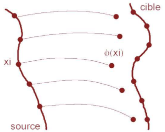
Geodesic equations and shooting algorithms for matching and template estimation

July 21, 2016

Finite dimensional setting for LDDMM

We come back to the LDDMM theory (following talk 1)

- In a discrete setting, shapes are often parametrized by a finite number of points (e.g. for curves or surface meshes : the vertices). So we consider data attachment terms which depend only on the final positions $\varphi_1^v(x_i)$: $A(\varphi_1^v) = \tilde{A}((\varphi_1^v(x_i))_{1 \leq i \leq n})$



- Denote $q_i(t) = \varphi_t^v(x_i)$ the trajectories of points x_i through the flow. The optimal vector fields must correspond at each time t to the optimal interpolation of vector $\dot{q}_i(t)$ at positions $q_i(t)$.

Finite dimensional setting for LDDMM

- $\Rightarrow$ at each time step t, the optimal vector fields depends on a finite number of vectors $p_i(t)$:

$$v_t(x) = \sum_{i=1}^{n} K_V(x, q_i(t))p_i(t), \quad \text{with } K_V(q(t), q(t))p(t) = \dot{q}(t)$$

  We call the $p_i(t)$ momentum vectors.
- Moreover, using the reproducing formula, we get

$$\|v_t\|_V^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle p_j(t) \, , K_V(q_j(t), q_i(t))p_i(t) \rangle$$

  or with matrix notations : $\|v_t\|_V^2 = p(t)^T K_V(q(t), q(t))p(t)$.
- Now since $\dot{q}(t) = K_V(q(t), q(t))p(t)$ (flow equation), we get also

$$\|v_t\|_V^2 = \dot{q}(t)^T K_V(q(t), q(t))^{-1} \dot{q}(t).$$

- $\Rightarrow \int_0^1 \|v_t\|_V^2 dt$ corresponds to the energy $E(q)$ of the path $q(t)$ for the Riemannian metric given by matrix $K_V(q(t), q(t))^{-1}$.

The landmark manifold

- Define

$$\mathcal{L}_n(\mathbb{R}^d) = \{q = (q_1, \ldots, q_n) \in (\mathbb{R}^d)^n,\ q_i \neq q_j,\ \forall i \neq j\}.$$

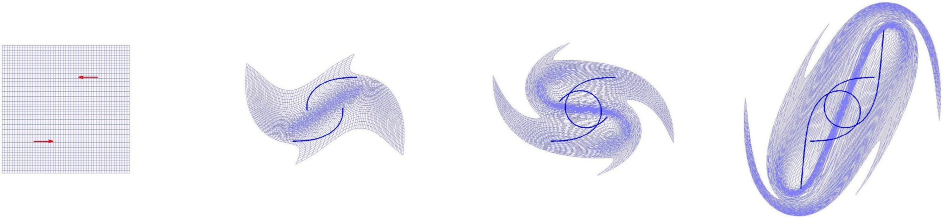
- $\mathcal{L}_n(\mathbb{R}^d)$ is a manifold as open set of $(\mathbb{R}^d)^n$.
- Consider on $\mathcal{L}_n(\mathbb{R}^d)$ the Riemannian metric whose matrix in the canonical coordinates is $K_V(q,q)^{-1}$.
- Optimal solution for matching problems correspond to geodesics in landmark space.
- We can derive the geodesic equations and use them in algorithms for optimizing matching problems.

The landmark manifold

- Geodesic equations can be written in Hamiltonian form :

$$\begin{cases} \dot{p} = -\frac{1}{2}\nabla_q \langle K_V(q,q)p \, , p\rangle \\ \dot{q} = K_V(q,q)p. \end{cases}$$

- Here is an example of solution : initial conditions are $q_1(0) = (0,0), q_2(0) = (1,1), p_1(0) = (1,0), p_2(0) = (-1,0)$, kernel is $K_V(x,y) = exp(-\|x-y\|^2/\sigma^2)\mathrm{id}$ with $\sigma = 1$.



$t = 0$ $t = 1/3$ $t = 2/3$ $t = 1$

Back to the matching functional

- For a matching functional the optimal trajectories must follow geodesics. So the optimal vector fields v_t depend only on the initial momentum vectors $p(0)$. So we rewrite the functional as

$$J(p(0)) = \gamma \left\langle K_V(q(0), q(0))p(0) \, , p(0) \right\rangle + A(q(1))$$

where $p(t)$ and $q(t)$ are constrained to follow the geodesic equations.

- The gradient of this functional writes

$$\nabla J(p(0)) = 2\gamma K_V(q(0), q(0))p(0) + \left(\frac{\partial q(1)}{\partial p(0)}\right)^T \nabla A(q(1))$$

The only difficult part is of course to compute $\left(\frac{\partial q(1)}{\partial p(0)}\right)^T$. This requires to differentiate the geodesic equations.

The adjoint equations

We have that

$$\left(\frac{\partial q(1)}{\partial p(0)}\right)^T \nabla A(q(1)) = \beta_p(0)$$

where $\beta(t) = (\beta_p(t), \beta_q(t)) \in \mathbb{R}^{dn} \times \mathbb{R}^{dn}$ is solution to the following adjoint equations :

$$\begin{cases} \dot{\beta}_p = \partial_q(K_V(q,q)p)\beta_p - K_V(q,q)\beta_q \\ \dot{\beta}_q = \frac{1}{2}\partial_q^2 \left\langle K_V(q,q)p \, , p \right\rangle \beta_p - (\partial_q(K_V(q,q)p)^T \beta_q. \end{cases}$$

with initial condition $\beta(1) = (0, \nabla A(q(1)))$.