An analysis of shape ensembles through modular diffeomorphisms

Mathematics of Shapes and Applications 2016

Barbara Gris

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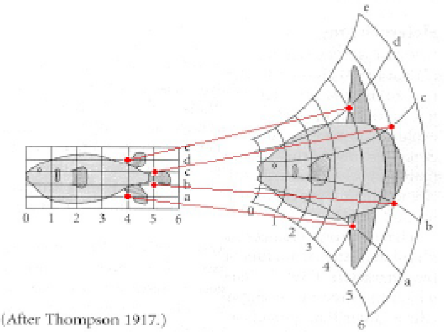
July 20, 2016

Sommaire

Framework : Large deformations

- Studying a population of shapes through diffeomorphisms deforming them.



(After Thompson 1917.)

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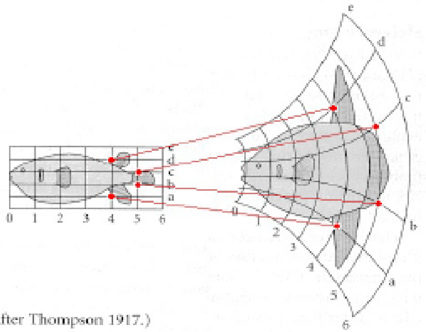

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- $\phi_0 = Id$, $\frac{d}{dt}\phi_t = v_t \circ \phi_t$ with $v_t \in V_{\phi_t}$.

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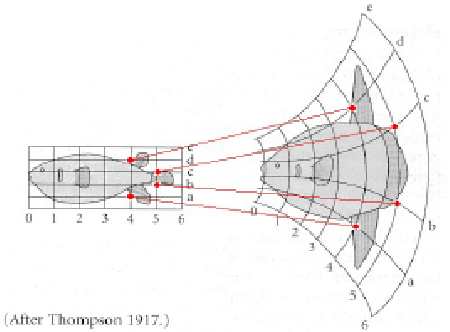

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$\longrightarrow$ Which trajectories of vector fields ?

Previous works

Non parametric frameworks : Shape spaces [S. Arguillere. Géométrie sous-riemannienne en dimension infinie et applications à l'analyse mathématique des formes . PhD thesis, 2014.]

- LDDMM [M. I. Miller, L. Younes, and A. Trouvé. Diffeomorphometry and geodesic positioning systems for human anatomy, 2014]
- Higher-order momentum [S. Sommer M. Nielsen, F. Lauze, and X. Pennec. Higher-order momentum distributions and locally affine lddmm registration. SIAM Journal on Imaging Sciences, 2013]
- Sparse LDDMM [S. Durrleman, M. Prastawa, G. Gerig, and S. Joshi. Optimal data-driven sparse parameterization of diffeomorphisms for population analysis. In Information Processing in Medical Imaging , pages 123-134. Springer, 2011]

"Is it possible to mechanize human intuitive understanding of biological pictures that typically exhibit a lot of variability but also possess characteristic structure ?"

Ulf Grenander

Hands : a Pattern Theoric Study of Biological Shapes, 1991

Previous works

Parametric models to model non linear patterns :

- GRID [U. Grenander , A. Srivastava , S. Saini. A pattern-theoric characerization of biological growth. IEEE, 2007]
- Poly-affine [C. Seiler , X. Pennec, and M. Reyes. Capturing the multiscale anatomical shape variability with polyaffine transformation trees. Medical image analysis, 2012]
- Diffeons [L. Younes. Constrained diffeomorphic shape evolution. Foundations of Computational Mathematics, 2012.]

Aim

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- Generic generators

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- Complex and chosen constraints

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- Generic generators
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- Metric on shape space taking into account constraints

Sommaire

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- Parametrized in small dimension

