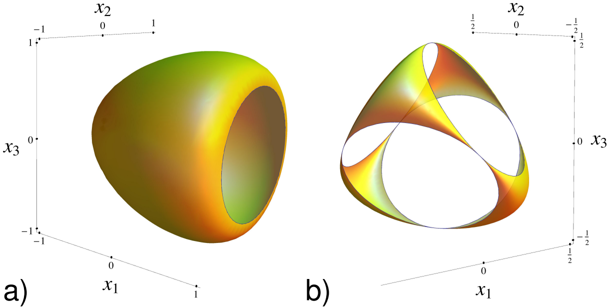
in dimension $\dim(W_3) = 3$ we obtain the following list of lattices $\mathcal{L}$; depicted are $\{$ coatoms of $\mathcal{L}$ $\} = \{$ large faces of W_3 $\}$



Classification of W_3 — examples

all classes of lattices $\mathcal{L}$ are indeed non-empty

Ex. a) $F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$



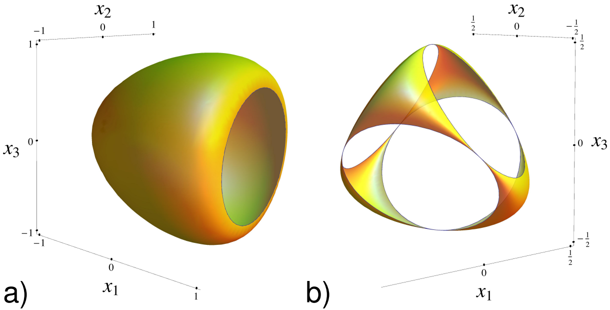
a) b)

equation of $S_3^\wedge(\mathbb{R})$ $-4x_1^3 - 4x_1^4 + 27x_2^2 + 18x_1x_2^2 - 13x_1^2x_2^2 - 32x_2^4 + 27x_3^2 + 18x_1x_3^2 - 13x_1^2x_3^2 - 64x_2^2x_3^2 - 32x_3^4 = 0$

Classification of W_3 — examples

all classes of determined by the graph embedding are indeed non-empty

Ex. b) $F_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $F_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$



equation of $S_3^{\wedge}(\mathbb{R})$ $\quad x_1x_2x_3 - x_1^2x_2^2 - x_1^2x_3^2 - x_2^2x_3^2 = 0$

Classification of W_3 — remaining examples

$e = 0$, $s = 0$: $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$e = 0, s = 1$: $F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $F_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 1 \\ -\mathrm{i} & 1 & 0 \end{pmatrix}$

$e = 2, s = 0$: $F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$e = 3, s = 0$: $F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 1 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$

Thank you for your attention!

Credits: Supported by a Brazilian Capes scholarship