Interpolation on Symmetric Spaces and Variational Discretization of Gauge Field Theories

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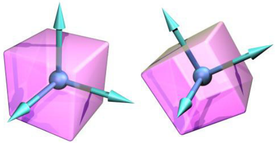
Manifold-Valued Data and Manifold-Valued Functions

Manifold-valued data and manifold-valued functions play an important role in a variety of applications:

- Mechanics



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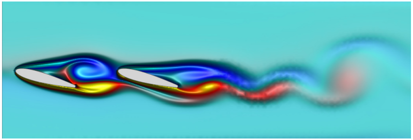
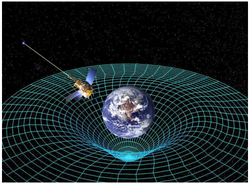
- Reduced-order modeling



- Numerical relativity



Gauge Field Theories

- A ***gauge symmetry*** is a continuous local transformation on the field variables that leaves the system physically indistinguishable.
- A consequence of this is that the Euler–Lagrange equations are ***underdetermined***, i.e., the evolution equations are insufficient to propagate all the fields.
- The ***kinematic fields*** have no physical significance, but the ***dynamic fields*** and their conjugate momenta have physical significance.
- The Euler–Lagrange equations are ***overdetermined***, and the initial data on a Cauchy surface satisfies a constraint (usually elliptic).
- These degenerate systems are naturally described using ***multi-Dirac*** mechanics and geometry.

Electromagnetism

- Let $\mathbf{E}$ and $\mathbf{B}$ be the electric and magnetic vector fields respectively.
- We can write Maxwell's equations in terms of the scalar and vector potentials ϕ and $\mathbf{A}$ by,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \nabla^2\phi + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = 0,$$

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \Box\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{\partial\phi}{\partial t}\right) = 0.$$

- The following transformation leaves the equations invariant,

$$\phi \to \phi - \frac{\partial f}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla f.$$

- The associated Cauchy initial data constraints are,

$$\nabla\cdot\mathbf{B}^{(0)} = 0, \qquad \nabla\cdot\mathbf{E}^{(0)} = 0.$$

Gauge conditions

- One often addresses the indeterminacy due to gauge freedom in a field theory through the choice of a ***gauge condition***.
- The ***Lorenz gauge*** is $\nabla \cdot \mathbf{A} = -\frac{\partial \phi}{\partial t}$, which yields,

$$\Box \phi = 0, \qquad \Box \mathbf{A} = 0.$$

- The ***Coulomb gauge*** is $\nabla \cdot \mathbf{A} = 0$, which yields,

$$\nabla^2 \phi = 0, \qquad \Box \mathbf{A} + \nabla \frac{\partial \phi}{\partial t} = 0.$$

- Given different initial and boundary conditions, some problems may be easier to solve in certain gauges than others. There is no systematic way of deciding which gauge to use for a given problem.

Noether's Theorem

Noether's Theorem

For every continuous symmetry of an action, there exists a quantity that is conserved in time.

- The simplest illustration of the principle comes from classical mechanics: a time-invariant action implies a conservation of the Hamiltonian, which is usually identified with energy.
- More precisely, if $S = \int_{t_a}^{t_b} L(q, \dot{q})dt$ is invariant under the transformation $t \to t + \epsilon$, then

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = \frac{dH}{dt} = 0$$

Noether's Theorem for Gauge Field Theories

Noether's Theorem for Gauge Field Theories

For every differentiable, local symmetry of an action, there exists a ***Noether current*** obeying a continuity equation. Integrating this current over a spacelike surface yields a conserved quantity called a ***Noether charge***.

- The action principle for electromagnetism is $S = \frac{1}{2}\int(\mathbf{B}^2 - \mathbf{E}^2)d^4x$. Applying Noether's theorem to the gauge symmetry yields the following currents:

$$j_0 = \mathbf{E}\cdot\nabla f \qquad\qquad \mathbf{j} = -\mathbf{E}\frac{\partial f}{\partial t} + (\mathbf{B}\times\nabla)f$$

Motivation for the approach we take

- Our long-term goal is to develop geometric structure-preserving numerical discretizations that systematically addresses the issue of gauge symmetries. Eventually, we wish to study discretizations of general relativity that address the issue of general covariance.
- Towards this end, we will consider ***multi-Dirac mechanics*** based on a ***Hamilton–Pontryagin variational principle for field theories*** that is well adapted to degenerate field theories.
- The issue of general covariance also leads us to avoid using a tensor product discretization that presupposes a slicing of spacetime, rather we will consider 4-simplicial complexes in spacetime.
- More generally, we will need to study discretizations that are invariant to some discrete analogue of the gauge symmetry group.

Pontryagin bundle and Hamilton–Pontryagin principle

- Consider the ***Pontryagin bundle*** $TQ \oplus T^*Q$, which has local coordinates (q, v, p).
- The ***Hamilton–Pontryagin principle*** is given by

$$\delta \int [L(q,v) - p(v - \dot{q})] = 0,$$

where we impose the second-order curve condition, $v = \dot{q}$ using Lagrange multipliers p.