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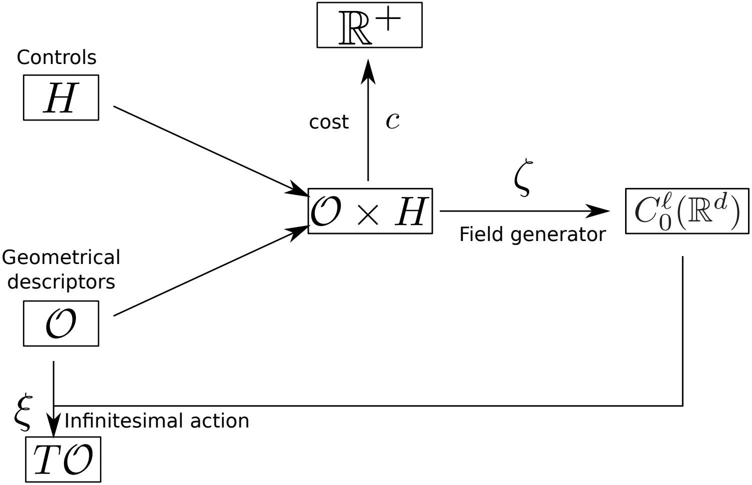
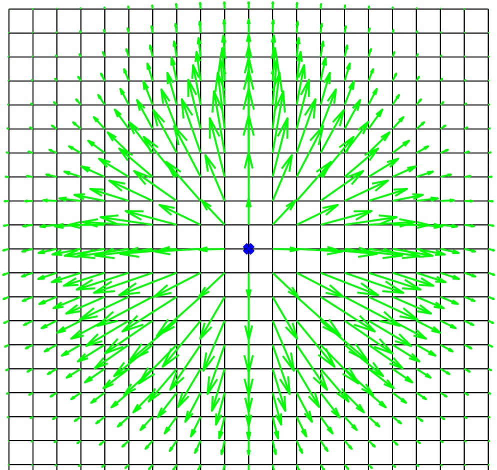


Local scaling of scale σ

Example of generated vector field



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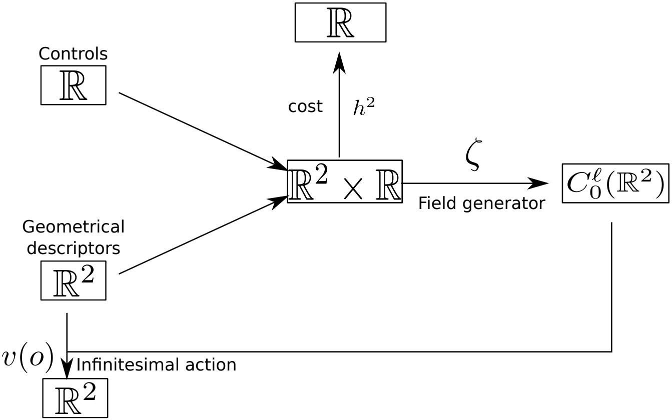

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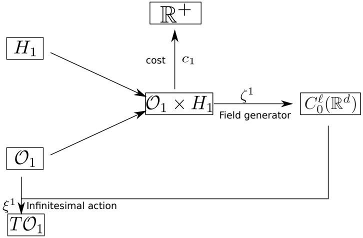
  $\longrightarrow$ Combine deformation modules


$\mathbb{R}^+$
H_1
cost
c_1
$\mathcal{O}_1 \times H_1$
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Field generator
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Infinitesimal action
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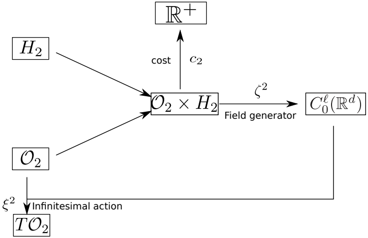
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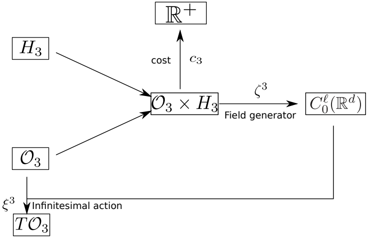
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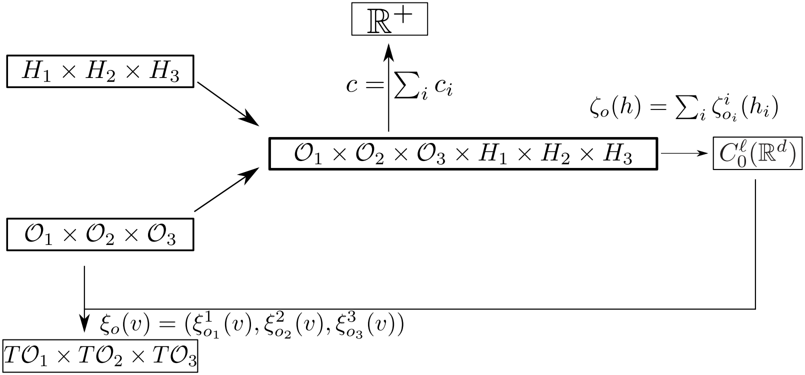
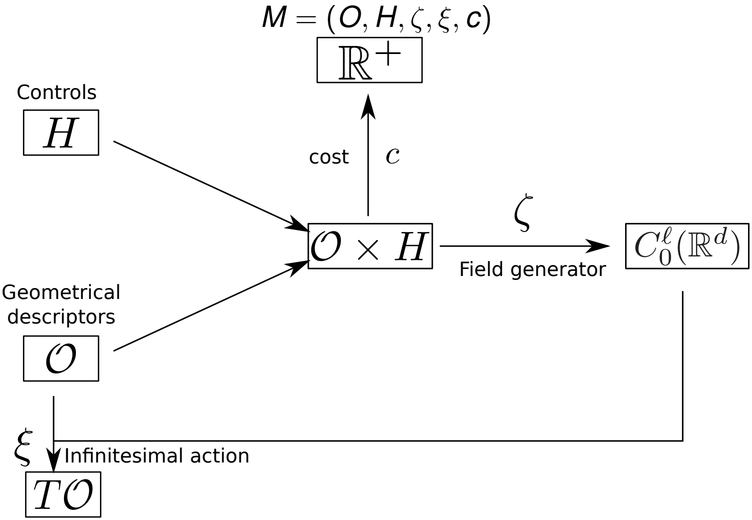
$$\mathcal{C}(M^l, l = 1 \cdots L)$$

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Uniform Embedding Condition

Definition

*Let $M = (\mathcal{O}, H, \zeta, \xi, c)$ be a C^k-deformation module of order ℓ. We say that M satisfies the **Uniform Embedding Condition (UEC)** if there exists a Hilbert space of vector fields V continuously embedded in $C_0^{\ell+k}(\mathbb{R}^d)$ and a constant $C > 0$ such that for all $o \in \mathcal{O}$ and for all $h \in H$, $\zeta_o(h) \in V$ and*

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Proposition

If M^l, $l = 1 \cdots L$, are C^k-deformation modules of order ℓ that satisfy UEC, then $\mathcal{C}(M^l, l = 1 \cdots L)$ satisfies UEC.

Definition (Finite energy controled paths on $\mathcal{O}$)

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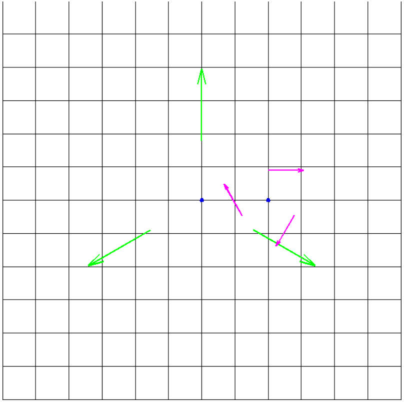
$\longrightarrow \varphi_{t=1}^{\zeta_o(h)}$ is a modular large deformation

An example



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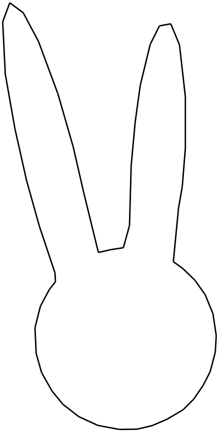