Implicit Lagrangian systems

- Taking variations in q, v, and p yield

$$\begin{aligned}\delta \int [L(q,v) - p(v-\dot{q})]dt \\ &= \int \left[\frac{\partial L}{\partial q}\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v-\dot{q})\delta p + p\delta\dot{q}\right] dt \\ &= \int \left[\left(\frac{\partial L}{\partial q} - \dot{p}\right)\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v-\dot{q})\delta p\right] dt,\end{aligned}$$

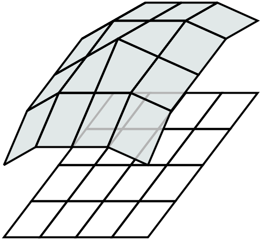
  where we used integration by parts, and the fact that the variation δq vanishes at the endpoints.

- This recovers the ***implicit Euler–Lagrange equations***,

$$\dot{p} = \frac{\partial L}{\partial q}, \qquad p = \frac{\partial L}{\partial v}, \qquad v = \dot{q}.$$

Multisymplectic Geometry

- ***Base space*** $\mathcal{X}$. $(n+1)$-spacetime.
- ***Configuration bundle***. Given by $\pi : Y \to \mathcal{X}$, with the fields as the fiber.
- ***Configuration*** $q : \mathcal{X} \to Y$. Gives the field variables over each spacetime point.
- ***First jet*** $J^1 Y$. The first partials of the fields with respect to spacetime.



- ***Lagrangian density*** $L : J^1 Y \to \Omega^{n+1}(\mathcal{X})$.
- ***Action integral*** given by, $\mathcal{S}(q) = \int_{\mathcal{X}} L(j^1 q)$.
- ***Hamilton's principle*** states, $\delta \mathcal{S} = 0$.

Hamilton–Pontryagin for Fields

- In coordinates, the Hamilton–Pontryagin principle for fields is

$$S(y^A, y^A_\mu, p^\mu_A) = \int_U \left[p^\mu_A \left(\frac{\partial y^A}{\partial x^\mu} - v^A_\mu \right) + L(x^\mu, y^A, v^A_\mu) \right] d^{n+1}x.$$

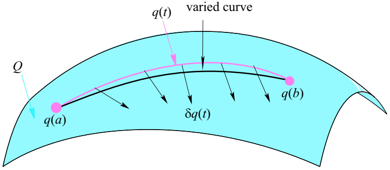
- By taking variations with respect to y^A, v^A_μ and p^μ_A (where δy^A vanishes on ∂U) we obtain the implicit Euler–Lagrange equations,

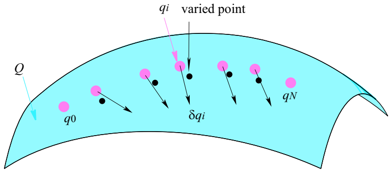
$$\frac{\partial p^\mu_A}{\partial x^\mu} = \frac{\partial L}{\partial y^A}, \quad p^\mu_A = \frac{\partial L}{\partial v^A_\mu}, \quad \text{and} \quad \frac{\partial y^A}{\partial x^\mu} = v^A_\mu.$$

- The ***covariant Legendre transform*** involves both the energy and momentum,

$$p^\mu_A = \frac{\partial L}{\partial v^A_\mu}, \qquad p = L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu.$$

Discrete Lagrangian Variational Principle





- ***Discrete Lagrangian***

$$L_d(q_0, q_1) \approx L_d^{\text{exact}}(q_0, q_1) \equiv \int_0^h L\left(q_{0,1}(t), \dot{q}_{0,1}(t)\right) dt,$$

  where $q_{0,1}(t)$ satisfies the Euler–Lagrange equations for L and the boundary conditions $q_{0,1}(0) = q_0$, $q_{0,1}(h) = q_1$.

- This is related to ***Jacobi's solution*** of the ***Hamilton–Jacobi equation***.

Discrete Lagrangian Variational Principle

- ***Discrete Hamilton's principle***

$$\delta \mathbb{S}_d = \delta \sum L_d(q_k, q_{k+1}) = 0,$$

where q_0, q_N are fixed.

- ***Discrete Euler-Lagrange equation***

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0.$$

- The associated discrete flow $(q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$ is automatically symplectic, since it is equivalent to,

$$p_k = -D_1 L_d(q_k, q_{k+1}), \quad p_{k+1} = D_2 L_d(q_k, q_{k+1}),$$

which is the characterization of a symplectic map in terms of a ***Type I generating function*** (discrete Lagrangian).

Main Advantages of Variational Integrators

- ***Discrete Noether's Theorem***
  If the discrete Lagrangian L_d is (infinitesimally) G-invariant under the diagonal group action on $Q \times Q$,
  $$L_d(gq_0, gq_1) = L_d(q_0, q_1)$$
  then the ***discrete momentum map*** $J_d : Q \times Q \to \mathfrak{g}^*$,
  $$\langle J_d(q_k, q_{k+1}), \xi \rangle \equiv \langle D_1 L_d(q_k, q_{k+1}), \xi_Q(q_k) \rangle$$
  is preserved by the discrete flow.

Main Advantages of Variational Integrators

- ***Variational Error Analysis***
  Since the exact discrete Lagrangian generates the exact solution of the Euler–Lagrange equation, the exact discrete flow map is *formally* expressible in the setting of variational integrators.
- This is analogous to the situation for B-series methods, where the exact flow can be expressed formally as a B-series.
- If a computable discrete Lagrangian L_d is of order r, i.e.,

$$L_d(q_0, q_1) = L_d^{\text{exact}}(q_0, q_1) + \mathcal{O}(h^{r+1})$$

  then the discrete Euler–Lagrange equations yield an order r accurate symplectic integrator.

Ritz Variational Integrators

- Consider an alternative expression for the exact discrete Lagrangian,

$$L_d^{\text{exact}}(q_0, q_1) \equiv \underset{\substack{q \in C^2([0,h],Q) \\ q(0)=q_0, q(h)=q_1}}{\text{ext}} \int_0^h L(q(t), \dot{q}(t)) dt,$$

  which is more amenable to discretization.
- Replace the infinite-dimensional function space $C^2([0, h], Q)$ with a ***finite-dimensional function space***.
- Replace the integral with a ***numerical quadrature formula***.

Ritz Variational Integrators

- A desirable property of a Ritz numerical method based on a finite-dimensional space $F_d \subset F$, is that it should exhibit ***optimal rates of convergence***, which is to say that the numerical solution $q_d \in F_d$ and the exact solution $q \in F$ satisfies,

$$\|q - q_d\| \leq c \inf_{\tilde{q} \in F_d} \|q - \tilde{q}\|.$$

- This means that the rate of convergence depends on the best approximation error of the finite-dimensional function space.

Ritz Variational Integrators

- Given a sequence of finite-dimensional function spaces $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \subset C^2([0,h],Q) \equiv \mathcal{C}_\infty$.
- For a correspondingly accurate sequence of quadrature formulas,

$$L_d^i(q_0,q_1) \equiv \operatorname*{ext}_{q\in\mathcal{C}_i} h \sum_{j=1}^{s_i} b_j^i L(q(c_j^i h), \dot{q}(c_j^i h)),$$

where $L_d^\infty(q_0,q_1) = L_d^{\text{exact}}(q_0,q_1)$.
- Proving $L_d^i(q_0,q_1) \to L_d^\infty(q_0,q_1)$, corresponds to Γ-convergence.
- For optimality, we require the bound,

$$L_d^i(q_0,q_1) = L_d^\infty(q_0,q_1) + c \inf_{\tilde{q}\in\mathcal{C}_i} \|q-\tilde{q}\|,$$

where we need to relate the rate of Γ-convergence with the best approximation properties of the family of approximation spaces.

Ritz Variational Integrators

Theorem (Optimality of Ritz Variational Integrators)

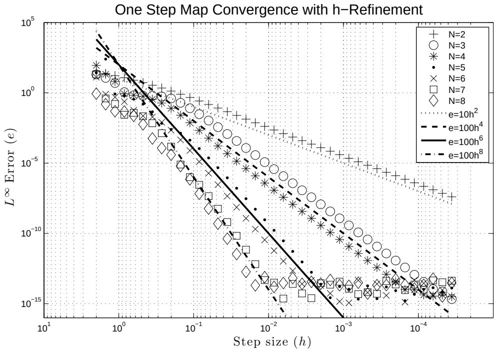
Under suitable technical hypotheses:

- Regularity of L in a closed and bounded neighboorhood;
- The quadrature rule is sufficiently accurate;
- The discrete and continuous trajectories *minimize* their actions;

the Ritz discrete Lagrangian has the same approximation error as the best approximation error of the approximation space.

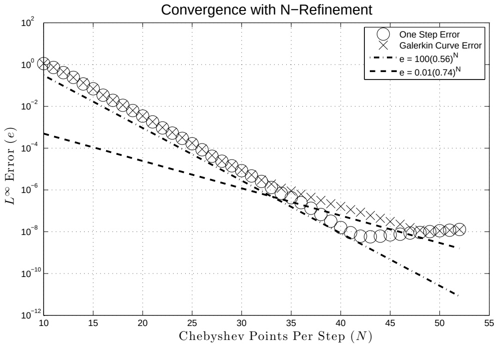
- The critical assumption is action minimization. For Lagrangians $L = \dot{q}^T M \dot{q} - V(q)$, and sufficiently small h, this assumption holds.
- Shows that Ritz variational integrators are ***order optimal***; spectral variational integrators are ***geometrically convergent***.