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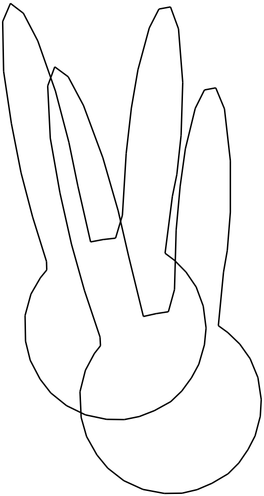
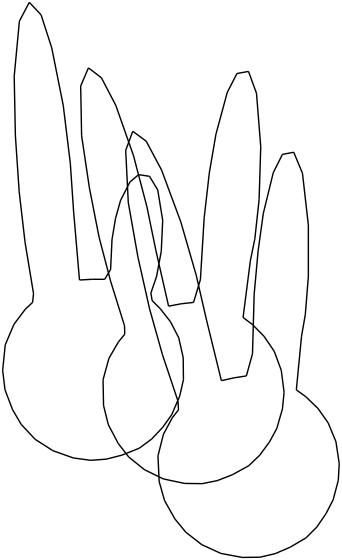
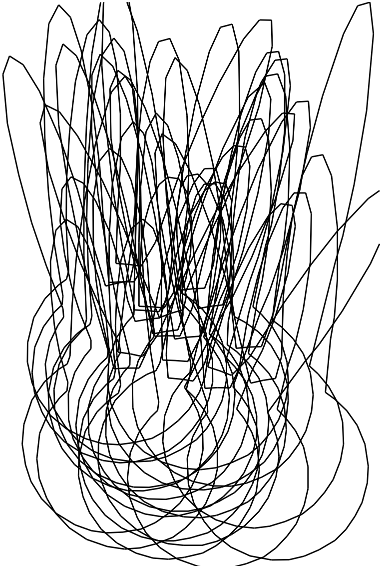