Order Optimal Convergence of Ritz variational integrators



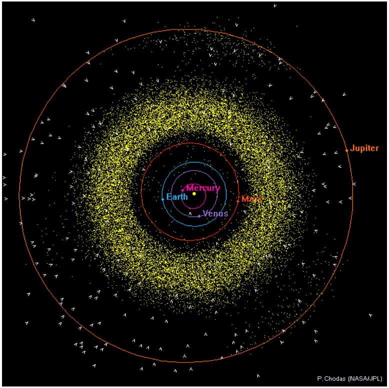
- Order optimal convergence of the Kepler 2-body problem with eccentricity 0.6 over 100 steps of $h = 2.0$.

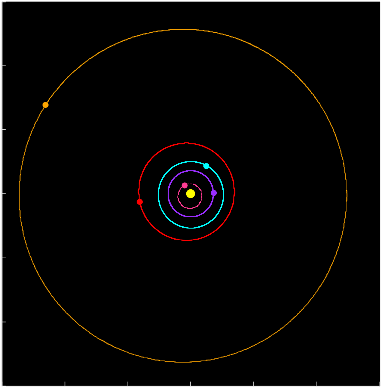
Geometric Convergence of Spectral variational integrators



- Geometric convergence of the Kepler 2-body problem with eccentricity 0.6 over 100 steps of $h = 2.0$.

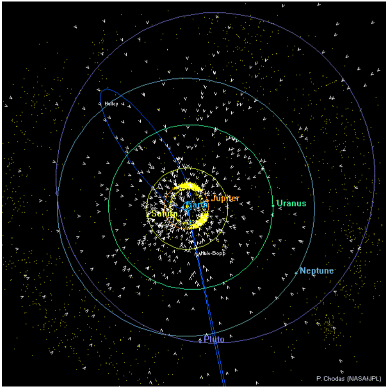
Numerical Experiments: Solar System Simulation

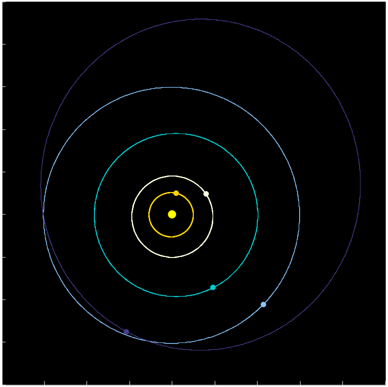




- Comparison of inner solar system orbital diagrams from a spectral variational integrator and the JPL Solar System Dynamics Group.
- $h = 100$ days, $T = 27$ years, 25 Chebyshev points per step.

Numerical Experiments: Solar System Simulation





- Comparison of outer solar system orbital diagrams from a spectral variational integrator and the JPL Solar System Dynamics Group. Inner solar system was aggregated, and $h = 1825$ days.

Multisymplectic Exact Discrete Lagrangian

- Recall the implicit characterization of a symplectic map in terms of generating functions:

$$\begin{cases} p_k = -D_1 L_d(q_k, q_{k+1}) \\ p_{k+1} = D_2 L_d(q_k, q_{k+1}) \end{cases} \qquad \begin{cases} p_k = D_1 H_d^+(q_k, p_{k+1}) \\ q_{k+1} = D_2 H_d^+(q_k, p_{k+1}) \end{cases}$$

- Symplecticity follows as a trivial consequence of these equations, together with $\mathbf{d}^2 = 0$, as the following calculation shows:

$$\begin{aligned} \mathbf{d}^2 L_d(q_k, q_{k+1}) &= \mathbf{d}(D_1 L_d(q_k, q_{k+1}) dq_k + D_2 L_d(q_k, q_{k+1}) dq_{k+1}) \\ &= \mathbf{d}(-p_k dq_k + p_{k+1} dq_{k+1}) \\ &= -dp_k \wedge dq_k + dp_{k+1} \wedge dq_{k+1} \end{aligned}$$

Analogy with the ODE case

- We consider a multisymplectic analogue of Jacobi's solution:

$$L_d^{\mathrm{exact}}(q_0, q_1) \equiv \int_0^h L\left(q_{0,1}(t), \dot{q}_{0,1}(t)\right) dt,$$

where $q_{0,1}(t)$ satisfies the Euler–Lagrange boundary-value problem.

- This is given by,

$$L_d^{\mathrm{exact}}(\varphi|_{\partial\Omega}) \equiv \int_\Omega L(j^1\tilde{\varphi})$$

where $\tilde{\varphi}$ satisfies the boundary conditions $\tilde{\varphi}|_{\partial\Omega} = \varphi|_{\partial\Omega}$, and $\tilde{\varphi}$ satisfies the Euler–Lagrange equation in the interior of Ω.

Multisymplectic Relation

- If one takes variations of the ***multisymplectic exact discrete Lagrangian*** with respect to the boundary conditions, we obtain,

$$\partial_{\varphi(x,t)} L_d^{\text{exact}}(\varphi|_{\partial\Omega}) = p_\perp(x,t),$$

  where $(x,t) \in \partial\Omega$, and $p_\perp$ is a codimension-1 differential form, that by Hodge duality can be viewed as the normal component (to the boundary $\partial\Omega$) of the multimomentum at the point (x,t).
- These equations, taken at every point on $\partial\Omega$ constitute a ***multisymplectic relation***, which is the PDE analogue of,

$$\begin{cases} p_k = -D_1 L_d(q_k, q_{k+1}) \\ p_{k+1} = D_2 L_d(q_k, q_{k+1}) \end{cases}$$

  where the sign comes from the orientation of the boundary.

Gauge Symmetries and Variational Discretizations

Theorem (Noether's Theorem)

For every continuous symmetry of an action, there exists a quantity that is conserved in time.

Theorem (Noether's Theorem for Gauge Field Theories)

For every differentiable, local symmetry of an action, there exists a ***Noether current*** obeying a continuity equation. Integrating this current over a spacelike surface yields a conserved quantity called a ***Noether charge***.

- Since gauge symmetries are associated with conserved quantities, we need finite-elements that are (approximately) group-equivariant.

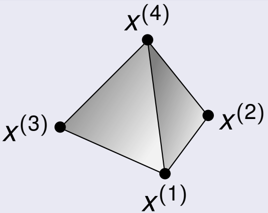
Motivating Example: Lorentzian Metrics

Let $\mathcal{L}$ denote the space of ***Lorentzian metric tensors***:

$$\mathcal{L} = \{L \in \mathbb{R}^{4\times 4} \mid L = L^T, \det L \neq 0, \text{signature}(L) = (3,1)\}.$$

Problem

Given $L^{(i)} \in \mathcal{L}$ at the vertices $x^{(i)}$ of a simplex Ω, find a continuous function $\mathcal{I}L : \Omega \to \mathcal{L}$ such that:



1. $\mathcal{I}L(x^{(i)}) = L^{(i)}$ for each i.
2. $\mathcal{I}L(x) \in \mathcal{L}$ for every $x \in \Omega$.
3. *(Frame invariance):* If $Q \in O(1,3)$ and $L^{(i)} \leftarrow QL^{(i)}Q^T$ for each i, then $\mathcal{I}L(x) \leftarrow Q\mathcal{I}L(x)Q^T$.

Here, $O(1,3)$ denotes the ***indefinite orthogonal group***:

$$O(1,3) = \{Q \in \mathbb{R}^{4\times 4} \mid QJQ^T = J\},$$

where $J = \text{diag}(-1,1,1,1)$.

Motivating Example: Lorentzian Metrics

Options:

1. Componentwise interpolation: Not signature-preserving, in general. For instance,

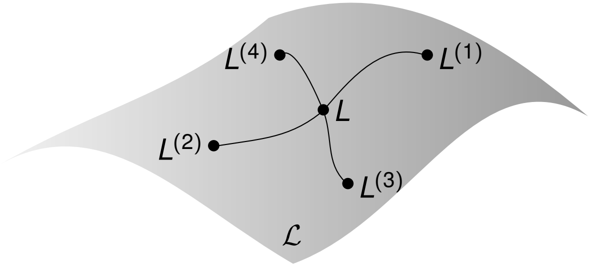
$$\frac{1}{2}\underbrace{\begin{pmatrix}0 & 4 & 0 & 0\\ 4 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}}_{\in\mathcal{L}\text{ since }\lambda=-4,1,1,4}+\frac{1}{2}\underbrace{\begin{pmatrix}2 & -4 & 0 & 0\\ -4 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}}_{\in\mathcal{L}\text{ since }\lambda=-2,1,1,6}=\underbrace{\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}}_{\notin\mathcal{L}\text{ since }\lambda=1,1,1,1}$$

Motivating Example: Lorentzian Metrics

1. ***Geodesic interpolation*** [Grohs, Sander]:

$$\mathcal{I}L(x) = \arg\min_{L\in\mathcal{L}} \sum_{i=1}^{m} \phi_i(x)\,\mathrm{dist}(L^{(i)}, L)^2,$$

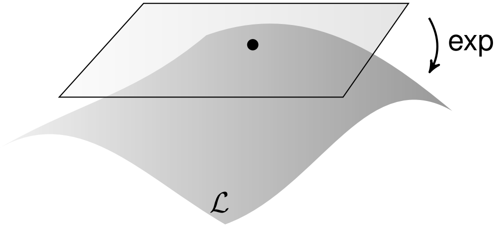
where $\{\phi_i\}_{i=1}^m$ are scalar-valued shape functions satisfying $\phi_i(x^{(j)}) = \delta_{ij}$. Also known as the weighted ***Riemannian mean***.



Motivating Example: Lorentzian Metrics

Our approach:

- Idea: If $\mathcal{L}$ were a Lie group, one could use the exponential map and perform all calculations on its Lie algebra, a linear space.



- In reality, $\mathcal{L}$ is not a Lie group (it is a ***symmetric space***). Nonetheless, a similar construction is available.

Motivating Example: Lorentzian Metrics

1. Notice that $\mathcal{L}$ is diffeomorphic to $GL_4(\mathbb{R})/O(1,3)$: The map

$$\bar{\varphi} : GL_4(\mathbb{R})/O(1,3) \to \mathcal{L}$$
$$[A] \mapsto AJA^T,$$

is a diffeomorphism, where $J = \text{diag}(-1,1,1,1)$.