Targets

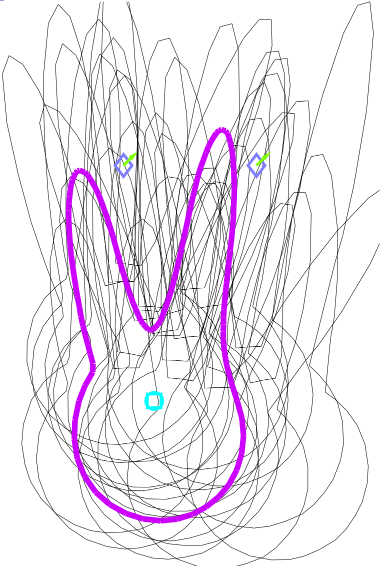


Targets

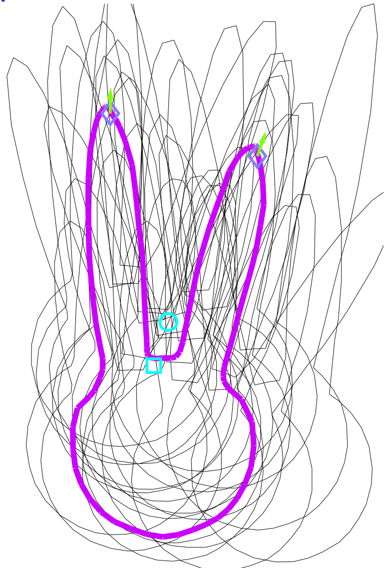


With previous rigid registration

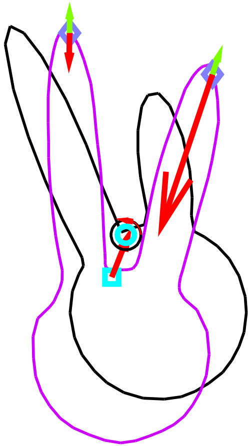
Before optimisation



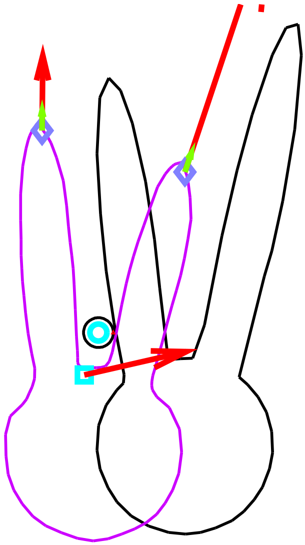
After optimisation



Example of trajectory



Example of trajectory

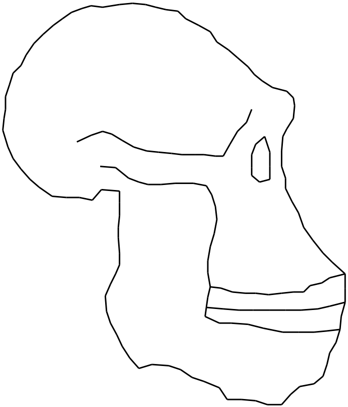


Weak prior

Targets

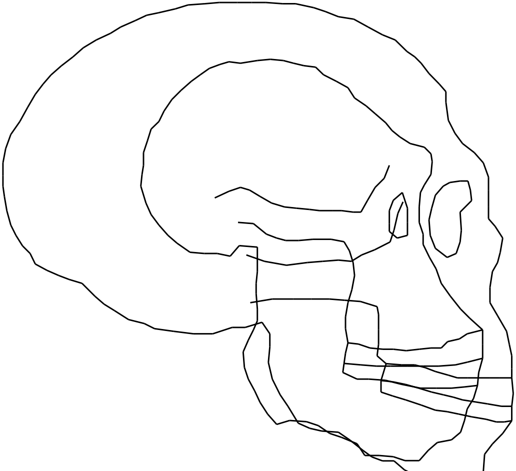
Weak prior

Targets



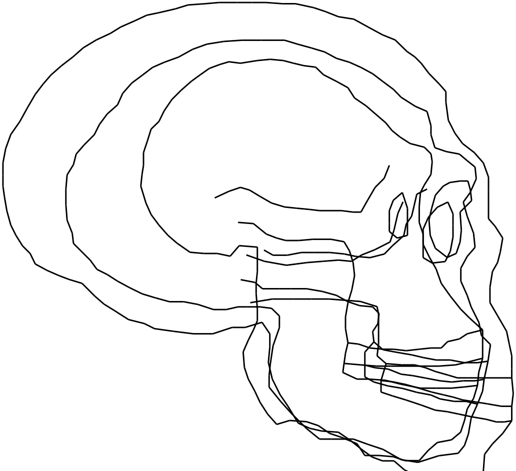
Weak prior

Targets



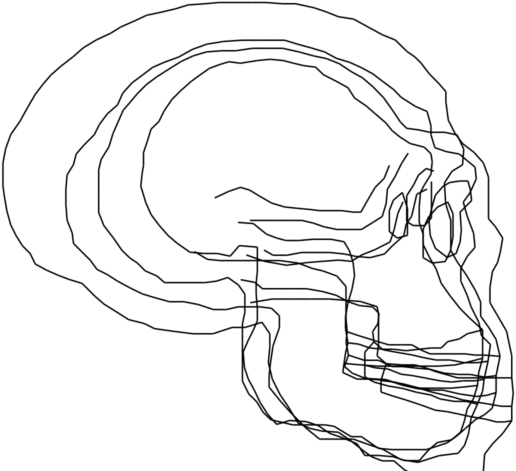
Weak prior

Targets



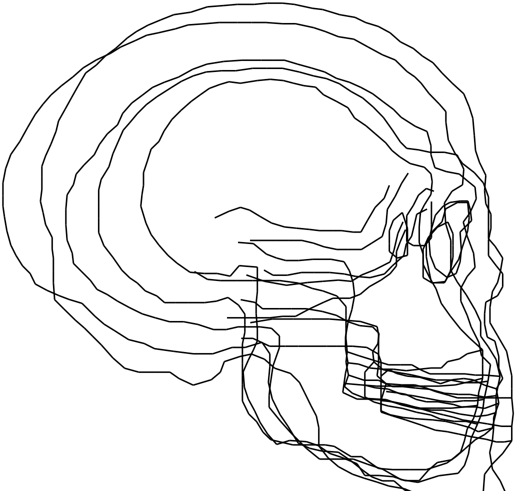
Weak prior

Targets



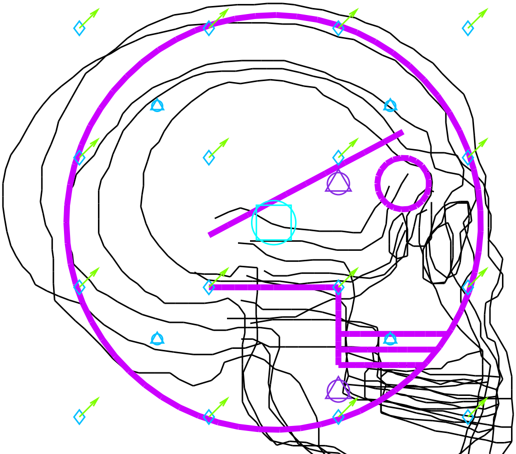
Weak prior

Targets



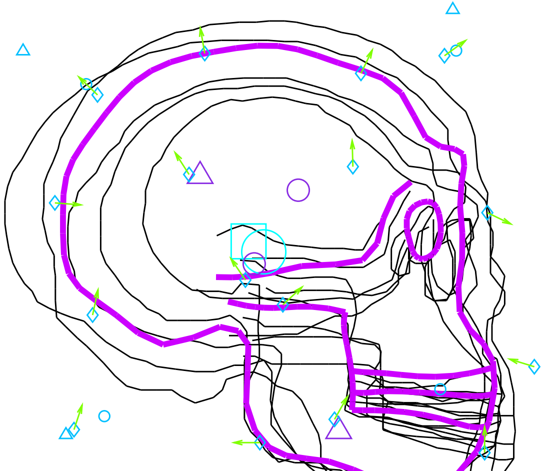
Weak prior

Before optimisation



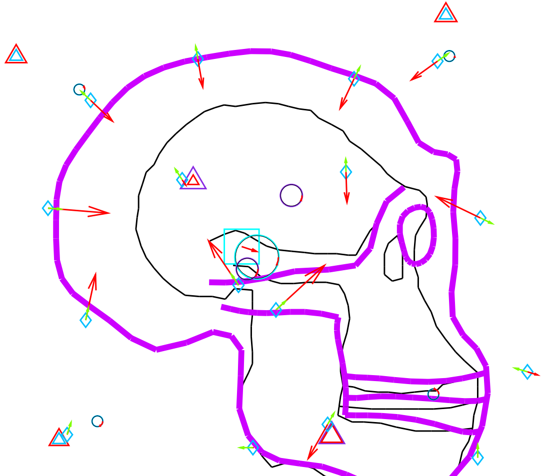
Weak prior

After optimisation



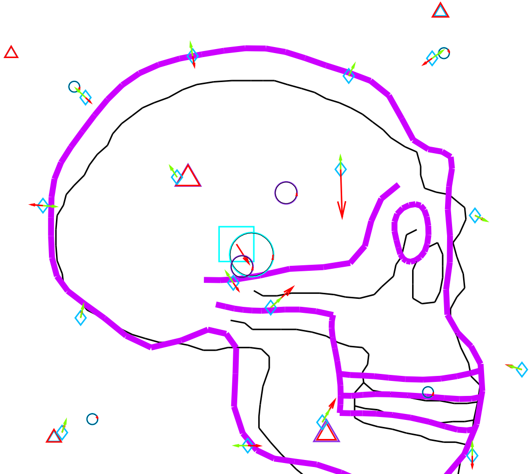