2. Every coset $[A]$ has a canonical representative Y by virtue of the ***generalized polar decomposition***:

$$A = YQ, \quad Y \in Sym_J(4),\ Q \in O(1,3),$$

where

$$Sym_J(4) = \{Y \in GL_4(\mathbb{R}) \mid YJ = JY^T\}.$$

3. $\log(Y)$ lives in a linear space called a ***Lie triple system***:

$$\log(Y) \in \mathfrak{sym}_J(4) = \{P \in \mathbb{R}^{4\times 4} \mid PJ = JP^T\}.$$

Motivating Example: Lorentzian Metrics

$$\begin{array}{ccccccc} & & & & GL_4(\mathbb{R}) & & \\ & & & \overset{\iota}{\nearrow} & \downarrow \pi & \overset{\varphi}{\searrow} & \\ \mathfrak{sym}_J(4) & \xrightarrow[\exp]{} & Sym_J(4) & \xrightarrow[\psi]{} & GL_4(\mathbb{R})/O(1,3) & \xrightarrow[\bar{\varphi}]{} & \mathcal{L} \\ \log(Y) & \longmapsto & Y & \longmapsto & [Y] & \longmapsto & YJY^T \end{array}$$

To summarize:

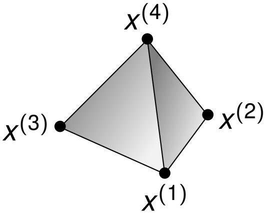
1. $\mathcal{L}$ is locally diffeomorphic to the ***Lie triple system***

$$\mathfrak{sym}_J(4) = \{P \in \mathbb{R}^{4\times 4} \mid PJ = JP^T\},$$

   which is a *linear space*.

2. Interpolation on a linear space is easy.

Motivating Example: Lorentzian Metrics



The resulting interpolation formula reads

$$\mathcal{I}L(x) = J \exp\left(\sum_{i=1}^{m} \phi_i(x) \log(JL^{(i)})\right),$$

where $J = \operatorname{diag}(-1, 1, 1, 1)$, and $\{\phi_i\}_{i=1}^m$ are scalar-valued shape functions satisfying $\phi_i(x^{(j)}) = \delta_{ij}$.

Motivating Example: Lorentzian Metrics

The interpolant so defined enjoys the following properties:

Signature preservation

The interpolant $\mathcal{I}L$ is signature-preserving; that is,

$$\mathcal{I}L(x) \in \mathcal{L}$$

for every $x \in \Omega$.

Frame invariance

Let $Q \in O(1,3)$. If $\tilde{L}^{(i)} = QL^{(i)}Q^T$, $i = 1, 2, \ldots, m$, and if Q is sufficiently close to the identity matrix, then

$$\mathcal{I}\tilde{L}(x) = Q\,\mathcal{I}L(x)\,Q^T$$

for every $x \in \Omega$.

Motivating Example: Lorentzian Metrics

Symmetry under inversion

If $\tilde{L}^{(i)} = (L^{(i)})^{-1}$, $i = 1, 2, \ldots, m$, then

$$\mathcal{I}\tilde{L}(x) = (\mathcal{I}L(x))^{-1}$$

for every $x \in \Omega$.

Determinant averaging

If $\sum_{i=1}^{m} \phi_i(x) = 1$ for every $x \in \Omega$, then

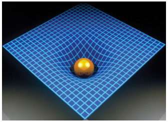
$$\det \mathcal{I}L(x) = \prod_{i=1}^{m} \left(\det L^{(i)}\right)^{\phi_i(x)}$$

for every $x \in \Omega$.

Motivating Example: Lorentzian Metrics

Numerical example: Interpolating the ***Schwarzschild metric***

$$-\left(1-\frac{1}{r}\right)dt^2+\left(1-\frac{1}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right).$$



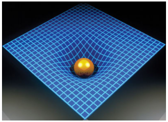
	Linear shape functions $\{\phi_i\}_i$			
N	L^2-error	Order	H^1-error	Order
2	$3.3\cdot10^{-3}$		$2.8\cdot10^{-2}$	
4	$8.4\cdot10^{-4}$	1.975	$1.4\cdot10^{-2}$	0.998
8	$2.1\cdot10^{-4}$	1.994	$7.1\cdot10^{-3}$	0.999
16	$5.3\cdot10^{-5}$	1.998	$3.6\cdot10^{-3}$	1.000

Error incurred when interpolating the Schwarzschild metric over the region $U=\{0\}\times[2,3]\times[2,3]\times[2,3]$ on a uniform $N\times N\times N$ grid of cubes, with shape functions $\{\phi_i\}_i$ on each cube given by tensor products of Lagrange polynomials of degree 1.

Motivating Example: Lorentzian Metrics

Numerical example: Interpolating the ***Schwarzschild metric***

$$-\left(1-\frac{1}{r}\right)dt^2+\left(1-\frac{1}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right).$$



	Quadratic shape functions $\{\phi_i\}_i$			
N	L^2-error	Order	H^1-error	Order
2	$1.7\cdot10^{-4}$		$2.5\cdot10^{-3}$	
4	$2.2\cdot10^{-5}$	3.001	$6.2\cdot10^{-4}$	1.993
8	$2.7\cdot10^{-6}$	3.000	$1.6\cdot10^{-4}$	1.998
16	$3.4\cdot10^{-7}$	3.000	$3.9\cdot10^{-5}$	1.999

Error incurred when interpolating the Schwarzschild metric over the region $U=\{0\}\times[2,3]\times[2,3]\times[2,3]$ on a uniform $N\times N\times N$ grid of cubes, with shape functions $\{\phi_i\}_i$ on each cube given by tensor products of Lagrange polynomials of degree 2.

Motivating Example: Lorentzian Metrics

$$\mathcal{I}L(x) = J \exp\left(\sum_{i=1}^{m} \phi_i(x) \log(JL^{(i)})\right)$$

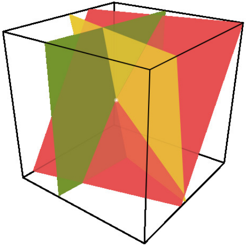
Remarks:

1. An alternative interpolant is obtained by defining $\mathcal{I}L(x)$ implicitly via
$$\mathcal{I}L(x) = \mathcal{I}L(x) \exp\left(\sum_{i=1}^{m} \phi_i(x) \log\left(\mathcal{I}L(x)^{-1} L^{(i)}\right)\right).$$
This interpolant is equivalent to the ***geodesic interpolant***.
2. Replacing $J = \mathrm{diag}(-1, 1, 1, 1)$ with the identity matrix, one recovers the weighted ***Log-Euclidean mean*** of symmetric positive-definite matrices [Arsigny et al.]:
$$\mathcal{I}L(x) = \exp\left(\sum_{i=1}^{m} \phi_i(x) \log(L^{(i)})\right)$$

Abstraction to Symmetric Spaces

This construction works if $\mathcal{L}$ is replaced by any ***symmetric space*** – a smooth manifold with an inversion symmetry (an involutive isometry) about every point. Examples include:

- Symmetric $n \times n$ matrices with signature $(p, n-p)$.
- Grassmannian $Gr(p, n)$ – space of p-dimensional linear subspaces of $\mathbb{R}^n$.



A key role in the construction is played by the ***generalized polar decomposition***.

Generalized Polar Decomposition

Generalized Polar Decomposition [Helgason]

Let G be a Lie group, and let $\sigma : G \to G$ be an involutive automorphism, i.e. $\sigma \neq \text{id.}$, $\sigma^2 = \text{id.}$, and $\sigma(gh) = \sigma(g)\sigma(h)$ for every $g, h \in G$. Then every $g \in G$ can be written as a product

$$g = pk, \quad p \in G_\sigma,\ k \in G^\sigma,$$

where

$$\begin{aligned} G^\sigma &= \{g \in G \mid \sigma(g) = g\}, \\ G_\sigma &= \{g \in G \mid \sigma(g) = g^{-1}\}. \end{aligned}$$

Moreover, this decomposition is locally unique.

Examples:

- $G = GL_n(\mathbb{R}), \sigma(A) = A^{-T} \implies G^\sigma = O(n), G_\sigma = Sym(n).$
- $G = GL_4(\mathbb{R}), \sigma(A) = JA^{-T}J \implies G^\sigma = O(1,3), G_\sigma = Sym_J(4).$

Abstraction to Symmetric Spaces

Abstract setting:

- $\mathcal{S}$ – smooth manifold — $\mathcal{L}$ (Lorentzian metrics)
- η – distinguished element of $\mathcal{S}$ — $J = \mathrm{diag}(-1, 1, 1, 1)$
- G – Lie group that acts transitively on $\mathcal{S}$ — $GL_4(\mathbb{R})$
- $\sigma : G \to G$ – involutive automorphism — $\sigma(A) = JA^{-T}J$
- $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ — $O(1,3)$
- $G_\sigma = \{g \in G \mid \sigma(g) = g^{-1}\}$ — $Sym_J(4)$

Key assumption: Isotropy subgroup of η coincides with the fixed set G^σ, i.e.

$$g \cdot \eta = \eta \iff \sigma(g) = g.$$

$$AJA^T = J \iff JA^{-T}J = A$$

Then $\mathcal{S}$ is diffeomorphic to G/G^σ (a ***symmetric space***), and every $[g] \in G/G^\sigma$ has a canonical representative $p \in G_\sigma$ by the ***generalized polar decomposition*** $g = pk$, $p \in G_\sigma$, $k \in G^\sigma$.