Weak prior

Example of trajectory



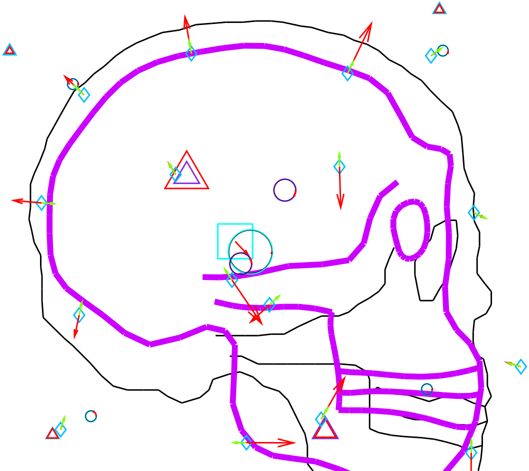
Weak prior

Example of trajectory



Weak prior

Example of trajectory



Sommaire

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- Easily incorporate complex constraints in deformation model

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- Metric taking into account constraints

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[B. G., S. Durrleman, A. Trouvé. A sub-Riemannian modular framework for diffeomorphism based analysis of shape ensembles, 2016]

Future work

- Better understanding parametrization of geodesics

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- Influence of cost

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- Influence of cost
- Choice of model

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Thank you for your attention !