Abstraction to Symmetric Spaces

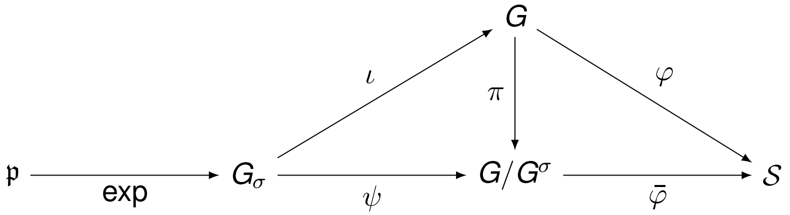
Abstract setting, continued:

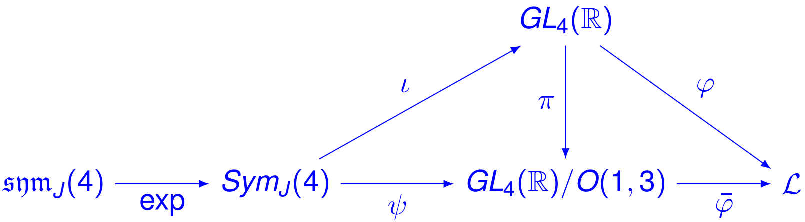
1. $\mathfrak{g}$ – Lie algebra of G $\qquad \mathbb{R}^{4\times 4}$
2. $\exp : \mathfrak{g} \to G$ – exponential map $\qquad \exp : \mathbb{R}^{4\times 4} \to GL_4(\mathbb{R})$
3. The preimage of G_σ under exp is the linear space

$$\mathfrak{p} = \{P \in \mathfrak{g} \mid d\sigma(P) = -P\} \subset \mathfrak{g}$$
$$= \{P \in \mathbb{R}^{4\times 4} \mid -JP^T J = -P\}$$

This space is a ***Lie triple system*** – it is closed under the double commutator $[\cdot, [\cdot, \cdot]]$, but not under $[\cdot, \cdot]$.

Abstraction to Symmetric Spaces





Abstraction to Symmetric Spaces

To summarize:

1. $\mathcal{S}$ is locally diffeomorphic to the Lie triple system $\mathfrak{p}$, which is a *linear space*.
2. Interpolation on a linear space is easy.
3. The resulting formula for interpolating $\{u^{(i)}\}_{i=1}^{m} \subset \mathcal{S}$ reads

$$\mathcal{I}u(x) = F\left(\sum_{i=1}^{m} \phi_i(x) F^{-1}(u^{(i)})\right),$$

where $\phi_i : \Omega \to \mathbb{R}$, $i = 1, 2, \dots, m$, are scalar-valued shape functions satisfying $\phi_i(x^{(j)}) = \delta_{ij}$, and

$$\begin{aligned} F : \mathfrak{p} &\to \mathcal{S} \\ P &\mapsto \exp(P) \cdot \eta. \end{aligned}$$

Abstraction to Symmetric Spaces

G^σ-equivariance

Let $g \in G^\sigma$. If $\tilde{u}^{(i)} = g \cdot u^{(i)}$, $i = 1, 2, \ldots, m$, and if g is sufficiently close to the identity, then

$$\mathcal{I}\tilde{u}(x) = g \cdot \mathcal{I}u(x)$$

for every $x \in \Omega$.

Symmetry under geodesic reflection

If $\tilde{u}^{(i)} = s_\eta(u^{(i)})$, $i = 1, 2, \ldots, m$, then

$$\mathcal{I}\tilde{u}(x) = s_\eta(\mathcal{I}u(x))$$

for every $x \in \Omega$, where $s_\eta : \mathcal{S} \to \mathcal{S}$ denotes the geodesic reflection about η.

Connection with Geodesic Interpolation

Interpolation formula:

$$\mathcal{I}u(x) = F\left(\sum_{i=1}^{m} \phi_i(x) F^{-1}(u^{(i)})\right),$$

where $F(P) = \exp(P) \cdot \eta$. Interpolation formula (generalized):

$$\mathcal{I}_{\bar{g}}u(x) = F_{\bar{g}}\left(\sum_{i=1}^{m} \phi_i(x) F_{\bar{g}}^{-1}(u^{(i)})\right),$$

where $F_{\bar{g}}(P) = \bar{g}\exp(P) \cdot \eta$.

1. By allowing $\bar{g}$ to vary with x, we may define $\bar{g}(x)$ implicitly via
$$\mathcal{I}_{\bar{g}(x)}u(x) = \bar{g}(x) \cdot \eta.$$
2. The resulting interpolant coincides with the ***geodesic interpolant*** [Grohs, Sander].
3. The geodesic interpolant has the advantage of being G-equivariant rather than being merely G^σ-equivariant.

Summary

- Gauge field theories exhibit gauge symmetries that impose Cauchy initial value constraints, and are also underdetermined.
- These result in degenerate field theories that can be described using multi-Dirac mechanics and multi-Dirac structures.
- Described a systematic framework for constructing and analyzing Ritz variational integrators, and the extension to Hamiltonian PDEs.
- Presented a local isomorphism between a Lie triple system and the associated symmetric space, which can be used to construct group-equivariant finite-element spaces that take values in a symmetric space.

Thank you!